

## Riemann's $\zeta$ -Function

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### Abstract

We provide an overview of the Riemann  $\zeta$ -function with an eye to a proof of the functional equation that is non-standard and more accessible than those proofs readily available in modern texts. This is a survey note with the goal of a more elegant and transparent presentation of the salient features surrounding the functional equation for the  $\zeta$ -function and the intimate connections with the Riemann hypothesis.

Strikingly, Riemann published exactly one paper [3] on number theory, appearing in 1859. This landmark paper changed the perspective of the subject immediately. His approach was elegant and profound in its approach to the distribution of prime numbers. He basically took Cauchy's theory of holomorphic functions and appealed to the process, discovered by Cauchy, of looking at each point of an analytic function as an average of its values at neighbouring points. This was calculated as a path integral along a closed curve in  $\mathbb{C}^2$ , which permitted deformations of the path of integration allowing an approximation of the integral.

Riemann took Euler's formula, described below, for primes and extended it to  $\mathbb{C}$ . This was accomplished via the Riemann  $\zeta$ -function given for  $s = a + bi \in \mathbb{C}$  with  $Re(s) = a > 1$  by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p=prime} (1 - p^{-s})^{-1}. \quad (1)$$

The last equality is known as the *Euler product*, which provides a fundamental relationship between the primes and the zeta function. The series on the left is absolutely convergent, which implies that  $\zeta(s)$  is analytic in the half plane  $Re(s) > 1$ . To show that  $\zeta(s)$  may be analytically continued to the

entire complex plane (with one singularity) this we may employ the following well-known result from any standard text in analysis. The symbol  $B_j$  denotes the  $j$ th Bernoulli number in what follows, where the Taylor series for  $x \in \mathbb{C}$  is given by

$$\frac{x}{e^x - 1} = \sum_{j=0}^{\infty} \frac{B_j x^j}{j!}.$$

Also,  $f^{(j)}(x)$  will denote the  $j$ th derivative of a given function  $f(x)$ . Some of the following is adapted from [2] and all relevant background material may be found in most texts on introductory number theory such as [1], for instance.

**Theorem 1** **The Euler-Maclaurin Sum Formula**

Let  $a < b$  are integers and  $n \in \mathbb{N}$ . If  $f(x)$  has a continuous derivative for  $x \in [a, b]$ , then

$$\begin{aligned} \sum_{j=a+1}^b f(j) &= \int_a^b f(x) dx + \sum_{i=1}^n (-1)^i \frac{B_i}{i!} (f^{(i-1)}(b) - f^{(i-1)}(a)) \\ &\quad + \frac{(-1)^{n-1}}{n!} \int_a^b B_n(x - [x]) f^{(n)}(x) dx. \end{aligned}$$

**Theorem 2** For  $s \in \mathbb{C}$  and  $\operatorname{Re}(s) > 1 - n$ , for  $n \in \mathbb{N}$ ,  $\zeta(s)$  is convergent, except at  $s = 1$ , and

$$\begin{aligned} \zeta(s) &= \frac{1}{s-1} + \frac{1}{2} + \sum_{j=2}^n \frac{B_j}{j!} s(s+1) \cdots (s+j-2) \\ &\quad - \frac{1}{n!} s(s+1) \cdots (s+n-1) \int_1^{\infty} B_n(t - [t]) t^{-s-n} dt. \end{aligned} \quad (2)$$

*Proof.* Let  $n \in \mathbb{N}$  and set  $f(x) = x^{-s}$ ,  $a = 1$ , and  $b = N$  in Theorem 1. Since

$$f^{(n)}(x) = (-1)^n s(s+1)(s+2) \cdots (s+n-1) x^{-s-n}$$

and

$$\zeta(s) = 1 + \lim_{N \rightarrow \infty} \sum_{j=2}^N f(j).$$

Therefore,

$$\zeta(s) - 1 = \lim_{N \rightarrow \infty} \left[ \int_1^N x^{-s} dx - \sum_{j=1}^N \frac{B_j}{j!} s(s+1) \cdots (s+j-2) (N^{-s+1-j} - 1) \right]$$

$$\begin{aligned}
 & \left[ -\frac{1}{n!} s(s+1) \cdots (s+n-1) \int_1^N B_n(x - [x]) x^{-s-n} dt \right] \\
 = & \lim_{N \rightarrow \infty} \left[ \frac{N^{1-s} - 1}{1-s} + \frac{N^{-s} - 1}{2} - \sum_{j=2}^n \frac{B_j}{j!} s(s+1) \cdots (s+j-2) (N^{-s-j+1} - 1) \right. \\
 & \left. - \frac{1}{n!} s(s+1) \cdots (s+n-1) \int_1^N B_n(x - [x]) x^{-s-n} dx \right].
 \end{aligned}$$

For  $Re(s) > 1$ , we may pull the limit through, so since

$$\lim_{N \rightarrow \infty} N^{1-s} = 0 = \lim_{N \rightarrow \infty} N^{-s},$$

we get (2), the right-hand side of which converges for  $Re(s) > 1 - n$ , except at  $s = 1$ . □

Now we turn to the relationship between the Riemann  $\zeta$ -function and the distribution of primes. We begin by reminding the reader that  $\pi(x)$  denotes the number of primes less than  $x$ . The first celebrated result is our starting point.

**Theorem 3** **The Prime Number Theorem (PNT)**

For  $x \in \mathbb{R}^+$ ,

$$\pi(x) \sim \frac{x}{\log_e x}.$$

The close relationship between the Riemann  $\zeta$ -function and  $\pi(x)$  is given by

$$\log_e \zeta(s) = s \int_2^\infty \frac{\pi(x)}{x(x^s - 1)} dx, \tag{3}$$

for  $Re(s) > 1$ .

It is noteworthy that the Euler product (1) for the Riemann  $\zeta$ -function tells us that since  $\zeta(s) \rightarrow \infty$  as  $s \rightarrow 1$  then there are infinitely many primes. To see this consider  $\zeta(s)$  for  $s \in \mathbb{R}^+$ . By the series expansion in (1),  $\zeta(s)$  diverges as  $s \rightarrow 1^+$  since the harmonic series

$$\sum_{j=1}^\infty 1/j$$

diverges. Actually, more can be said. Since

$$\log_e \left( \prod_{p=prime} (1 - p^{-s})^{-1} \right) = - \sum_{p=prime} \log_e(1 - p^{-s})$$

$$= \sum_{p=\text{prime}} p^{-s} + O(1) < \sum_{p=\text{prime}} p^{-1} + O(1) \text{ when } s > 1,$$

then as  $s \rightarrow 1^+$ , the sum  $\sum_{p=\text{prime}} p^{-1}$  diverges.

Understanding the sums  $\sum_{p \leq x} p^{-1}$  is implicit in the development of Theorem 3. Indeed, the following predates the PNT, and follows from it.

**Theorem 4** Merten's Theorem

$$\sum_{p \leq x} \frac{1}{p} = \log_e \log_e x + M + o(1),$$

where  $\gamma$  is Euler's constant and

$$M = \gamma + \sum_{p=\text{prime}} \left( \log_e \left( 1 - \frac{1}{p} \right) + \frac{1}{p} \right),$$

called Merten's constant.

Note that Theorem 4 is equivalent to the asymptotic relationship

$$\prod_{p \leq x} \left( 1 - \frac{1}{p} \right) \sim \frac{e^{-\gamma}}{\log_e x}.$$

There is an equivalent formulation of Theorem 3 via the following function called *Merten's function*.

$$M(x) = \sum_{n \leq x} \mu(x),$$

for any  $x \in \mathbb{R}$ , where  $\mu$  is the Möbius function. It can be shown that Theorem 3 is equivalent to the following.

**Theorem 5** Merten's Equivalence to the PNT

$$M(x) = o(x).$$

Even more, Theorem 3 is also equivalent to the following.

**Theorem 6** Möbius' Equivalence to the PNT

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0.$$

The relationship between the Riemann  $\zeta$ -function and Merten's function is given by

$$\frac{1}{\zeta(s)} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^s} = s \int_1^{\infty} \frac{M(x)}{x^{s+1}} dx,$$

for  $\operatorname{Re}(s) > 1$ . This brings us to one of the most important and celebrated outstanding problems.

**Conjecture 1** **The Riemann Hypothesis (RH)**

*All of the zeros of  $\zeta(s)$  in the critical strip  $0 < \operatorname{Re}(s) < 1$  lie on the critical line  $\operatorname{Re}(s) = 1/2$ .*

In terms of the Merten's function we may reformulate Conjecture 1 as being equivalent to the following.

**Conjecture 2** **Merten's Equivalence to the RH**

$$M(x) = O_{\varepsilon} \left( x^{\frac{1}{2} + \varepsilon} \right),$$

for any fixed  $\varepsilon > 0$ , where  $O_{\varepsilon}$  means that, in the big  $O$  notation, the constant depends on  $\varepsilon$  only.

Also, Riemann conjecture the following in 1859, which is also equivalent to Conjecture 1.

**Conjecture 3** **Integral Equivalence to the RH**

$$\pi(x) = \operatorname{li}(x) + O \left( \sqrt{x} \log_e x \right),$$

where

$$\operatorname{li}(x) = \int_2^{\infty} \frac{dt}{\log_e t},$$

called the logarithmic integral.

Conjecture 3 will hold if and only if the Riemann  $\zeta$ -function does not vanish on the half plane  $\operatorname{Re}(s) > 1/2$ . In other words, Conjecture 1 is equivalent to the statement that the error which occurs, when  $\pi(x)$  is estimated by  $\operatorname{li}(x)$ , is  $O(\sqrt{x} \log_e x)$ .

Now we are in a position to establish a fundamental equation, which puts the above more in focus. Indeed, the following shows the central role that the Riemann  $\zeta$ -function plays in analytic number theory via the *functional equation*,  $\zeta(s) = f(s)\zeta(1-s)$ , where we define  $f(s)$  below.

Let  $n = 3$  in (2) to get,

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \frac{B_2 s}{2} - \frac{s(s+1)(s+2)}{6} \int_1^\infty B_3(t - [t])t^{-s-3} dt,$$

for  $\operatorname{Re}(s) > -2$ . An exercise involving integration by parts three times and the Bernoulli relationship

$$B'_{n+1}(x) = (n+1)B_n(x),$$

yields

$$\frac{s(s+1)(s+2)}{6} \int_0^1 B_3(t - [t])t^{-s-3} dt = -\frac{B_2 s}{2} - \frac{1}{2} - \frac{1}{s-1}, \quad (4)$$

so

$$\zeta(s) = -\frac{s(s+1)(s+2)}{6} \int_0^\infty B_3(t - [t])t^{-s-3} dt.$$

Replacing  $B_3(t - [t])$  by the Fourier series,

$$B_{2k-1}(x - [x]) = (-1)^k 2(2k-1)! \sum_{j=1}^\infty \frac{\sin(2\pi j x)}{(2\pi j)^{2k-1}}, \text{ for } k \geq 2,$$

we get

$$\zeta(s) = -\frac{s(s+1)(s+2)}{6} \sum_{j=1}^\infty 12 \int_0^\infty \frac{\sin(2\pi j t)}{(2\pi j)^3} t^{-s-3} dt$$

and by setting  $x = 2\pi j t$ , this equals,

$$-2s(s+1)(s+2) \sum_{j=1}^\infty \frac{1}{(2\pi j)^{1-s}} \int_0^\infty x^{-s-3} \sin x \, dx.$$

Since

$$\sin x = \sum_{i=0}^\infty (-1)^i \frac{x^{2i+1}}{(2i+1)!}$$

is the analytic continuation of the usual trigonometric function, and converges for all  $x \in \mathbb{C}$ , then we may interchange the sum and integral above so the latter equals

$$-2^{-s} \pi^{s-1} s(s+1)(s+2) \left( \int_0^\infty x^{-s-3} \sin x \, dx \right) \sum_{j=1}^\infty \frac{1}{j^{1-s}}.$$

Hence,

$$\zeta(s) = -2^{-s} \pi^{s-1} s(s+1)(s+2) \left( \int_0^\infty x^{-s-3} \sin x \, dx \right) \cdot \zeta(1-s). \quad (5)$$

In order to complete the derivation of the functional equation, we need the following concept due to Euler.

**Definition 1** **The Gamma Function**

For  $s \in \mathbb{C}$  and  $\operatorname{Re}(s) > 0$  the gamma function is given by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt.$$

We will employ two well-known formulas for the gamma function given as follows.

For  $0 < \operatorname{Re}(z) < 1$ , the *Wolfskehl equation* is given by

$$\left(\sin \frac{\pi z}{2}\right) \cdot \Gamma(1 - z) = z \int_0^\infty y^{-z-1} \sin y \, dy, \tag{6}$$

a formula known since 1886—see [5], and

$$(-z)\Gamma(-z) = \Gamma(1 - z). \tag{7}$$

Now we are ready for the functional equation.

**Theorem 7** **Riemann's Functional Equation for  $\zeta(s)$**

For  $s \in \mathbb{C}$ ,

$$\zeta(s) = 2^s \pi^{-s} \Gamma(1 - s) \zeta(1 - s) \cdot \left(\sin \frac{\pi s}{2}\right).$$

*Proof.* From (5), we have to show only that

$$-s(s + 1)(s + 2) \int_0^\infty x^{-s-3} \sin x \, dx = \left(\sin \frac{\pi s}{2}\right) \Gamma(1 - s).$$

To this end, we employ (6)–(7) as follows,

$$\begin{aligned} -s(s + 1)(s + 2) \int_0^\infty x^{-s-3} \sin x \, dx &= s(s + 1) \left(\sin \frac{\pi s}{2}\right) \Gamma(1 - (s + 2)) \\ &= s(s + 1) \left(\sin \frac{\pi s}{2}\right) \Gamma(-1 - s) = -s \left(\sin \frac{\pi s}{2}\right) (-(s + 1)\Gamma(-(s + 1))) \\ &= -s \left(\sin \frac{\pi s}{2}\right) \Gamma(-s) = \left(\sin \frac{\pi s}{2}\right) (-s\Gamma(-s)) = \left(\sin \frac{\pi s}{2}\right) \Gamma(1 - s), \end{aligned}$$

and we have our functional equation. □

**Remark 1** *The functional equation is valid for all complex numbers  $s$  where both sides are defined. We know that  $\zeta(s)$  has no zeros for  $\Re(s) \geq 1$  and has only trivial zeros for  $\Re(s) \leq 0$ , which correspond to poles of  $\Gamma(s/2)$ , and has infinitely many zeros on the critical strip  $0 < \Re(s) < 1$ . We may define a related function, which shows symmetry properties more readily than does Theorem 7. If we define*

$$\xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s),$$

*then it may be shown that*

$$\xi(s) = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) = \xi(1-s) \quad (8)$$

*showing that  $\xi(s)$  is symmetric about the critical line  $\Re(s) = 1/2$ . Note that  $\xi(s)$  is analytic on the whole plane (such functions are called entire), since the factor of  $s-1$  eliminates the pole of  $\zeta(s)$  at  $s=1$ . (Often  $\xi(s)$  is called the completed zeta function.) The functional equation given in Theorem 7 shows that if  $s$  is a zero in the critical strip, then so is  $1-s$ , since by Theorem 2, zeros occur in complex conjugate pairs. So if it were to be (incredibly) that the Riemann hypothesis is false, then zeros in the critical strip that are not on the critical line would occur in four-tuples corresponding to vertices of rectangles in the complex plane.*

*The zeros of the  $\zeta$ -function are intimately connected with the distribution of primes. If  $\mathfrak{U}$  denotes the upper bound of the real parts of the zeros of  $\zeta(s)$ , with  $1/2 \leq \mathfrak{U} \leq 1$ , then  $|\pi(x) - li(x)| \leq cx^{\mathfrak{U}} \log_e x$  for a constant  $c \in \mathbb{R}^+$ . The Riemann hypothesis is tantamount to  $\mathfrak{U} = 1/2$ .*

*Furthermore, as discussed on page ?? in reference to transcendence, the values of  $\zeta(2n+1)$  are largely a mystery, with the exception of a notable—Apéry's constant*

$$\zeta(3) = \frac{5}{2} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j^3 \binom{2j}{j}} \notin \mathbb{Q}.$$

**Remark 2** *Another rather interesting application of the  $\zeta$ -function is the following. It can be shown that the probability of two randomly selected integers being relatively prime is approximately equal to  $1/\zeta(2) = 6/\pi^2 = 0.608\dots$ . This is also the probability that a randomly selected integer is squarefree. The reason the latter is true, in terms of the Riemann  $\zeta$ -function given by the Euler product in (1), is that for a number to be squarefree it must not be divisible by the same prime more than once. In other words, either it is not divisible by  $p$  or it is divisible by  $p$  but not divisible by it again. Thus, the probability that an integer is not divisible by the square of a prime  $p$  equals*

$$\left(1 - \frac{1}{p}\right) + \frac{1}{p} \left(1 - \frac{1}{p}\right) = 1 - \frac{1}{p^2},$$

and taking the product over all primes (assuming the independence of the divisibility by different primes) the probability then that an integer is squarefree tends to

$$\prod_{p=\text{prime}} (1 - p^{-2}) = \zeta(2)^{-1}.$$

This has a generalization, which can be shown by the same reasoning as for  $n = 2$ , namely that the probability that  $n$  randomly selected integers are coprime is

$$P_n \sim \zeta(n)^{-1} \tag{9}$$

Thus, we may calculate  $P_3 \sim 0.832$ ,  $P_4 \sim 90/\pi^2 \sim 0.9239$ , and so forth. Again using similar reasoning to the above, the probability that a randomly selected integer is cube-free, or fourth-power free, etc., is also given by (9). Thus, the probability that a randomly selected integer is cube-free equals roughly 83%, and that an integer is fourth-power free is roughly 92%.

A more general question still, and one that is not resolved at the time of this writing is the following. What is the probability that  $n$  randomly selected integers have greatest common divisor equal to  $g$ ? Let this probability be denoted by  $P_n(g)$ —see [4, page 48].

#### Conjecture 4

$$P_n(g) \sim \frac{1}{g^n \zeta(n)}.$$

We have demonstrated the validity of Conjecture 4 for  $g = 1$ , which is (9). The problem remains open.

The above is a modest journey through the great depth of the contributions by Riemann to analytic number theory, but provides an overview of some rather pleasant connections from a slightly different and perhaps more accessible perspective.

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