Fundamental Units and Continued Fractions

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1 Introduction

The purpose of this paper is to give a completely general description of methods designed to explicitly yield fundamental units of real quadratic orders as well as parameterized forms for the discriminant. This generalizes and reveals what lies below the surface of some special cases of this phenomenon which have appeared such as the paper [7].

In [7], the author looks at field discriminants $D \equiv 1 \pmod 4$ such that the period length of the simple continued fraction expansion of $\sqrt{D}$ is 3, and gives explicit parametric representations of both the discriminant and its fundamental unit. In this paper, we present methods for such explicit representations of the discriminant and fundamental unit for *any* real quadratic *order*. Furthermore, we show that these results follow from classical results in the literature.

2 Preliminaries

We begin by defining some fundamental sequences arising from the theory of simple continued fraction expansions. First let $\alpha = \langle q_0; q_1, \ldots, q_k, \ldots \rangle$, with $q_j \in \mathbb{R}$ for $j \geq 0$ and $q_j > 0$ for $j > 0$, denote the continued fraction expansion of $\alpha \in \mathbb{R}$. We will be concerned primarily with *simple* continued fraction expansions, namely those for which $q_j \in \mathbb{Z}$ for $j \geq 0$. In particular, suppose that α is a quadratic irrational, namely a real number of the form

$$\alpha = \frac{P + \sqrt{D}}{Q}, \ (P, Q \in \mathbb{Z}),$$

where $q \neq 0$ and $P^2 \equiv D \pmod Q$. Then α is known to have a *periodic* simple continued fraction expansion, which means that there exists an integer $n \geq 0$ and $\ell \in \mathbb{N}$ such that $q_k = q_{k+\ell}$ for all $k \geq n$. We use the notation

$$\alpha = \langle q_0; q_1, \ldots, q_{n-1}, \overline{q_n, q_{n+1}, \ldots, q_{\ell+n-1}} \rangle,$$

as an abbreviation. If ℓ is the smallest such natural number, then $\ell = \ell(\alpha)$ is called the *period length* of α. If $n = 0$, then in the above, then α is said to be *purely periodic*. One may view the above from the perspective of a pair of

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recursively defined sequences as follows. Set $P = P_0$, $Q = Q_0$, $\alpha = \alpha_0$, and define for $j \geq 0$:

$$\alpha_j = \frac{P_j + \sqrt{D}}{Q_j},$$

$$q_j = \lfloor \alpha_j \rfloor, \tag{2.1}$$

$$P_{j+1} = q_j Q_j - P_j, \tag{2.2}$$

and

$$Q_{j+1} = \frac{D - P_{j+1}^2}{Q_j}. \tag{2.3}$$

Then $\alpha = \langle q_0; q_1, \ldots, \rangle$. For instance,

$$\sqrt{D} = \langle q_0; \overline{q_1, q_2, \ldots, q_{\ell-1}, 2q_0} \rangle, \tag{2.4}$$

and if $D \equiv 1 \pmod 4$, then

$$\frac{1+\sqrt{D}}{2} = \langle q_0; \overline{q_1, q_2, \ldots, q_{\ell-1}, 2q_0 - 1} \rangle, \tag{2.5}$$

where $q_j = q_{\ell-j}$ for $j \in \mathbb{N}$, (see [2, Corollary 5.3.1, p.242]).

We may view the convergents of continued fractions from another pair of sequences. Define two sequences $\{A_j\}$ and $\{B_j\}$ by

$$A_{-2} = 0,\ A_{-1} = 1,\ A_j = q_j A_{j-1} + A_{j-2}, \tag{2.6}$$

and

$$B_{-2} = 1,\ B_{-1} = 0,\ B_j = q_j B_{j-1} + B_{j-2}. \tag{2.7}$$

Then $C_j = A_j / B_j$ is the j^{th} convergent in the simple continued fraction expansion of α (see [2, Theorem 5.1.2, p. 224]). We will also have need of the following fact. If $\alpha = \langle q_0; q_1, \ldots \rangle$ is an infinite simple continued fraction expansion and $\alpha_k \in \mathbb{R}^+$, then

$$\alpha = \langle q_0; q_1, \ldots, q_{k-1}, \alpha_k \rangle = \frac{\alpha_k A_{k-1} + A_{k-2}}{\alpha_k B_{k-1} + B_{k-2}}, \tag{2.8}$$

(see [2, Exercise 5.2.19, p. 238]).

3 Kraitchik's Methods

The results on parameterization of discriminants $D \equiv 1 \pmod 4$ with $\sqrt{D}$ having simple continued fraction expansion period length 3, given in [7], appeared six years earlier in [4]. Moreover, period lengths 4 and 5 were handled in [5]-[6], where we looked at some related class number one problems. However, these parameterizations themselves are not new. In fact, they are simple consequences of far more general results, which follow from methods developed by Kraitchik [1] in the 1920's. In this section, we describe Kraitchik's methods, since they are

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By (2.4)-(2.5),

$$\omega_D = \langle q_0;$$

where

$$\sigma = \{$$

Therefore,

$$\omega_D - q_0 = \langle 0; \overline{q_1, q_2, \ldots, q_{\ell-}}$$

so by (2.8), if

then

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$$q_0$$

and by comparing the coeffic

and

$$(q_0(\omega_D + \omega_D') - \omega$$

$$q_0$$

From (3.1)-(3.2), we get

$$D = \frac{\sigma^2}{B_{\ell-1}}(2q_0 A_{\ell-1} + (1 -$$

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$$\frac{{}_j + \sqrt{D}}{Q_j},$$

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$$Q_j - P_j, \tag{2.1}$$

$$\frac{- P_{j+1}^2}{Q_j}. \tag{2.2}$$

(2.3)

$$\overline{\cdots, q_{\ell-1}, 2q_0}\rangle, \tag{2.4}$$

$$\overline{\cdots, q_{\ell-1}, 2q_0 - 1}\rangle, \tag{2.5}$$

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ot widely known, but clearly there is a need for such knowledge in the current iterature.

Let $D \in \mathbb{N}$, which is not a perfect square. We begin by setting:

$$\omega_D = \begin{cases} (1 + \sqrt{D})/2 & \text{if } D \equiv 1 \pmod 4, \\ \sqrt{D} & \text{otherwise.} \end{cases}$$

$\omega_D' = \frac{1-\sqrt{D}}{2}$ or $-\sqrt{D}$

Ne now show how Kraitchik's method allows for the parameterization of D.

By (2.4)-(2.5),

$$\omega_D = \langle q_0; \overline{q_1, q_2, \ldots, q_{\ell-1}, 2q_0 - \sigma + 1}\rangle,$$

where

$$\sigma = \begin{cases} 2 & \text{if } D \equiv 1 \pmod 4, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore,

$$\omega_D - q_0 = \langle 0; \overline{q_1, q_2, \ldots, q_{\ell-1}, 2q_0 - \sigma + 1}\rangle = \langle 0; \overline{q_1, q_2, \ldots, q_{\ell-1}, q_0 - \omega_D'}\rangle,$$

so by (2.8), if

$$\frac{A_j}{B_j} = \langle 0; q_1, \ldots, q_j\rangle,$$

then

$$\omega_D - q_0 = \frac{(q_0 - \omega_D')A_{\ell-1} + A_{\ell-2}}{(q_0 - \omega_D')B_{\ell-1} + B_{\ell-2}}.$$

By cross-multiplying and rearranging, we get

$$(q_0(\omega_D + \omega_D') - \omega_D\omega_D' - q_0^2)B_{\ell-1} - q_0 B_{\ell-2} + B_{\ell-2}\omega_D =$$
$$q_0 A_{\ell-1} + A_{\ell-2} - \omega_D' A_{\ell-1},$$

and by comparing the coefficients of $\sqrt{D}$ and 1 we get,

$$B_{\ell-2} = A_{\ell-1}, \tag{3.1}$$

and

$$(q_0(\omega_D + \omega_D') - \omega_D\omega_D' - q_0^2)B_{\ell-1} + \frac{\sigma - 1}{\sigma}B_{\ell-2} - q_0 B_{\ell-2} =$$
$$q_0 A_{\ell-1} + \frac{1 - \sigma}{\sigma}A_{\ell-1} + A_{\ell-2}. \tag{3.2}$$

From (3.1)-(3.2), we get

$$D = \frac{\sigma^2}{B_{\ell-1}}(2q_0 A_{\ell-1} + (1 - \sigma)A_{\ell-1} + A_{\ell-2}) + \sigma^2 q_0^2 - 2(\sigma - 1)\sigma q_0 + \sigma - 1. \tag{3.3}$$

To see how the representation of D given in (3.3) is sufficient to yield a parameterization for any simple continued fraction expansion, we look at the case considered in [7].

Example 3.4 *Consider $D \in \mathbb{N}$ squarefree such that*

$$\omega_D = \frac{1+\sqrt{D}}{2} = \langle q_0; \overline{q, q, 2q_0 - 1} \rangle,$$

Here, $q_0 = \lfloor \sqrt{D} \rfloor$. $D = (2q_0 - 1)^2 + b$, *for some positive* $b \leq 2a$, $\ell(\omega_D) = \ell = 3$, *and* $\sigma = 2$. *Thus,*

$$\omega_D - q_0 = \langle 0; q_1, q_1, q_0 - \omega'_D \rangle = \langle 0; q_1, q_1, q_0 - 1 + \omega_D \rangle.$$

We may now use (2.1)–(2.3), *and* (2.6)–(2.7) *to tabulate the following.*

Table 3.5

i	-2	-1	0	1	2	3
P_j			1	$2q_0 - 1$	$q_1 Q_1 - 2q_0 + 1$	$2q_0 - 1$
Q_j			2	Q_1	Q_1	2
q_j			q_0	q_1	q_1	$2q_0 - 1$
A_j	0	1	0	1	q_1	$(2q_0 - 1)q_1 + 1$
B_j	1	0	1	q_1	$q_1^2 + 1$	$(2q_0 - 1)(q_1^2 + 1) + q_1$

We may use (2.3) *to determine* Q_1 *as follows,*

$$D = (2q_0 - 1)^2 + 2Q_1 = (q_1 Q_1 - 2q_0 + 1)^2 + Q_1^2,$$

so by simplifying and rearranging the second equality, we get

$$Q_1 = \frac{2}{q_1^2 + 1}(2q_0 q_1 - q_1 + 1)$$

We may now invoke (3.3) *to get:*

$$D = Q_0 Q_1 + P_1^2 = 4\left(\frac{2q_0 q_1 - q_1 + 1}{q_1^2 + 1}\right) + (2q_0 - 1)^2. \tag{3.6}$$

Thus, $(q_1^2 + 1) \mid (2q_0 q_1 - q_1 + 1)$. *Set* $(2q_0 q_1 - q_1 + 1)/(q_1^2 + 1) = c \in \mathbb{Z}$. *Then it follows that*

$$r = \frac{2q_0 - 1 - q_1}{q_1^2 + 1} = 2q_0 - 1 - q_1 c \in \mathbb{Z}. \tag{3.7}$$

By plugging this into (3.6), *and rearranging, we get*

$$D = (q_1^2 + 1)^2 r^2 + 2q_1(q_1^2 + 3)r + q_1^2 + 4, \tag{3.8}$$

which is precisely the parameterization found in [7]. *In fact, the methods used in* [7] *are just versions of Kraitchik's methods for this special case. The case where* $\lfloor \sqrt{D} \rfloor$ *is even is similarly handled, and it too is known* (see [3, Theorem 3.2.1, p. 78]).

In the next section, we show how classical methods can be used to explicitly find the fundamental unit.

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$D \in \mathbb{N}$ *squarefree such that*

$$= \frac{1+\sqrt{D}}{2} = \langle q_0; \overline{q, q, 2q_0 - 1} \rangle,$$

$q_0 - 1)^2 + b$, *for some positive* $b \leq 2a$, $\ell(\omega_D) = \ell = 3$,

$$q_1, q_1, q_0 - \omega'_D \rangle = \langle 0; q_1, q_1, q_0 - 1 + \omega_D \rangle.$$

.3), and (2.6)–(2.7) to tabulate the following.

Table 3.5

1	2	3
$2q_0 - 1$	$q_1Q_1 - 2q_0 + 1$	$2q_0 - 1$
Q_1	Q_1	2
q_1	q_1	$2q_0 - 1$
1	q_1	$(2q_0 - 1)q_1 + 1$
q_1	$q_1^2 + 1$	$(2q_0 - 1)(q_1^2 + 1) + q_1$

rmine Q_1 *as follows,*

$$1)^2 + 2Q_1 = (q_1Q_1 - 2q_0 + 1)^2 + Q_1^2,$$

anging the second equality, we get

$$= \frac{2}{q_1^2+1}(2q_0q_1 - q_1 + 1)$$

get:

$$= 4\left(\frac{2q_0q_1 - q_1 + 1}{q_1^2 + 1}\right) + (2q_0 - 1)^2. \qquad (3.6)$$

1). Set $(2q_0q_1 - q_1 + 1)/(q_1^2 + 1) = c \in \mathbb{Z}$. *Then*

$$\frac{1 - q_1}{+1} = 2q_0 - 1 - q_1c \in \mathbb{Z}. \qquad (3.7)$$

rearranging, we get

$$^2r^2 + 2q_1(q_1^2 + 3)r + q_1^2 + 4, \qquad (3.8)$$

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w classical methods can be used to explicitly

4 Explicit Fundamental Units

We need one more recursively defined sequence to augment the ones given in Section Two. Let $\alpha = (P_0 + \sqrt{D})/Q_0$ be a quadratic irrational. For $j \geq -1$, define

$$G_{j-1} = Q_0A_{j-1} - P_0B_{j-1},$$

where the A_j and B_j arise from the simple continued fraction expansion of α.

Suppose that $\alpha = \omega_D$, and $\ell(\alpha) = \ell$. Then from the classical theory of continued fractions (for instance, see [3, p. 48]), the fundamental unit ε_D of the real quadratic order $\mathbb{Z}[\omega_D]$ is given by:

$$\varepsilon_D = \left| \frac{G_{(\ell-1)/2} + B_{(\ell-1)/2}\sqrt{D}}{B_{(\ell-3)/2}\sqrt{D} - G_{(\ell-3)/2}} \right|, \qquad (4.1)$$

when ℓ is odd, and

$$\varepsilon_D = \left| \frac{G_{\ell/2-1} - B_{\ell/2-1}\sqrt{D}}{G_{\ell/2-1} + B_{\ell/2-1}\sqrt{D}} \right|,$$

if ℓ is even. Now let us apply this to Example 3.4. Since $\ell = 3$ in that case, it is easy to verify that $G_1 = q_1(2q_0 - 1) + 2$, $B_1 = q_1$, $B_0 = 1$ and $G_0 = 2q_0 - 1$. Thus, for such D,

$$\varepsilon_D = \frac{G_1 + B_1\sqrt{D}}{B_0\sqrt{D} - G_0} = \frac{q_1(2q_0 - 1) + 2 + q_1\sqrt{D}}{\sqrt{D} - 2q_0 + 1} =$$

$$\frac{(q_1(2q_0 - 1) + 2 + q_1\sqrt{D})(\sqrt{D} + 2q_0 - 1)}{D - (2q_0 - 1)^2} =$$

$$\frac{(2q_0 - 1)^2q_1 + 2(2q_0 - 1) + q_1D + (2q_1(2q_0 - 1) + 2)\sqrt{D}}{D - (2q_0 - 1)^2},$$

and by using (3.7)-(3.8), this equals:

$$\frac{(q_1^2 + 1)^2r + q_1(q_1^2 + 3) + (q_1^2 + 1)\sqrt{D}}{2},$$

which is the parameterization given in [7], wherein the author cites results from the last quarter century. However, this is misleading since, as shown above, all of this was already known by the 1920's. The author of [7] also lists those field discriminants with class number one and period length 3. However, this and much more is known and tabulated up to period length 24 (see [3, Table A1, pp. 271–272]).

We conclude with an explicit example to illustrate the above.

Example 4.2 *Let* $D = 317$ *for which* $\sqrt{317} = \langle 9; \overline{2, 2, 17} \rangle$, *so* $\ell = 3$. *We calculate that* $A_0 = 9$, $A_1 = 19$, $B_0 = 1$, $B_1 = 2$, $G_0 = 17$ *and* $G_1 = 36$. *Thus, by* (4.1),

$$\varepsilon_D = \frac{G_1 + B_1\sqrt{D}}{B_0\sqrt{D} - G_0} = \frac{36 + 2\sqrt{317}}{\sqrt{317} - 17} = \frac{89 + 5\sqrt{317}}{2}.$$

Moreover, the reader may verify the above parameterizations for D and ε_D using the values $q_1 = 2$ and $r = 3$.

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