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5 **ON AN ELEMENTARY APPROACH TO THE  
 LEBESGUE–NAGELL EQUATION**

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15 We discuss the feasibility of an elementary solution to the Diophantine equation of the  
 form  $x^2 + D = y^n$ , where  $D > 1$ ,  $n \geq 3$  and  $x > 0$ , called the Lebesgue–Nagell equation,  
 17 which has recently been solved for  $1 \leq D \leq 100$  in [1].

*Keywords:* Diophantine equations; class numbers; quadratic orders.

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**1. Introduction**

21 The equation  $x^2 + D = y^n$  has a long and celebrated history which is nicely outlined  
 in [1], where the equation has been solved for  $1 \leq D \leq 100$  as noted in the abstract.  
 23 We look at this equation from a more elementary point of view than has been con-  
 sidered in the literature thus far in the hopes of providing an alternative approach  
 25 that may prove to be valuable, at least in some special situations. In any case, we  
 show how the solutions to this equation can be obtained via these elementary meth-  
 27 ods, but leave open the means for verifying that this is all of them, since we cannot  
 prove that the list is exhaustive. Also, we provide a new and natural approach to  
 29 the Lebesgue–Nagell equation (see Theorem 2.2 below) that allows solutions of the  
 equation to be found, in certain cases, by elementary means involving class numbers  
 31 of complex quadratic orders.

33 We will need the following elementary results in the next section, so we present  
 them here for convenience.

**Theorem 1.1.** *Let  $n \in \mathbb{N}$  (the natural numbers). Then  $n$  has a primitive represen-*  
 35 *tation as a difference of two squares of natural numbers if and only if either  $n$  is*  
*odd or  $n \equiv 0 \pmod{8}$ .*

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1 **Proof.** See [10, Theorem 6.2.3, p. 296] for instance.  $\square$

3 **Theorem 1.2 (Number of Primitive Differences of Squares).** *Suppose that*  
 *$n = 2^a \prod_{j=1}^k p_j^{a_j}$  is the canonical prime factorization of the natural number  $n$ , where*  
 *$p_j > 2$ , and  $a = 0$  or  $a \geq 3$ . Then the number of primitive representations  $n =$*   
 *$x^2 - y^2$  is*

$$\begin{cases} 2^{k-1} & \text{if } a = 0 \text{ and } k \geq 1, \\ 2^k & \text{if } a \geq 3 \text{ and } k \geq 1, \\ 1 & \text{if } k = 0. \end{cases}$$

7 **Proof.** See [10, Theorem 6.2.4, p. 297] for instance.  $\square$

9 Note that throughout the paper  $D > 0$  is not a perfect square, and we employ the notation  $E = \pm D$ .

## 2. Lebesgue–Nagell and Class Numbers

11 The reader unfamiliar with terminology surrounding general quadratic orders should  
 consult [9, Sec. 1.5] for terminology and background. In what follows,  $f_{4E}$  signifies  
 13 the conductor of the quadratic order  $\mathbb{Z}[\sqrt{E}]$  where  $4E$  is the *discriminant* of  $\mathbb{Z}[\sqrt{E}]$ .  
 This means that if  $E = d^2 D_0$ , where  $D_0$  is squarefree,  $d \geq 1$ , then

$$15 \quad f_{4E} = \begin{cases} d & \text{if } D_0 \not\equiv 1 \pmod{4} \\ 2d & \text{if } D_0 \equiv 1 \pmod{4}. \end{cases}$$

$D_0$  is often called the *fundamental radicand*. In our case, since we are always con-  
 sidering the order  $\mathbb{Z}[\sqrt{E}]$ , even when  $E = D_0$ , we are not necessarily considering  
 the *maximal order* (also called the *ring of integers* of  $\mathbb{Q}(\sqrt{E})$ ), since when  $D_0 \equiv$   
 19  $1 \pmod{4}$ , we are looking at the non-maximal order  $\mathbb{Z}[\sqrt{D_0}]$ , which has conductor  
 $f_{4D_0} = 2$ , whereas the maximal order  $\mathbb{Z}[(1 + \sqrt{D_0})/2]$  has conductor  $f_{D_0} = 1$ . This  
 21 is what sets Lemma 2.1, presented below, apart from earlier considerations and  
 allows us to tackle situations for Diophantine equations that previously required  
 23 more sophisticated methods.

The following generalizes [8, Lemma 2, p. 178] (see also [9]) to arbitrary orders  
 25  $\mathbb{Z}[\sqrt{E}]$ . The symbol  $I \sim J$  will refer to equivalence of ideals in the class group  
 of the order  $\mathbb{Z}[\sqrt{E}]$ , with 1 being the principal class and  $h_\Delta$  will denote the class  
 27 number of the quadratic order with discriminant  $\Delta$ . The notation  $I = [a, b + \sqrt{E}]$   
 will denote a primitive ideal in the order  $\mathbb{Z}[\sqrt{E}]$ , namely that  $\mathbb{Z}$ -module with norm  
 29  $a$  where  $b^2 \equiv E \pmod{a}$ .

**Lemma 2.1.** *Suppose that  $I = [a, b + \sqrt{-D}]$  is a primitive, proper ideal in the*  
 31 *order  $\mathbb{Z}[\sqrt{-D}]$ , with  $b^2 < D^2 - D$ . Then  $I \sim 1$  if and only if either  $a = 1$  or*  
 *$a = N(b + \sqrt{-D})$ .*

33 **Proof.** Clearly, if  $a = 1$ , then  $I \sim 1$ , and if  $a = N(b + \sqrt{-D})$ , then  $I =$   
 $(b + \sqrt{-D}) \sim 1$ .

1        Conversely, assume that  $I \sim 1$  and  $N(b + \sqrt{-D}) = ac$ . Since  $b^2 < D^2 - D$ , then  
 2         $N(b + \sqrt{-D}) < D^2$ . Thus, either  $a < D$  or  $c < D$ . Since  $I \sim 1$ , then there exists  
 3         $x, y \in \mathbb{Z}$  such that  $\alpha = ax + (b + \sqrt{-D})y$  and  $N(\alpha) = a$ .

4        Therefore, if  $a < D$ , then  $D > a = (ax + by)^2 + Dy^2 > D$  for  $y \neq 0$ ,  
 5        a contradiction that forces  $y = 0$  and  $a = 1$ . If  $c < D$ , then  
 6         $(b + \sqrt{-D}) = [ac, b + \sqrt{-D}] = [a, b + \sqrt{-D}][c, b + \sqrt{-D}] \sim 1$ . Since  $I = [a, b + \sqrt{-D}] \sim$   
 7         $1$ , then  $[c, b + \sqrt{-D}] \sim 1$ . We may employ the same argument as above to show  
 8        that  $c = 1$ , so we are done.  $\square$

9        Lemma 2.1 will be crucial in proving our first main result. The following gen-  
 10        eralizes [9, Theorem 3.1.1, p. 67]. The notation  $[x]$  refers to the greatest integer  
 11        function.

12        **Theorem 2.2.** *Suppose that  $-D = x^2 - y^n < 0$  where  $x, y, n \in \mathbb{Z}$ ,  $n > 1$ ,  $y > 1$ ,  
 13        and  $x > 0$  with  $\gcd(y, f_{-4D}) = 1$ , and  $x \neq [y^{n/2}]$  when  $n$  is odd. Then  $n \mid h_{-4D}$ .*

14        **Proof.** We first verify that the hypothesis of Lemma 2.1 is satisfied, namely that  
 15         $x^2 < D^2 - D$ . Suppose to the contrary that  $x^2 \geq D^2 - D$ . Then  $x^2 > (D - 1)^2$ ,  
 16        so  $x > D - 1 = y^n - x^2 - 1$ . Therefore,  $(x + 1)^2 > y^n$ , and we deduce that  
 17         $y^{n/2} > x > y^{n/2} - 1$ . This is impossible if  $n$  is even, so  $n$  is odd and  $x = [y^{n/2}]$ ,  
 18        contradicting the hypothesis.

19        Hence, we may invoke Lemma 2.1 to conclude that if  $I = [y, x + \sqrt{-D}]$ , then  
 20         $I^c \sim 1$  for  $1 \leq c \leq n$ , if and only if  $c = n$ . However, if  $g = \gcd(n, h_{-4D})$ , then  
 21         $I^g \sim 1$ . To see this, observe that there exist  $u, v \in \mathbb{Z}$  such that  $g = nu + vh_{-4D}$   
 22        and  $I^g \sim (I^n)^u (I^{h_{-4D}})^v \sim (I^{h_{-4D}})^v \sim 1$ . Yet,  $g \leq n$ , so  $g = n$  is forced, namely  
 23         $n \mid h_{-4D}$ .  $\square$

24        **Example 2.3.** Consider

$$25 \qquad -15 = [4^{3/2}]^2 - 4^3 = 7^2 - 4^3,$$

26        where  $h_{-4 \cdot 15} = 2$  and  $f_{-4 \cdot 15} = 2$ , so  $\gcd(f_{4D}, y) = 2$ .

27        Also, consider,

$$28 \qquad -D = -341 = 2759646^2 - 377^5,$$

29        where  $h_{-4D} = 28$  and  $2759646 = [377^{5/2}]$ . This last example is interesting since  
 30        it represents the only case where  $-D = x^2 - y^p$  for an odd prime  $p$  with  $p$  **not**  
 31        dividing the class number  $h_{-4D}$  when  $-D \equiv -1 \pmod{4}$  is squarefree. This was  
 32        shown in [2, Corollary 7, p. 62], as a generalization of a result by this author from  
 33        [9]. Also, see [12] where we corrected some errors in [2].

34        Theorem 2.2 has numerous applications that we now explore. In particular, when  
 35         $-D = -2^k$  for an integer  $k \geq 1$ , and  $n \geq 3$ , several authors have worked on finding  
 36        solutions using, for instance, elliptic curve techniques (see [13]). We now show how  
 37        this equation relates to Theorem 2.2

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1 **Corollary 2.4.** *If  $-2^k = \lfloor y^{n/2} \rfloor^2 - y^n$  for  $k \geq 1$ , odd  $y > 1$ , and odd  $n \geq 3$ , then*  
 (1, 5, 3, 3).

3 **Proof.** Since  $n > 1$  is odd, Theorem 2.2 tells us that  $x = \lfloor y^{n/2} \rfloor$  since  $h_{-2^{k+2}}$  is a  
 power of 2 (see [9, Footnote 1.5.9, p. 25]). When  $k$  is odd, Cohn has shown in [3]  
 5 that the only possibility is that given above. When  $k$  is even Siksek has shown in  
 [13] that there are no possibilities.  $\square$

7 **Remark 2.5.** The case where  $n$  is even in Corollary 2.4 is easily shown to boil  
 down to one solution as follows. Since  $-2^k = x^2 - y^n = x^2 - (y^{n/2})^2$ ,  $x = 2^{k-2} - 1$   
 9 and  $y^{n/2} = 2^{k-2} + 1$  (see [10, Exercise 6.2.3], for instance). Since  $n \geq 4$ , then  
 we can apply the latter again to  $2^{k-2} = (y^{n/4})^2 - 1$  to get  $y^{n/4} = 2^{k-4} + 1$  and  
 11  $1 = 2^{k-4} - 1$ . It follows that  $k = 5$  and  $y^n/2 = 9$ , so  $y = 3$ ,  $n = 4$ , and  $x = 7$ .  
 Notice that  $-D = -2^5 = 7^2 - 3^4 = x^2 - y^n$  and  $4 \mid h_{-4 \cdot 2^5}$ . Indeed,  $h_{-4 \cdot 2^5} = 4$ , and  
 13  $\gcd(f_{-4D}, y) = \gcd(4, 3) = 1$  so this is an application of Theorem 2.2.

Cohn actually showed more in his paper [3] since he was not interested in prim-  
 15 itive solutions ( $y$  odd) as we have seen here. (See also [4, 5].) He proved that *all*  
 solutions of  $x^2 - y^n = -2^k$  for odd  $k$  and odd  $n$  are given by

$$17 \quad (k, x, y, n) \in \{(6\alpha + 1, 5 \cdot 2^{3\alpha}, 3 \cdot 2^{2\alpha}, 3), (10\alpha + 5, 11 \cdot 2^{5\alpha+3}, 3 \cdot 2^{2\alpha+1}, 5)\},$$

and for odd  $k$  and even  $n$  only

$$19 \quad (k, x, y, n) \in \{(4\alpha + 5, 7 \cdot 2^{2\alpha}, 3 \cdot 2^\alpha, 4)\}.$$

However, note that merely dividing the equation through by  $2^{2\alpha}$  in the first case  
 21 we get the result in Corollary 2.4. Dividing the equation through by  $2^{10\alpha} + 5$  in the  
 second case, we get the trivial equation  $-1 = 11^2 \cdot 2 - 3^5$ . (Notice however, that if  
 23 we set  $-D = -11^2 \cdot 2 = 1 - 3^5$ , then  $5 \mid h_{-4D} = 10$ , an application of Theorem 2.2.)  
 In the case where  $n$  is even, dividing through by  $2^{4\alpha}$  we get the result in the first  
 25 paragraph of this remark.

Also, Siksek found more in [13] since he too was concerned with all possible  
 27 solutions, not just the primitive ones when  $k$  is even. He found that if  $k = 2m$ ,  
 then one family of solutions is given by  $x = 2^m$ . Dividing the equation through by  
 29  $2^{2m}$  yields a trivial equation. The other case is given by  $n = 3$ .  $x = 11 \cdot 2^{3M}$  where  
 $m = 3M + 1$ , again a non-primitive family. Dividing through by appropriate powers  
 31 of 2 does not yield any non-trivial primitive solutions.

Observe that in the two cases from Corollary 2.4,  $-2^2 = 11^2 - 5^3$  where  $11 =$   
 33  $\lfloor 5^{3/2} \rfloor$  and  $-2 = 5^2 - 3^3$  where  $5 = \lfloor 3^{3/2} \rfloor$ . In the case of even  $n$ , we have  $-2^5 =$   
 $7^2 - 3^4$  where  $4 \mid h_{-2^7}$ . Indeed  $h_{-2^7} = 4$ .

35 Since Diophantine equations of the form  $x^2 + D = y^n$  are called Lebesgue–Nagell  
 equations, then the above are special cases thereof. Recently Bugeaud, Mignotte and  
 37 Siksek have completely solved this equation for all positive integers  $D \leq 100$  in [1].  
 We may now show how Theorem 2.2 is related.

Table 1.

$D$	$h_{-4D}$	$(x, y, 2m)$
17	4	(8, 3, 4)
32	4	(7, 3, 4)
45	4	(6, 3, 4)
49	4	(24, 5, 4)
53	6	(26, 3, 6)
56	8	(5, 3, 4)
65	8	(4, 3, 4)
72	4	(3, 3, 4)
77	8	(2, 3, 4)
80	8	(1, 3, 4)
96	8	(23, 5, 4)
97	4	(48, 7, 4)

1 **Theorem 2.6.** *If the Diophantine equation given by*

$$x^2 + D = y^{2m}, \quad (2.1)$$

3 *has integer solutions for  $m \geq 2$ ,  $100 \geq D > 1$ ,  $y > 1$  odd, and  $x > 0$ , then  $(x, y, 2m)$  is given in Table 1.*

**Proof.** By Theorem 2.2, if  $\gcd(f_{-4D}, y) = 1$ , then  $(2m) \mid h_{-4D}$ . The list of class numbers within the range of  $D$  given that they are even, at least 4 (and which are not of the form  $D \equiv 2 \pmod{4}$  or  $D \equiv 4 \pmod{8}$ ) by Theorem 1.1) are:

$$D \in \{17, 21, 24, 29, 32, 33, 35, 39, 40, 41, 45, 48, 49, 51, 53, 55, 56, 57, 61, 63, 65, 69, 72, 73, 75, 77, 80, 81, 85, 87, 88, 89, 93, 95, 96, 97, 99\}.$$

5 If  $D = 17$  then as a difference of squares  $17 = y^4 - x^2$  can be expressed in exactly one way by Theorem 1.2. Since  $8^2 + 17 = 3^4$ , we are done with this case. If  $D = 21$ , then  
7 by Theorem 1.2 there are exactly two primitive representations of  $D$  has exactly two representations, and they are  $21 = 5^2 - 2^2 = 11^2 - 10^2$ , neither of which satisfy  
9  $21 = y^4 - x^2$ , given that  $h_{-4D} = 4$ .

11 For the balance, we will be accessing Theorems 1.1 and 1.2 without explicit reference. If  $D = 24$ , then  $h_{-4D} = 4$ . There are exactly two representations of 24 as a difference of squares and they are given by  $24 = 5^2 - 1^2 =$   
13  $7^2 - 5^2$ , neither of which satisfy our hypothesis that  $x^2 + 24 = y^4$ , so we eliminate it from the list. By a similar argument, we can also eliminate  $D =$   
15  $29, 33, 35, 40, 41, 51, 57, 61, 69, 73, 75, 85, 88, 89, 93, 95, 99$ .

17 If  $D = 32$ , then there is exactly one representation as a difference of squares given by  $32 = 3^4 - 7^2$  which satisfies our requirement given that  $h_{-4D} = 4$ . If  
19  $D = 39$ , then there are exactly two representations of it as a difference of squares. They are given by  $39 = 2^6 - 5^2 = 20^2 - 19^2$ , but  $\gcd(f_{-4D}, y) = \gcd(2, y) = 2$   
21 for  $y = 20, 2$ , which do not satisfy the hypothesis that  $y$  is odd. Similarly, we may eliminate  $D = 48, 55, 63, 87$ . The value 81 is eliminated since its only representation is  $81 = 0^2 + 3^4$  and we require that  $x > 0$ .

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1 If  $D = 45$ , then there are exactly two representations of  $D$  as a difference  
of squares:  $45 = 3^4 - 6^2$ , which satisfies our criterion since  $h_{-4D} = 4$ , and  
3  $45 = 23^2 - 22^2$ , which we reject. Similarly, we achieve the balance of the values in  
Table 1.  $\square$

5 **Remark 2.7.** The case where  $D = 49$  is a special instance of a more general  
phenomenon where we may consider  $x^2 + p^2 = y^n$  where  $p$  is prime, which has been  
7 studied by such authors as Ljunggren [7], and Kutsuna [6]. In particular, if  $D = p^2$   
where  $p$  is an odd prime, then it may be shown (see [9, Footnote 1.5.9, pp. 25–26],  
9 for instance) that

$$h_{-4D} = \begin{cases} (p+1)/2 & \text{if } p \equiv -1 \pmod{4} \\ (p-1)/2 & \text{if } p \equiv 1 \pmod{4}. \end{cases}$$

11 Thus, if  $n$  is odd,  $\gcd(p, y) = 1$ , then either  $n \mid (p \pm 1)/2$  (which coincides with  
results found by Ljunggren in [7]), or  $x = \lfloor y^{n/2} \rfloor$ . Moreover, in [6] Kutsuna used  
13 his result to prove that for  $p = 7$  the only solution is  $(x, y, n) = (24, 5, 4)$ , which we  
demonstrated above.

15 The values in Table 1 agree exactly with the values in [1]. Moreover, the authors  
therein agree that the even exponent case is easily solved using differences of squares  
17 as we have demonstrated above. What is perhaps more interesting is how we can  
achieve the odd exponents as follows.

19 If we wish to solve

$$x^2 + D = y^n, n \geq 3 \text{ odd}, x > 0, y > 1, \text{ with } \gcd(y, f_{-4D}) = 1 \quad (2.2)$$

21 then we may employ Theorem 2.2 to conclude that when  $n$  does not divide  $h_{-4D}$   
(as is the case with the vast majority of the values) then  $x = \lfloor y^{n/2} \rfloor$ .

Using properties of the ceiling and floor functions (see [10, Exercises  
1.3.1–1.3.2, p. 40] for instance), we get since  $y$  cannot be a perfect square

Table 2.

$D$	$h_{-4D}$	$f(y, n)$	$(x, y, n)$
7	1	$f(2, 3) \approx -4.7$	(1, 2, 3)
7	1	$f(32, 3) \approx -6$	(181, 32, 3)
7	1	$f(2, 5) \approx -7.3$	(5, 2, 5)
7	1	$f(8, 5) \approx -6$	(181, 8, 5)
7	1	$f(2, 7) \approx -6.6$	(11, 2, 7)
7	1	$f(2, 15) \approx -6$	(181, 2, 15)
100	4	$f(5, 3) \approx -3.4$	(5, 5, 3)
100	4	$f(10, 3) \approx -39.2$	(30, 10, 3)
100	4	$f(34, 3) \approx -99.5$	(198, 34, 3)
100	4	$f(5, 5) \approx -100.8$	(55, 5, 5)
$9409 = 97^2$	48	$f(12545, 3) \approx -9408$	(1405096, 12545, 3)
$18225 = 135^2$	72	$f(2701, 3) \approx -18224.1$	(140374, 2701, 3)

in this case),

$$\begin{aligned} -D &= \lfloor y^{n/2} \rfloor^2 - y^n = (\lceil y^{n/2} \rceil - 1)^2 - y^n \\ \lceil y^{n/2} \rceil^2 - 2\lceil y^{n/2} \rceil + 1 - y^n &< \lceil y^{n/2} \rceil^2 - 2y^{n/2} + 1 - y^n. \end{aligned}$$

1 If we set  $f(y, n) = \lceil y^{n/2} \rceil^2 - 2y^{n/2} - y^n$ , then we have

$$-D - 1 < f(y, n) < 0. \quad (2.3)$$

3 Thus, to get all of the solutions of Eq. (2.2), we need only find the values of  $y$  and  
 5  $n$  satisfying the inequality (2.3), and test them on the equation. For instance, let us  
 7 look at the values for the celebrated Ramanujan–Nagell equation where  $D = 7$  and  
 9 for the top value  $D = 100$ , considered in [1]. There are also the values of  $D = a^2$   
 for  $3 \leq a \leq 501$  found recently by Tengely [14, Corollary 2, p. 302], which includes  
 the case  $a = 7$  in [1], as well as two others for values of  $a^2 > 100$  with  $\gcd(x, y) = 1$   
 and  $x \geq a^2$ , which are the conditions placed on Eq. (2.2) in [14].

11 These are, as proved in [1], all of the values with  $n$  odd for  $D = 7$  and all the  
 values for  $D = 100$ . The one with  $n$  even for  $D = 7$  is given by  $(x, y, n) = (3, 2, 4)$ ,  
 13 which is easily found given that  $\gcd(f_{-4.7}, y) = \gcd(2, y) = 2$  must hold. Also,  
 the case above where  $D = 7$ ,  $\gcd(f_{-4D}, y) = \gcd(2, y) = 2$ , yet only in the case  
 15  $(x, y, n) = (1, 2, 3)$  do we have  $x \neq \lfloor y^{n/2} \rfloor$ . Also note that in both cases  $D = 97^2$   
 and  $D = 135^2$ , we have that  $n = 3$  divides  $h_{-4D}$  as well as  $x = \lfloor y^{n/2} \rfloor$ . In any  
 17 case, if we could determine a bound beyond which we knew that  $f(y, n)$  is not in  
 the range dictated by inequality (2.3), then we would have an elementary method  
 19 for solving the general Lebesgue–Nagell equation because we can find them all by  
 this method but cannot **verify** that this is indeed all of them. This is due to the  
 21 fact that the behavior of  $f(y, n)$  for fixed  $n$  say, is like that of a sharp-toothed  
 saw (when graphed as  $f(x) = \lceil x \rceil - 2x - x^2$ ), so it is not clear how to look at  
 23 this behavior to determine suitable bounds for which one could make an exhaustive  
 search possible. Nevertheless, it is worth having made this observation given the  
 potential for solving a famed equation, albeit perhaps only in some special cases,  
 25 without all the massive machinery that has been aimed at it thus far.

27 It is also worth observing that the bound in which  $f(y, n)$  must sit are as sharp  
 as possible. For instance, when  $D = 7$  above, we must have  $-8 < f(y, n) < 0$   
 where  $f(2, 5) \approx -7.3$ ; for  $D = 100$  we must have  $-101 < f(y, n) < 0$ , where  
 29  $f(5, 5) \approx -100.8$ ; for  $D = 9409$ , we must have  $-9410 < f(y, n) < 0$ , where  
 $f(12545, 3) \approx -9408$ ; and for  $D = 18225$  we must have  $-18226 < f(y, n) < 0$   
 31 where  $f(2701, 3) \approx -18224.1$ .

33 It only remains for the reader to determine how (if it is at all possible) to control  
 $f(y, n)$  in order to prove, not only that we have these solutions as has been done  
 here, but that this is indeed **all** of them. That would be a very worthy endeavor.

35 We conclude with a mechanism for generating infinite families of solutions to  
 Eq. (2.2) using  $f(y, n)$  as follows. If  $y$  is a square, then  $f(y, n) = -2y^{n/2}$ . Thus  
 37 infinitely many solutions are given by  $D = 2y^{n/2} - 1$ ,  $x = -1 + y^{n/2}$ . For instance,

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Table 3.

$x$	$y$	$D$	$h_{-4D}$
7	4	15	2
26	9	53	6
63	16	127	5
124	25	249	12
215	36	431	21
342	49	685	12
511	64	1023	16
728	81	1457	24
999	100	1999	27
1330	121	2661	48
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 4.

$x$	$n$	$D$	$h_{-4D}$
26	3	53	6
242	5	485	20
2186	7	4373	70
19682	9	39365	180
177146	11	354293	352
$\vdots$	$\vdots$	$\vdots$	$\vdots$

1 if we fix odd  $n$ , say  $n = 3$ , then we can generate a family as follows, where we list only  
 2 the first ten. Note that when  $y$  is odd, then  $3|h_{-4D}$  as predicted by Theorem 2.2.

3 Similarly, we can fix  $y$  and vary  $n$  to get other infinite classes. We fix  $y = 9$  and  
 4 list only the first five. Note that  $n|h_{-4D}$  in each case as told by Theorem 2.2.

5 It is the aim of this work to have presented an elementary approach that might  
 6 inspire an interest in a variety of exponential Diophantine equations (without having  
 7 to resort to the deeper machinery) in this introductory exploration.

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