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Palindromy and Ambiguous Ideals Revisited

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The purpose of this article is to revisit the relationship between ambiguous ideals and palindromy in the simple continued fraction expansions of quadratic irrationals begun by the first author and A. J. van der Poorten (1995, *Bull. Austral. Math. Soc.* **51**, 215–233). We present simpler proofs of known results, new interrelationships, and correct some misinterpretations. We do this via the infrastructure of real quadratic fields. The conclusion is that palindromy is ambiguity, when properly viewed.

1. NOTATION AND PRELIMINARIES

Let $D_0 \neq 1$ be a square-free integer, and set

$$\Delta_0 = \begin{cases} D_0 & \text{if} \quad D_0 \equiv 1 \pmod 4, \\ 4D_0 & \text{otherwise.} \end{cases}$$

Then Δ_0 is called a fundamental discriminant with associated fundamental radicand D_0. Let $f_\Delta \in \mathbb{N}$, and set $\Delta = f_\Delta^2 \Delta_0$. Then

$$\Delta = \begin{cases} D & \text{if} \quad D_0 \equiv 1 \pmod 4) \text{ and } f_1 \text{ is odd}, \\ 4D & \text{otherwise,} \end{cases}$$

is a discriminant with conductor f_Δ, and associated radicand

$$D = \begin{cases} f_\Delta^2 D_0 & \text{if} \quad D_0 \not\equiv 1 \pmod 4 \text{ or } f_\Delta \text{ is odd}, \\ (f_\Delta/2)^2 D_0 & \text{otherwise,} \end{cases}$$

having underlying fundamental discriminant Δ_0 with associated fundamental radicand D_0.

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Let Δ be a discriminant with associated radicand D. Then

$$\omega_\Delta = \begin{cases} (1+\sqrt{D})/2 & \text{if} \quad \Delta = D, \\ \sqrt{D} & \text{otherwise,} \end{cases}$$

is the *principal surd* associated with Δ. This will provide the canonical basis element for certain rings that we now define.

Let $[\alpha, \beta] = \alpha\mathbb{Z} + \beta\mathbb{Z}$ be a $\mathbb{Z}$-module. Then

$$\mathcal{O}_\Delta = [1, \omega_\Delta],$$

is an *order* in $K = \mathbb{Q}(\sqrt{\Delta}) = \mathbb{Q}(\sqrt{D_0})$ with conductor f_Δ. If $f_\Delta = 1$, then $\mathcal{O}_\Delta$ is called the *maximal order in K*.

Now we bring ideal theory into the picture. Let $I = [a, b + d\omega_\Delta]$, with $a > 0$. The following tells us when such a module is an ideal (see [2, Exercise 1.2.1(a), p. 12]).

Let Δ be a discriminant, and let $I \neq (0)$ be a $\mathbb{Z}$-submodule of $\mathcal{O}_\Delta$. Then I has a representation of the form

$$I = [a, b + c\omega_\Delta],$$

where $a, c \in \mathbb{N}$ and $b \in \mathbb{Z}$ with $0 \leqslant b < a$. Furthermore, I is an ideal of $\mathcal{O}_\Delta$ if and only if this representation satisfies $c \mid a$, $c \mid b$, and $ac \mid N(b + c\omega_\Delta)$. (For convenience, we call I an $\mathcal{O}_\Delta$-ideal.) If $c = 1$, then I is called primitive, and I has a canonical representation as

$$I = [a, (b + \sqrt{\Delta})/2],$$

with $-a \leqslant b < a$.

If $I = [a, b + \omega_\Delta]$ is a primitive $\mathcal{O}_\Delta$-ideal, then a is the least positive rational integer in I, denoted $N(I) = a$, called the *norm of I*.

An $\mathcal{O}_\Delta$-ideal I is called *reduced* if there does *not* exist any element $\alpha \in I$ such that both $|\alpha| < N(I)$ and $|\alpha'| < N(I)$, where α' denotes the *algebraic conjugate* of $\alpha \in \mathcal{O}_\Delta$, namely if $\alpha = (x + y\sqrt{\Delta})/2$, then $\alpha' = (x - y\sqrt{\Delta})/2$. On the other hand, the conjugate of the ideal I is $I' = [a, b + \omega'_\Delta]$. When $I = I'$, I is said to be an *ambiguous* ideal.

It is convenient to have easily verified conditions for reduction (see [2, Exercise 1.5.9, p. 29]).

THEOREM 1.1. *Suppose that $\Delta > 0$ is a discriminant and $I = [a, b + \omega_\Delta]$ is an $\mathcal{O}_\Delta$-ideal. Then each of the following hold.*

(1) *If $N(I) < \sqrt{\Delta}/2$, then I is reduced.*
(2) *If I is reduced, then $N(I) < \sqrt{\Delta}$.*

(3) *If* $0 \leqslant b < a < \sqrt{\Delta}$ *and* $a > \sqrt{\Delta}/2$, *then I is reduced if and only if*

$$a - \omega_\Delta < b < -\omega'_\Delta.$$

(4) *If I is an ambiguous ideal, and* $N(I) < \sqrt{\Delta}$, *then either I is reduced or* $4 \mid \Delta$ *and* $\sqrt{\Delta}/2 \in I$.

Now we give an elucidation of the theory of continued fractions as it pertains to the above. Continued fraction expansions will be denoted

$$\langle a_0; a_1, a_2, ..., a_l, ...\rangle,$$

where $a_i \in \mathbb{R}$ are called the *partial quotients* of the continued fraction expansion. If $a_i \in \mathbb{Z}$, and $a_i > 0$ for all $i > 0$, then the continued fraction is called an *infinite simple continued fraction* (which is equivalent to being an irrational number), whereas if the expression terminates, then it is called a finite simple continued fraction (which is equivalent to being a rational number).

We will be discussing *quadratic irrationals* which are real numbers γ associated with a radicand D such that γ can be written in the form

$$\gamma = (P + \sqrt{D}\,)/Q,$$

where $P, Q, D \in \mathbb{Z}$, $D > 0$, $Q \neq 0$, and $P^2 \equiv D \pmod{Q}$. The following is a setup for our discussion of the continued fraction algorithm.

Suppose that $I = [a, b + \omega_\Delta]$ is a primitive ideal in $\mathcal{O}_\Delta$. Then we define the following for the quadratic irrational $\gamma = (b + \omega_\Delta)/a$,

$$(P_0, Q_0) = \begin{cases} (2b+1, 2a) & \text{if} \quad D \equiv 1 \pmod 4, \\ (b, a) & \text{otherwise,} \end{cases} \tag{1.1}$$

and (for $i \geqslant 0$),

$$D = P_{i+1}^2 + Q_i Q_{i+1}, \tag{1.2}$$

$$P_{i+1} = a_i Q_i - P_i, \tag{1.3}$$

and

$$a_i = \lfloor (P_i + \sqrt{D})/Q_i \rfloor, \tag{1.4}$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x, namely the *floor* of x. Therefore, $\gamma = \langle a_0; a_1, ..., a_i, ...\rangle$ is the simple continued fraction expansion of γ.

Remark 1.1 The *simple* continued fraction expansion of a quadratic irrational γ is called *purely periodic*, provided that there is an integer $l \in \mathbb{N}$

such that $\gamma = \langle a_0; \overline{a_1, a_2, ..., a_l} \rangle = \langle \overline{a_0; a_1, a_2, ..., a_{l-1}} \rangle$. The value $l = l(\gamma)$ is called the *period length* of the simple continued fraction expansion of γ. Furthermore, quadratic irrationals are purely periodic if and only if they are *reduced*, namely a quadratic irrational γ is purely periodic if and only if $\gamma > 1$ and $-1 < \gamma' < 0$ (see [2, Theorem 2.1.1, pp. 42–43] for details).

In what follows we need the notion of equivalence of ideals. Two ideals I and J of $\mathcal{O}_\Delta$ are *equivalent* (denoted by $I \sim J$) if there exist non-zero $\alpha, \beta \in \mathcal{O}_\Delta$ such that $(\alpha) I = (\beta) J$ (where (x) denotes the principal ideal generated by x). For a discriminant Δ, the *class group* of $\mathcal{O}_\Delta$ determined by these equivalence classes is denoted $\mathcal{C}_\Delta$, with order h_Δ, the *class number* of $\mathcal{O}_\Delta$. The following is fundamental to the discussion (see [1, Theorem 5.5.2, pp. 261–266]). The following relationship between the ideals and continued fractions was dubbed the *infrastructure* of a real quadratic field by Dan Shanks.

THEOREM 1.2. (The Continued Fraction Algorithm). *Let $\Delta > 0$ be a discriminant, and let $I = I_1 = [a, b + w_\Delta]$ be a primitive ideal in the order $\mathcal{O}_\Delta$. Set $P = P_0$ and $Q = Q_0$, as defined in Eq.* (1.1), *and let P_i and Q_i for $i > 0$ be defined by Eqs.* (1.2)–(1.4) *in the simple continued fraction expansion of $\gamma = \gamma_0 = (P + \sqrt{D})/Q$. Let $\sigma = 2$ if $D \equiv 1 \pmod 4$, and $\sigma = 1$ otherwise. If $I_i = [Q_{i-1}/\sigma_i (P_{i-1} + \sqrt{D})/\sigma]$, then $I_1 \sim I_i$ for all $i \geqslant 1$, and there exists a least value $m \geqslant 1$ such that I_{m+i} is reduced for all $i \geqslant 0$.*

There is also another useful fact that we will exploit in the next section.

THEOREM 1.3. *Suppose that $D > 0$ is a radicand, and $\ell(\sqrt{D}) = \ell$ with the Q_j defined for the simple continued fraction expansion of $\sqrt{D}$ as in Equations* 1.1–1.4. *Then $Q_j \mid 2D$ with $0 < j < \ell$ if and only if $j = \ell/2$. Furthermore, if D is even, then $Q_j \mid D$ with $0 < j < \ell$ if and only if $j = \ell/2$. In either case, $a_{\ell/2} = 2P_{\ell/2}/Q_{\ell/2}$, and $I_{\ell/2}$ is ambiguous.*

Proof. See [2, Theorem 6.1.4, p. 193]. ▌

2. RESULTS

In [3], we explored the relationship between palindromy of the partial quotients in the simple continued fraction expansion of a reduced quadratic irrational, and the related ambiguous ideals in its cycle as given by Theorem 1.2. This was given an in-depth analysis in [2, Chap. 6, pp. 187–221]. However, only one type of palindromy was named in [2, Definition 6.1.4, p. 194] (see Definition 2.2 below). We now name the other type of palindromy discussed in [2, Remark 6.1.3, p. 195], and determine its interrelationship with ambiguous ideals. The notation of Section 1 is in force throughout.

DEFINITION 2.1. A reduced quadratic irrational γ is said to have *pure skew-symmetric* period if

$$\gamma = \langle \overline{q_0; q_1, ..., q_{\ell-1}} \rangle,$$

where

$$q_j = q_{\ell-j} \qquad \text{for} \quad j = 1, 2, ..., \ell - 1.$$

In other words,

$$q_1 q_2 \cdots q_{\ell-1}$$

is a palindrome.

THEOREM 2.1. *Let $\gamma = (P + \sqrt{D})/Q$ be a reduced quadratic irrational where $D > 0$ is a radicand and $l = l(\gamma)$. Then the following are equivalent.*

(1) *γ has pure skew-symmetric period.*

(2) *For all $j \in \mathbb{N}$ with $j \leqslant \ell - 1$,*

$$\gamma_{\ell-j+1} \gamma'_j = -1.$$

(3) $\gamma\gamma'_1 = -1$.

(4) *The ideal $[\gamma]$ is ambiguous.*

(5) *If $P = P_0$ and $Q = Q_0$, in the simple continued fraction expansion of γ, then $D = P_0^2 + Q_0 Q_1$.*

Proof. If γ has pure skew-symmetric period, then for all natural numbers $j \leqslant \ell - 1$, we have that $q_j = q_{\ell-j}$. Thus, for such j,

$$\begin{aligned} \gamma_j &= \langle \overline{q_j, q_{j+1} ..., q_{\ell-1}, q_0, q_1 ..., q_{j-1}} \rangle \\ &= \langle \overline{q_{\ell-j}, q_{\ell-j-1}, ..., q_1, q_0, q_{\ell-1}, ..., q_{\ell-j+2}, q_{\ell-j+1}} \rangle. \end{aligned}$$

Also, by [2, Corollary 2.1.1, p. 44],

$$-\frac{1}{\gamma'_j} = \langle \overline{q_{\ell-j+1}; q_{\ell-j+2}, ..., q_0, q_1, ..., q_{\ell-j}} \rangle = \gamma_{\ell-j+1}.$$

Hence, for all such j,

$$\gamma_{\ell-j+1}\gamma'_j = -1,$$

so part (1) implies part (2).

Next, assume part (2). Then, in particular, for $j=1$ we have part (3), since $\gamma_\ell = \gamma_0 = \gamma$.

Now assume part (3) holds. Thus,

$$-1 = \gamma_0\gamma'_1 = \left(\frac{P_0+\sqrt{D}}{Q_0}\right)\left(\frac{P_1-\sqrt{D}}{Q_1}\right).$$

Multiplying numerator and denominator by $P_1-\sqrt{D}$ and using Eq. (1.2), we get

$$-1 = \left(\frac{P_0+\sqrt{D}}{Q_0Q_1}\right)\left(\frac{P_1^2-D}{P_1+\sqrt{D}}\right) = -\left(\frac{P_0+\sqrt{D}}{P_1+\sqrt{D}}\right),$$

so $P_0 = P_1$. Therefore, by [2, Exercise 4.2.9, p. 144], the ideal $[\gamma]$ must satisfy

$$[\gamma]' = [Q_0/\sigma, (P_1+\sqrt{D})/\sigma] = [Q_0/\sigma, (P_0+\sqrt{D})/\sigma] = [\gamma],$$

namely $[\gamma]$ is ambiguous, so part(3) implies part (4).

Next, assume that $[\gamma]$ is ambiguous. Then by [2, Lemma 6.1.1, p. 188], for any $j \in \mathbb{Z}$ with $0 \leqslant j \leqslant \ell$, we have

$$q_j = q_{\ell-j}, \qquad Q_j = Q_{\ell-j}, \qquad \text{and} \qquad P_{j+1}P_{\ell-j}. \tag{2.5}$$

Thus, in particular, $P_0 = P_\ell = P_1$, so by Eq. (1.2),

$$D = P_1^2 + Q_0Q_1 = P_0^2 + Q_0Q_1,$$

which is part (5).

Finally, assume part (5). Then $D = P_0^2 + Q_0Q_1 = P_1^2 + Q_0Q_1$, by Eq. (1.2), so $P_0 = P_1$. Therefore, as above, $[\gamma]$ is ambiguous, and so by Eq. (2.2), $q_j = q_{\ell-j}$ for $1 \leqslant j \leqslant \ell - 1$, namely γ has pure skew-symmetric period. Hence, part (5) implies part (1), and the circle of equivalences is complete. ∎

EXAMPLE 2.1. If $\gamma = (5+\sqrt{145})/2$, then

$$\gamma = \langle \overline{1; 1, 2, 2, 1} \rangle,$$

so γ has pure skew-symmetric period. The full continued fraction expansion data for γ is given in Table I.

TABLE I

j	0	1	2	3	4
P_j	5	5	7	9	7
Q_j	10	12	8	8	12
q_j	1	1	2	2	1

Notice that

$$\gamma\gamma_1' = \left(\frac{5+\sqrt{145}}{10}\right)\left(\frac{5-\sqrt{145}}{12}\right) = -1.$$

The reader may verify that $\gamma_{\ell-j+1}\gamma_j' = \gamma_{6-j}\gamma_j' = -1$, as well for natural numbers $j \leqslant \ell - 1 = 4$. Furthermore, the ideal

$$[\gamma] = [5, (5+\sqrt{145})/2]$$

is ambiguous. Finally, $D = P_0^2 + Q_0 Q_1 = 5^2 + 10 \cdot 12$.

EXAMPLE 2.2. Let $D = 385$ and $\gamma = (7+\sqrt{385})/14$. Then the simple continued fraction data for γ is given in Table II.

Since

$$q_1 q_2 \cdots q_9 = 196232691$$

is a palindrome, where $\ell = 10$, then γ has pure skew-symmetric period. Observe as well that the ideal

$$[\gamma] = [7, (7+\sqrt{385})/2]$$

is ambiguous, and $D = 385 = P_0^2 + Q_0 Q_1 = 7^2 + 14 \cdot 24$. Also,

$$\gamma\gamma_1' = \left(\frac{7+\sqrt{385}}{14}\right)\left(\frac{7-\sqrt{385}}{24}\right) = \frac{7^2 - 385}{336} = -1,$$

and the reader may verify that $\gamma_{\ell-j+1}\gamma_j' = -1$ for any natural number $j \leqslant 9$.

TABLE II

j	0	1	2	3	4	5	6	7	8	9
P_j	7	7	17	19	17	15	15	17	19	17
Q_j	14	24	4	6	16	10	16	6	4	24
q_j	1	1	9	6	2	3	2	6	9	1

DEFINITION 2.2. Let $\gamma = \langle \overline{q_0; q_1, ..., q_\ell} \rangle$ be a reduced quadratic irrational, where $D > 0$ is a radicand. Then γ is said to have *pure symmetric period* if

$$q_j = q_{\ell - j - 1} \qquad \text{for all integers } j \text{ with} \quad 0 \leqslant j \leqslant \ell - 1.$$

In other words,

$$q_0 q_1 \cdots q_{\ell - 1}$$

is a palindrome.

THEOREM 2.2. *Let $\gamma = (P + \sqrt{D})/Q = \langle \overline{q_0; q_1, ..., q_{\ell-1}} \rangle$ be a reduced quadratic irrational with radicand $D > 0$. Then the following are equivalent.*

(1) *γ has pure symmetric period.*

(2) $\gamma\gamma' = -1$.

(3) *The ideal class of $[\gamma]$ has at most one ambiguous ideal in it.*

(4) $D = P^2 + Q^2$.

(5) *For any $j \in \mathbb{Z}$ with $0 \leqslant j \leqslant \ell - 1$*

$$\gamma'_j \gamma_{\ell - j} = -1.$$

Proof. The equivalence of parts (1)–(3) is [2, Theorem 6.1.5, p. 194], and the equivalence of parts (3) and (4) follows from [2, Lemmas 6.1.2, p. 190, and 6.1.4, p. 194]. To show the equivalence of parts (1) and (5), we assume first that γ has pure symmetric period. Then for any $j \in \mathbb{Z}$ with $0 \leqslant j \leqslant \ell - 1$,

$$\begin{aligned} \gamma_j &= \langle \overline{q_j; q_{j+1}, ..., q_{\ell-1}, q_0 \cdot q_1, ..., q_{j-2}, q_{j-1}} \rangle \\ &= \langle \overline{q_{\ell-j-1}; q_{\ell-j-2}, ..., q_0, q_{\ell-1}, q_{\ell-2}, ..., q_{\ell-j+1}, q_{\ell-j}} \rangle. \end{aligned}$$

Also, by [2, Corollary 2.1.1, p. 44],

$$-\frac{1}{\gamma'_j} = \langle \overline{q_{\ell-j}; q_{\ell-j+1}, ..., q_{\ell-1}, q_0, q_1, ..., q_{\ell-j-1}} \rangle = \gamma_{\ell-j},$$

so $\gamma'_j \gamma_{\ell-j} = -1$. This is part (5).

Now assume that part (5) holds. In particular, it holds for $j = 0$, namely $\gamma\gamma' = -1$. Hence, by part (2), part (1) holds, and the circle of equivalences is complete. ∎

EXAMPLE 2.3. Let $D = 145$, and $\gamma = (9 + \sqrt{145})/8$. Then the simple continued fraction data for γ is given in Table III.

TABLE III

j	0	1	2	3	4
P_j	9	7	5	5	7
Q_j	8	12	10	12	8
q_j	2	1	1	1	2

Thus, $\gamma = \langle \overline{2; 1, 1, 1, 2} \rangle$ has pure symmetric period. Notice as well that

$$\gamma\gamma' = \left(\frac{9+\sqrt{145}}{8}\right)\left(\frac{9-\sqrt{145}}{8}\right) = -1,$$

and $D = 145 = 9^2 + 8^2$. Also, the only ambiguous ideal in the class of $[\gamma]$ is

$$[\gamma_{(\ell-1)/2}] = [\gamma_2] = [5, (5+\sqrt{145})/2],$$

and $\gamma_{(\ell-1)/2}$ has pure skew-symmetric period.

EXAMPLE 2.4. *Let* $D = 221$, *and* $\gamma = (11+\sqrt{221})/10$. *Then the simple continued fraction data for* γ *is given in Table* IV.

Thus, $\gamma = \langle \overline{2; 1, 1, 2} \rangle$ *has pure symmetric period. Furthermore,*

$$\gamma\gamma' = \left(\frac{11+\sqrt{221}}{10}\right)\left(\frac{11-\sqrt{221}}{10}\right) = -1,$$

and $D = 221 = 11^2 + 10^2$. *Also, there are no ambiguous ideals in the class of* $[\gamma]$, *and*

$$\gamma_{\ell/2} = \gamma_2 = \langle \overline{1; 2, 2, 1} \rangle$$

has pure symmetric period. (*Complete details concerning the case where ideal classes have no ambiguous ideals are covered in* [2, *Chap.* 6, *pp.* 187–199].)

Examples 2.3–2.4 *suggest the following.*

TABLE IV

j	0	1	2	3
P_j	11	9	5	9
Q_j	10	14	14	10
q_j	2	1	1	2

COROLLARY 2.1. *Let γ be a reduced quadratic irrational with $\ell(\gamma)=\ell$. Then the following are equivalent.*

(1) *γ has pure symmetric period.*

(2) (a) *If ℓ is even, then $\gamma_{\ell/2}$ has pure symmetric period.*

(b) *If ℓ is odd, then $\gamma_{(\ell-1)/2}$ has pure skew-symmetric period.*

Proof. If γ has pure symmetric period, then by part (5) of Theorem 2.2

$$\gamma_j \gamma_{\ell-j} = -1 \qquad \text{for any natural number} \quad j \leqslant \ell - 1.$$

In particular, if ℓ is even, then

$$\gamma_{\ell/2} \gamma'_{\ell/2} = -1,$$

so by part (2) of Theorem 2.2, $\gamma_{\ell/2}$ has pure symmetric period. If ℓ is odd, then

$$\gamma_{(\ell-1)/2} \gamma_{(\ell+1)/2} = -1.$$

Therefore, by part (3) of Theorem 2.1, $\gamma_{(\ell-1)/2}$ has pure skew-symmetric period.

Conversely, assume that part (2) holds. In the case where ℓ is even,

$$\gamma_{\ell/2} = \langle \overline{q_{\ell/2}; q_{\ell/2+1}, ..., q_{\ell-2}, q_{\ell-1}, q_0, q_1, ..., q_{\ell/2-2}, q_{\ell/2-1}} \rangle.$$

Therefore,

$$q_{\ell/2} = q_{\ell/2-1}, q_{\ell/2+1} = q_{\ell/2-2}, ..., q_{\ell-2} = q_1, q_{\ell-1} = q_0,$$

namely $q_j = q_{\ell-j-1}$ for all integers $j = 0, 1, 2, ..., \ell-1$. In other words, γ has pure symmetric period.

If ℓ is odd, then

$$\gamma_{(\ell-1)/2} = \langle \overline{q_{(\ell-1)/2}, q_{(\ell+1)/2}, q_{(\ell+3)/2}, ..., q_{\ell-2},} \\ \overline{q_{\ell-1}, q_0, q_1, ..., q_{(\ell-5)/2}, q_{(\ell-3)/2}} \rangle.$$

Thus,

$$q_{(\ell+1)/2} = q_{(\ell-3)/2}, q_{(\ell+3)/2} = q_{(\ell-5)/2}, ..., q_{\ell-2} = q_1, q_{\ell-1} = q_0.$$

In other words, $q_j = q_{\ell-j-1}$ for all $j = 0, 1, 2, ..., \ell-1$, so γ has pure symmetric period. ∎

COROLLARY 2.2. *Let γ be a reduced quadratic irrational with $\ell(\gamma)=\ell$. Then the following are equivalent.*

(1) *γ has pure skew-symmetric period.*

(2) (a) *If ℓ is even, then $\gamma_{\ell/2}$ has pure skew-symmetric period.*

(b) *If ℓ is odd, then $\gamma_{(\ell+1)/2}$ has pure symmetric period.*

Proof. This is proved in the same fashion as Corollary 2.1. ∎

EXAMPLE 2.5. Looking at Example 2.1 again, we see that

$$\gamma_{(\ell+1)/2} = \gamma_3 = \langle \overline{2; 1, 1, 1, , 2} \rangle,$$

has pure symmetric period, and in Example 2.2,

$$\gamma_{\ell/2} = \gamma_5 = \langle \overline{3; 2, 6, 9, 1, 1, 1, 9, 6, 2} \rangle,$$

has pure skew-symmetric period.

Remark 2.1. In a given ideal class of $[\gamma]$, where γ is a reduced quadratic irrational, there can be at most two ambiguous reduced ideals (see [2, Theorem 6.1.4, p. 193]). Suppose that two ambiguous ideals exist in the class of $[\gamma]$, and set $\ell = \ell(\gamma)$. Then there cannot exist any $j \in \mathbb{Z}$ with $0 \leqslant j \leqslant \ell - 1$ with $\gamma_j \gamma_j' = -1$, by Theorem 2.2. Hence, by Corollary 2.2, if γ has pure skew-symmetric period, and ℓ is odd, then there does not exist a $j \in \mathbb{Z}$ with $0 \leqslant j \leqslant \ell - 1$ such that $\gamma_j \gamma_j' = -1$. For instance, in Example 2.2, there are two ambiguous ideals, but there is no γ_j such that $\gamma_j \gamma_j' = -1$.

A natural question is to ask if there are reduced quadratic irrationals that have *both* pure symmetric and pure skew-symmetric periods. The answer is yes, but they are very special. By Theorems 2.1–2.2, if both occur simultaneously for a given reduced quadratic irrational γ, then we must have

$$D = P_0^2 + Q_0^2 = P_1^2 + Q_0 Q_1,$$

so

$$P_0 = P_1, \qquad Q_0 = Q_1, \qquad \text{and} \qquad q_0 = q_1.$$

Hence, $\ell(\gamma) = 1$.

EXAMPLE 2.6. Let $D = 650 = 2 \cdot 5^2 \cdot 13$ and $\gamma = (25 + \sqrt{650})/5$. Then the reduced quadratic irrational γ has $\ell = 1$ with $\gamma = \langle \overline{10} \rangle$, and

$$D = 25^2 + 5^2 = P_0^2 + Q_0^2.$$

EXAMPLE 2.7. Let $D = 65 = 8^2 + 1 = P_0^2 + Q_0^2$, and $\gamma = 8 + \sqrt{65}$. Then γ is reduced and $\gamma = \langle \overline{16} \rangle$.

EXAMPLE 2.8. Let $D = 26 = 5^2 + 1$. Then $\gamma = 5 + \sqrt{26} = \langle \overline{10} \rangle$.

Remark 2.2. In [3, Remark, p. 229], it is stated that, "Amongst many other helpful remarks the referee observes that it is not at all clear whether an ambiguous class without ambiguous reduced ideals does or does not contain ambiguous ideals." This is an unfortunate comment (added at the last minute by one of the authors) that requires clarification, since it is indeed *clear* that the non-existence of reduced ambiguous ideals in a given class of a quadratic irrational implies the non-existence of any ambiguous ideal.

THEOREM 2.3. *Let $\Delta > 0$ be a discriminant, and let $\mathscr{C}$ be an ambiguous class of $\mathscr{C}_\Delta$. If $\mathscr{C}$ has an ambiguous ideal in it, then $\mathscr{C}$ has an ambiguous* reduced *ideal in it.*

Proof. Let $I = [a, (b + \sqrt{\Delta})/2] = I'$ in $\mathscr{C}$, and assume that I is not reduced.

Case 2.1. D is divisible by a.

Let $D = an$ for some $n \in \mathbb{N}$, and set

$$J = [n, (n + \sqrt{\Delta})/2] = J'.$$

Since I is not reduced, then $a > \sqrt{D}$, by Theorem 1.1. Hence, $n < \sqrt{D}$. Thus, by Theorem 1.1 J is reduced. Finally, $I \sim J$ since $IJ = (\omega_\Delta)$. This completes Case 2.1.

Case 2.2. D is not divisible by a.

Claim 2.1. D is odd.

Let $D = 2^{\alpha_1} d_1$ with $\alpha_1 \geqslant 0$, d_1 odd, and $a = 2^{\alpha_2} d_2$ where d_2 is odd and $\alpha_2 > \alpha_1$ (since $a \nmid D$, but $a \mid \Delta$).

We note that $|b| = a$ by observing that $I = I'$ forces $at + b/2 = -b/2$ for some $t \in \mathbb{Z}$. Therefore, $at = -b$, so $a \mid b$. If $b = 0$, then $a \mid D$ a contradiction, so $|b| = a$. Since $\Delta \equiv b^2 \equiv a^2 \pmod{4a}$, then $a^2 - \Delta = 4as$ for some $s \in \mathbb{Z}$. Thus,

$$2^{2\alpha_2} d_2^2 - 2^{\alpha_1 + 2} d_1 = 2^{\alpha_2 + 2} s d_2.$$

Therefore,

$$2^{\alpha_1 + 2}(2^{2\alpha_2 - \alpha_1 - 2} d_2^2 - d_1) = 2^{\alpha_2 + 2} s d_2.$$

Since $2\alpha_2 - \alpha_1 - 2 \geqslant \alpha_1$, then if D is even, $2^{2\alpha_2 - \alpha_1 - 2} d_2^2 - d_1$ is odd, so $2^{\alpha_2 + 2} \mid 2^{\alpha_1 + 2}$. Thus, $\alpha_1 + 2 \geqslant \alpha_2 + 2$, a contradiction. This is Claim 2.1.

Claim 2.2. $(a/2) \mid D$.

As above, $a^2 \equiv \Delta \pmod{4a}$, so if $4 \mid a$, then $16 \mid \Delta = 4D$, a contradiction. Hence, $(a/2) \mid D = \Delta/4$, observing that the proof of Claim 2.1 implies that $\alpha_2 \leqslant 2$ and Case 2.2 implies $\alpha_2 \geqslant 1$.

Since $\Delta = an$, for some $n \in \mathbb{Z}$, then n is even and $(n/2) \mid D$. Set

$$H = [a/2, \sqrt{D}] = H' \qquad \text{and} \qquad J = [n/2, \sqrt{D}] = J'.$$

Then $H \sim J$ since $D = (a/2)(n/2)$. Set

$$Q = [2, 1 + \sqrt{D}],$$

then

$$I = HQ \sim JQ = L.$$

If $a < \sqrt{\Delta}$, then by part (4) of Theorem 1.1, $\sqrt{D} \in I$ thereby forcing $a \mid D$, a contradiction (observing that $\sqrt{D} \in I$ implies $I \mid (\sqrt{D})$, which in turn implies that $a \mid D$). Hence, $a > \sqrt{D}$ forcing $n < \sqrt{\Delta}$. If $\sqrt{D} \in L$, then $n \mid D$, a contradiction so L is reduced by part (4) of Theorem 1.1. ∎

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