

On the Cyclotomic Polynomial

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For a given positive integer m and an algebraic number field K necessary and sufficient conditions for the m th cyclotomic polynomial to have K -integral solutions modulo a given integer of K are given. Among applications thereof are: that the solvability of the cyclotomic polynomial mod an integer yields information about the class number of related number fields; and about representation of integers by binary quadratic forms. The latter extends previous work of the author. Moreover some information is obtained pertaining to when an integer of K is the norm of an integer in a given quadratic extension of K . Finally an explicit determination of the pq th cyclotomic polynomial for distinct primes p and q is provided, and known results in the literature as well as generalizations thereof are obtained.

1. NOTATION AND PRELIMINARIES

The symbol \mathbb{Q} will denote the rational number field, and \mathbb{Z} will denote the rational integers. For a given algebraic number field K , O_K will denote the ring of integers of K , and $h(K)$ will denote its class number. For a given positive integer m we let ε_m denote a primitive m th root of unity. The symbol \prod is used to denote a product and when confusion cannot arise we will eliminate the indexing variable for convenience sake. Finally $(*/*)$ denotes the Legendre symbol.

2. SOLVABILITY MODULO INTEGERS

The minimum polynomial of ε_m over \mathbb{Q} is $\phi_m(x) = \prod_{(k,m)=1} (x - \varepsilon_m^k)$, which is the m th cyclotomic polynomial. Moreover $x^m - 1 = \prod_{d|m} \phi_d(x)$.

First we provide a result which is of independent interest in that for a given positive $m \in \mathbb{Z}$ and a given number field K it provides necessary and sufficient conditions for $\phi_m(x)$ to have K -integral solutions modulo a given $\alpha \in O_K$.

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THEOREM 2.1. *Let K be an algebraic number field with $\alpha \in O_K$ such that $(\alpha) = \mathfrak{q}_1^{b_1} \mathfrak{q}_2^{b_2} \cdots \mathfrak{q}_s^{b_s}$ for distinct K -primes \mathfrak{q}_i , where $b_i > 0$. Suppose $m = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, where $p_1 < p_2 < \cdots < p_r$ are distinct rational primes; and if $m_0 = m/p_r^{a_r} > 1$, then ε_{m_0} is not in K . We assume that $a_r \geq f_i$ for all $i = 1, 2, \dots, s$, where f_i is the inertial degree of \mathfrak{q}_i in K/\mathbb{Q} . Furthermore if $p_r = 2$, then we assume that 2 is unramified in K . Then the following are equivalent:*

- (1) $\phi_m(\beta) \equiv 0 \pmod{(\alpha)}$ for some $\beta \in O_K$.
- (2) Each \mathfrak{q}_i for $i = 1, 2, \dots, s$, is either completely split or ramified in $K(\varepsilon_m)$. In the latter case $\mathfrak{q}_i \mid p_r$, \mathfrak{q}_i is completely split in $K(\varepsilon_{m_0})$, and $b_i = 1$ for $p_r > 2$ (respectively, $a_r = 1$ or $b_i = 1$ for $p_r = 2$).

Proof. Assume $\phi_m(\beta) \equiv 0 \pmod{(\alpha)}$ for some $\beta \in O_K$. Then $\phi_m(\beta) \equiv 0 \pmod{\mathfrak{q}_i^{b_i}}$ for each $i = 1, 2, \dots, s$. Let $d = p_1^{c_1} p_2^{c_2} \cdots p_r^{c_r}$ be the order of β modulo $\mathfrak{q}_i^{b_i}$. By the generalized Euler criterion we have that d divides $\mathfrak{q}_i^{f_i(b_i-1)}(\mathfrak{q}_i^{f_i} - 1)$, where $\mathfrak{q}_i \cap \mathbb{Z} = q_i$. If $p_j \neq q_i$ for any $j = 1, 2, \dots, r$, then we have $\mathfrak{q}_i^{f_i} \equiv 1 \pmod{d}$. If $c_j < a_j$ for any $j = 1, 2, \dots, r$, then consider: $\phi_m(\beta) \prod \phi_k(\beta) = (\beta^m - 1)/(\beta^t - 1) = \beta^{t(m/t-1)} + \beta^{t(m/t-2)} + \cdots + \beta^t + 1$, where the product ranges over all $k \neq m$ such that $p_j^{c_j+1}$ divides k ; and $t = m/p_j^{a_j-c_j}$. But $\beta^t \equiv 1 \pmod{\mathfrak{q}_i^{b_i}}$ so $(\beta^m - 1)/(\beta^t - 1) \equiv p_j^{a_j-c_j} \pmod{\mathfrak{q}_i^{b_i}}$. Thus $\mathfrak{q}_i \mid p_j$; which implies $p_j = q_i$, a contradiction. Hence $\mathfrak{q}_i^{f_i} \equiv 1 \pmod{m}$; that is, \mathfrak{q}_i is completely split in $K(\varepsilon_m)$.

Now, if $\mathfrak{q}_i = p_j$ for some $j = 1, 2, \dots, r$, then by the same argument as above we obtain that $c_k = a_k$ for all $k \neq j$. If $j < r$, then since $\mathfrak{q}_i^{f_i} = p_j^{f_i} \equiv 1 \pmod{\prod_{k \neq j} p_k^{a_k}}$ we have $p_r^{f_i} > p_j^{f_i} > p_r^{a_r}$, a contradiction since $a_r \geq f_i$ by hypothesis. Therefore $j = r$; that is, \mathfrak{q}_i is completely split in $K(\varepsilon_{m/p_r^{a_r}})$.

Now, if $\mathfrak{q}_i \mid p_r = 2$, then we claim that either $b_i = 1$ or $a_r = 1$. If not, then $\phi_{2a_r}(\beta) = \beta^{2a_r-1} + 1 \equiv 0 \pmod{\mathfrak{q}_i^{b_i}}$. Therefore -1 is a square modulo \mathfrak{q}_i^2 . This implies by [6, 6C, p. 278] and the fact that 2 is unramified in K , that \mathfrak{q}_i is unramified in $K(\sqrt{-1})$. Therefore we have that $\hat{\mathfrak{q}}_i$ is unramified in $K(\sqrt{-1})$ over \mathbb{Q} , where $\hat{\mathfrak{q}}_i$ is a $K(\sqrt{-1})$ -prime above \mathfrak{q}_i . But $\hat{\mathfrak{q}}_i \cap \mathbb{Q}(\sqrt{-1})$ is ramified over \mathbb{Q} , a contradiction. Therefore $b_i = 1$ or $a_r = 1$.

If $K(\varepsilon_m) = K$, then trivially \mathfrak{q}_i is completely split in $K(\varepsilon_m)$ so we assume henceforth that $K(\varepsilon_m) \neq K$ and $\mathfrak{q}_i \mid p_r > 2$. If $\beta = \varepsilon_{m/p_r}^k$ for some $k \in \mathbb{Z}$, then as previously $p_r \equiv (\beta^m - 1)/(\beta^t - 1) \equiv 0 \pmod{\mathfrak{q}_i^{b_i}}$, where $t = m/p_r$. However, since $p_r \in \mathfrak{q}_i$, then $p_r^{b_i} \equiv 0 \pmod{\mathfrak{q}_i^{b_i}}$ so $b_i = 1$. We assume now that $\beta \neq \varepsilon_{m/p_r}^k$ for any $k \in \mathbb{Z}$. Now since $\phi_m(\beta) = \phi_{m_0}(\beta^{p_r^{a_r}})/\phi_{m_0}(\beta^{p_r^{a_r}-1})$, where $m_0 = m/p_r^{a_r}$, then it suffices to show that if $\phi_{m_0}(\beta^{p_r^{a_r}-1})$ is divisible by exactly $\mathfrak{q}_i^{l_1}$, say, then $\phi_{m_0}(\beta^{p_r^{a_r}})$ is exactly divisible by $\mathfrak{q}_i^{l_1+1}$. Since $\phi_{m_0}(\beta^{p_r^{a_r}-1})$ is divisible by the same power of \mathfrak{q}_i as $\beta^{m/p_r} - 1 \neq 0$, then $(\beta^{m/p_r} - 1) \in \mathfrak{q}_i^{l_1}$. Thus

$$\begin{aligned}
 (\beta^{m/p_r} - 1)^{p_r} &= \beta^m - \beta^{m/p_r(p_r-1)} \binom{p_r}{1} + \beta^{m/p_r(p_r-2)} \binom{p_r}{2} \\
 &\quad - \dots + \beta^{m/p_r} \binom{p_r}{p_r-1} - 1 \\
 &= \beta^m + \alpha \beta^{m/p_r} p_r - 1,
 \end{aligned}$$

where $\binom{\cdot}{\cdot}$ denotes the binomial coefficient, where α is relatively prime to \mathfrak{q}_i since $p_r > 2$. Thus $\beta^m - 1$ is exactly divisible by \mathfrak{q}_i^{l+1} . Since $\phi_{m_0}(\beta^{p_r^{a_r}})$ is divisible by exactly the same power of \mathfrak{q}_i as $\beta^m - 1$ we have accomplished that $b_i = 1$. Thus we have shown that (1) implies (2).

Conversely assume (2). By the Chinese remainder theorem it suffices to show that $\phi_m(\beta_i) \equiv 0 \pmod{\mathfrak{q}_i}$ for $i = 1, 2, \dots, s$, where $\beta_i \in O_K$. If \mathfrak{q}_i is completely split in $K(\varepsilon_m)$, then we may choose an element of order m modulo $\mathfrak{q}_i^{b_i}$. By the choice of β_i we have $\phi_k(\beta_i) \not\equiv 0 \pmod{\mathfrak{q}_i}$ for any proper divisor k of m . Therefore $\phi_m(\beta_i) \equiv 0 \pmod{\mathfrak{q}_i^{b_i}}$. If $\mathfrak{q}_i \mid p_r = 2$, then $a_r = 1$ or $b_i = 1$. If $a_r = 1$, then $\phi_m(-1) \equiv 0 \pmod{\mathfrak{q}_i^{b_i}}$. If $b_i = 1$, then $\phi_m(1) \equiv 0 \pmod{\mathfrak{q}_i}$. Now assume $\mathfrak{q}_i \mid p_r > 2$. Choose $\beta_i \in O_K$ of order $m_0 = m/p_r^{a_r}$ modulo \mathfrak{q}_i . We have $\phi_m(\beta_i) = \phi_{m_0}(\beta_i^{p_r^{a_r}}) / \phi_{m_0}(\beta_i^{p_r^{a_r-1}})$. If $\phi_{m_0}(\beta_i^{p_r^{a_r-1}}) = 0$, then $\beta_i^{p_r^{a_r-1}} = \varepsilon_{m_0}$ which implies ε_{m_0} is in K . Therefore by hypothesis $m_0 = 1$. In this case $\phi_m(\beta_i) = p_r \equiv 0 \pmod{\mathfrak{q}_i}$, and since $b_i = 1$ we have the result. If $\phi_{m_0}(\beta_i^{p_r^{a_r-1}}) \neq 0$, then $\phi_m(\beta_i) \equiv 0 \pmod{\mathfrak{q}_i}$ since $b_i = 1$ and $\phi_{m_0}(\beta_i^{p_r^{a_r-1}})$ is divisible by exactly one lower power of \mathfrak{q}_i than $\phi_{m_0}(\beta_i^{p_r^{a_r}})$. Q.E.D.

The following result which is immediate from Theorem 2.1 yields [5, Theorem 2.4] as a special case.

COROLLARY 2.2. *Let $n = q_1^{b_1} q_2^{b_2} \dots q_s^{b_s}$ for distinct rational primes q_i . Then the following are equivalent:*

- (1) $\phi_m(x) \equiv 0 \pmod{n}$ for some $x \in \mathbb{Z}$.
- (2) All q_i are such that $q_i = p_r \equiv 1 \pmod{m_0}$ and $b_i = 1$ for $p_r > 2$ (respectively, $a_r = 1$ or $b_i = 1$ for $p_r = 2$); or $q_i \equiv 1 \pmod{m}$.

Continuing to maintain the above notation we obtain information in the following result, pertaining to the class number of $\mathbb{Q}(\varepsilon_{mq})$ for each $q \mid n$ with $q \nmid m$. This generalizes [5, Corollary 2.5].

COROLLARY 2.3. *If $\phi_m(r) \equiv 0 \pmod{n}$ for some $r \in \mathbb{Z}$, $m > 2$, and if $q_i > 2$ does not divide m , then for each $j = 1, 2, \dots, r$, we have that $p_j^{\phi(m)/2-1}$ divides $h(\mathbb{Q}(\varepsilon_{mq_j}))$ and if $p_j = 2$, then $2^{\phi(m)/2-1}$ divides $h(\mathbb{Q}(\varepsilon_m, \sqrt{q_j^*}))$, where $q_i^* = (-1)^{(q_i-1)/2} q_i$.*

Proof. By Theorem 2.1 we have that the hypothesis of [5, Theorem 1.1] is satisfied. The result follows. Q.E.D.

We note that if 2 ramifies in K , then the theorem fails. For example, if

$K = \mathbb{Q}(\sqrt{7})$, then $\phi_4(\sqrt{7}) \equiv 0 \pmod{\varphi^2}$, where $(2) = \varphi^2$ in K . However, $a_r > 1$ and $b_i > 1$ contradicting Theorem 2.1.

Also if $a_r < f_i$ for some $i = 1, 2, \dots, s$, then the theorem fails. For example, if $K = \mathbb{Q}(\varepsilon_5)$, then $\phi_{15}(\varepsilon_5) \equiv 0 \pmod{\varphi}$, where $(3) = \varphi$ in K . If $m = 15$, then $a_r = 1 < f = 4$, where f is the inertial degree of 3 in K . However, $\varphi \nmid 5 = p_r$, contradicting Theorem 2.1.

The stage is now set to provide a generalization of [5, Theorem 2.6]. In what follows $m = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r} > 2$ for distinct primes p_i with $p_1 < p_2 < \dots < p_r$ and

$$\begin{aligned} m^* &= \pm \prod_{i=1}^r p_i, & \text{if } p_1 > 2, \\ &= \pm \prod_{i=2}^r p_i, & \text{if } p_1 = 2 \text{ and } r > 1, \\ &= -1, & \text{if } p_1 = 2, r = 1 \text{ and } a_1 = 2, \\ &= -2, & \text{if } p_1 = 2, r = 1 \text{ and } a_1 > 2. \end{aligned}$$

The “ \pm ” signs for the first two values of m^* indicate that either sign is admissible in the theorem.

THEOREM 2.3. *Let $n = q_1^{b_1} q_2^{b_2} \dots q_s^{b_s} > 1$, where the q_i 's are distinct primes such that $b_i \equiv 0 \pmod{2}$ whenever $q_i \equiv 3 \pmod{4}$. If $h(\mathbb{Q}(\sqrt{m^*})) = 1$, then whenever $\phi_m(x) \equiv 0 \pmod{n}$ for some $x \in \mathbb{Z}$, then $n = a^2 - m^*b^2$ for some $a, b \in \mathbb{Z}$.*

Proof. It suffices to show that $q_i = a^2 - m^*b^2$ for $i = 1, 2, \dots, s$ since we have that $(c^2 - m^*d^2)(e^2 - m^*f^2) = (ce - m^*df)^2 - m^*(de - cf)^2$. For $q_i \equiv 3 \pmod{4}$ we have $q_i = (q_i^{a_i/2})^2 - m^* \cdot 0^2$ so we assume $q_i \equiv 1 \pmod{4}$ or $q_i = 2$. Since $\phi_m(x) \equiv 0 \pmod{n}$ has an integer solution then by Theorem 2.1 we have $q_i \equiv 1 \pmod{m}$ or $q_i \mid m$. If $q_i \equiv 1 \pmod{m}$, then for $p_1 > 2$ or $r > 1$ we have that $(m^*/q_i) = \prod_j (p_j/q_i) = \prod_j (q_i/p_j) = 1$. If $p_1 = 2$ and $r = 1$, then $(m^*/q_i) = (-1/q_i) = 1$, whenever $a_1 = 2$; and if $a_1 > 2$, then $(m^*/q_i) = (-2/q_i) = 1$. Thus q_i is completely split in $\mathbb{Q}(\sqrt{m^*})$ or q_i ramifies in $\mathbb{Q}(\sqrt{m^*})$. In either case $q_i = N(\varphi_i)$, where φ_i is a $\mathbb{Q}(\sqrt{m^*})$ -prime above q_i . Moreover since $h(\mathbb{Q}(\sqrt{m^*})) = 1$, then $q_i = N(\varphi_i) = N(a + \sqrt{m^*} b)$, where $a + \sqrt{m^*} b$ is an element of $O_{\mathbb{Q}(\sqrt{m^*})}$. Hence $q_i = a^2 - m^*b^2$. Now if $m^* \equiv 2, 3 \pmod{4}$, then a and b are integers. If $m^* \equiv 1 \pmod{4}$, then $2a$ and $2b$ are integers. In this case let $a = c/2$ and $b = d/2$, where c and d are integers. Therefore if $q_i \equiv 1 \pmod{m}$, then we have $4 \equiv 4q_i \equiv c^2 \pmod{4m^*}$ which implies that c is even and so d is even. In any case we have the result for any $q_i \equiv 1 \pmod{m}$. If $q_i \mid m$, then $q_i = p_r$, where $p_r > p_i$ for all $i < r$, by

Theorem 2.1 and $q_i \equiv 1 \pmod{\prod_{i=1}^{r-1} p_i}$. Therefore if $r > 1$, then as in the above argument we have the result. The only remaining case is $r = 1$ and $p_r = q_i$. We are assuming that either $q_i = 2$ or $q_i \equiv 1 \pmod{4}$. If $q_i \equiv 1 \pmod{4}$, then there are integers a and b such that $b^2 - q_i a^2 = -1$. Therefore $q_i = (q_i a)^2 - q_i b^2$. If $q_i = 2$ and $a_1 = 2$, then $2 = 1^2 + 1^2$. If $q_i = 2$ and $a_1 > 2$, then $2 = 0^2 + 2 \cdot 1^2$. Q.E.D.

We note that if we remove the restriction $h(\mathbb{Q}(\sqrt{m^*})) = 1$ in Theorem 2.3, then the theorem fails to hold. For example, if $m = 79 = m^*$ and $n = 317$, then $h(\mathbb{Q}(\sqrt{79})) = 3$. Since $317 \equiv 1 \pmod{79}$, then by Theorem 2.1 we have $\phi_{79}(x) \equiv 0 \pmod{317}$ has an integer solution. Moreover $317 \equiv 1 \pmod{4}$. However, it can be shown that $317 \neq a^2 - 79b^2$ for any $a, b \in \mathbb{Z}$.

Furthermore, if we remove the restriction on n that all q_i dividing n appear to an even power if $q_i \equiv 3 \pmod{4}$, then the theorem fails to hold. For example, if $n = 23$ and $m = 11$, then $m^* = 11$ and $h(\mathbb{Q}(\sqrt{11})) = 1$. Moreover since $23 \equiv 1 \pmod{11}$, then $\phi_{11}(x) \equiv 0 \pmod{23}$ has a solution $x \in \mathbb{Z}$. However, since 23 is inert in $\mathbb{Q}(\sqrt{11})$, then $23 \neq a^2 - 11b^2$ for any $a, b \in \mathbb{Z}$.

However, what is of interest here is that if we relax our demands on the conclusion we can generalize the result. Suppose we only want that n is a norm of an integer from $\mathbb{Q}(\sqrt{m^*})$. Then as we shall see not only may we eliminate the restriction on n but we may also proceed from \mathbb{Q} to any number field K . We do however have to restrict m^* somewhat. We define $m = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} > 2$, where $p_1 < p_2 < \cdots < p_r$ are distinct primes and we define

$$\begin{aligned} m^* &= \prod_1^r p_i, & \text{if } \prod_1^r p_i \equiv 1 \pmod{4} \text{ or } p_1 = 2 \text{ and } a_1 > 2, \\ &= -\prod_1^r p_i, & \text{if } \prod_1^r p_i \equiv 3 \pmod{4}, \\ &= \prod_2^r p_i, & \text{if } p_1 = 2, a_1 \leq 2 \text{ and } \prod_2^r p_i \equiv 1 \pmod{4}, \\ &= -\prod_2^r p_i, & \text{if } p_1 = 2, a_1 \leq 2 \text{ and } \prod_2^r p_i \equiv 3 \pmod{4}. \end{aligned}$$

THEOREM 2.4. *Let K be an algebraic number field and let $\alpha \in O_K$ with $(\alpha) = \varphi_1^{b_1} \varphi_2^{b_2} \cdots \varphi_s^{b_s}$, where $\varphi_i | q_i$ in K/\mathbb{Q} . If $m_0 = m/p_r^{a_r} > 1$, then ε_{m_0} is not in K , and we assume that $a_r \geq f_i$ for all $i = 1, 2, \dots, s$, where m, a_r , and f_i are as above. Furthermore if $p_r = 2$, then we assume that 2 is unramified in K . Now if $h(K(\sqrt{m^*})) = 1 = h(K)$, then whenever $\phi_m(\beta) \equiv 0 \pmod{(\alpha)}$ for some*

$\beta \in O_K$, then $\alpha = N(\gamma)$ for some $\gamma \in O_{K(\sqrt{m^*})}$, where N is the norm map in $K(\sqrt{m^*})/K$.

Proof. By the fact that $h(K) = 1$, then $q_i = (\alpha_i)$ for some $\alpha_i \in O_K$ for $i = 1, 2, \dots, s$. By the multiplicativity of the norm map it suffices to show that $\alpha_i = N(\gamma_i)$ for some $\gamma_i \in O_{K(\sqrt{m^*})}$. Since $h(K(\sqrt{m^*})) = 1$, then if we can show that each q_i is completely split or ramified in $K(\sqrt{m^*})$, then we have that $N((\gamma_i)) = N(\hat{q}_i) = q_i = (\alpha_i)$, where $\hat{q}_i | q_i$ in $K(\sqrt{m^*})/K$. But by hypothesis we may invoke Theorem 2.1 to get that each q_i is completely split or ramified in $K(\varepsilon_m)$. Moreover by the choice of m^* we have $K(\sqrt{m^*}) \subseteq K(\varepsilon_m)$. This secures the theorem. Q.E.D.

We note that it is important to have conditions, even under as restrictive a hypothesis as Theorem 2.4, to determine when an algebraic integer is the norm of an algebraic integer from a given quadratic extension. It is possible, even in the simplest cases, to have an algebraic integer which is a norm from a quadratic extension but *not* the norm of an algebraic integer. One may for example use the product formula for the local norm residue symbol to determine whether or not an integer is a norm, but we cannot determine by those methods whether or not it is the norm of an integer.

For example, if $F = \mathbb{Q}(\sqrt{79})$, then $N((19 + 2\sqrt{79})/3) = 5$. However, using the norm residue symbol we get $N(\alpha) \neq 5$ for any $\alpha \in O_F$. One reason for this is that the F -ideals above 5 are not principal.

We now return to a discussion of Theorem 2.3 which has implications for the theory of representations of numbers by binary quadratic forms, as the following applications indicate.

Applications of Theorem 2.3

(1) By taking $m = 8$ we get that all primes of the form $8k + 1$ are representable in the form $x^2 + 2y^2$ for $x, y \in \mathbb{Z}$. Fermat proved this result in 1654.

More generally we have that all integers of the form $n = q_1^{b_1} q_2^{b_2} \dots q_s^{b_s}$, where all $q_i \equiv 1 \pmod{8}$, or $n = 2$, are representable in the form $x^2 + 2y^2$.

(2) We have Fermat's two-square theorem; viz. if $\phi_4(x) \equiv 0 \pmod{n}$ is solvable for $x \in \mathbb{Z}$, then $n = a^2 + b^2$. In other words if $n = q_1^{b_1} \dots q_s^{b_s}$, where $b_i \equiv 0 \pmod{2}$ if $q_i \equiv 3 \pmod{4}$, and $q_i \leq 1$ if $q_i = 2$, then $n = a^2 + b^2$.

(3) If p is an odd prime, then $x^2 - py^2$ represents all primes of the form $1 + 4pt$ (e.g., $x^2 - 3y^2$ represents all $q \equiv 1 \pmod{12}$).

The converse of Theorem 2.3 fails, as the following examples indicates. If $m^* = m = 7$ and $n = 53$, then $53 = 9^2 - 7 \cdot 2^2$. Moreover $h(\mathbb{Q}(\sqrt{7})) = 1$ and $53 \equiv 1 \pmod{4}$. However, $53 \not\equiv 1 \pmod{7}$ and so by Theorem 2.1

$\phi_7(x) \not\equiv 0 \pmod{53}$ for any $x \in \mathbb{Z}$. The converse even fails for $m = 2, 3$. For example, if $n = 2^2 + 2^2$, and $m = 4$, then by Theorem 2.1, $\phi_4(x) \not\equiv 0 \pmod{8}$ for any $x \in \mathbb{Z}$. Also if $n = 4 = 4^2 - 3 \cdot 2^2$ and $m = 3$, then $\phi_3(x) \not\equiv 0 \pmod{4}$ since $2 \not\equiv 1 \pmod{3}$. However, with minor restrictions on a given $n \in \mathbb{Z}$ we can get the converse of Theorem 2.3 to hold for $m = 2, 3$ as shown in [5, Lemma 2.7].

The latter result leads us into an interesting conjecture made by Chowla in [3]; viz. If $g(x, y), h(x, y) \in \mathbb{Z}[x, y]$ are primitive irreducible polynomials which represent the same numbers, then g and h are equivalent by a unimodular transformation. However, Schinzel in [7] gave the following counterexample: $f(x, y) = x^2 + 3y^2$ and $g(t_1, t_2) = t_1^2 + t_1 t_2 + t_2^2$ have the same set of values but they are not equivalent by a unimodular transformation. Now in [5] we showed that given $n = a^2 + 3b^2$ with $(a, b) = 1$ and $2 \mid a$, then $\phi_3(x) \equiv 0 \pmod{n}$ is solvable for $x \in \mathbb{Z}$. Moreover we have an explicit determination of those integer solutions.¹ We now give a more palatable analog of this method using the form of $g(t_1, t_2)$. Let $e = a - b$ and $f = 2b$, then $n = a^2 + 3b^2 = e^2 + ef + f^2$ with $(e, f) = 1$. Thus there exist $c, d \in \mathbb{Z}$ such that $fd - ce = 1$. Now consider the following matrix $A = \begin{pmatrix} f & -e \\ -c & d \end{pmatrix}$ and the following matrix product:

$$\begin{aligned} A \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} A^{-1} &= \begin{pmatrix} -df - ed - fc & -ef - e^2 - f^2 \\ cd + d^2 + c^2 & ce + ed + fc \end{pmatrix} \\ &= \begin{pmatrix} -df - ed - fc & -ef - e^2 - f^2 \\ cd + d^2 + c^2 & -1 + df + ed + fc \end{pmatrix} \\ &= \begin{pmatrix} k & -n \\ h & -1 - k \end{pmatrix}. \end{aligned}$$

However, the determinant of the latter matrix must clearly be 1. Therefore $-k(1 + k) + nh = 1$; that is, $k^2 + k + 1 \equiv 0 \pmod{n}$. Thus $k = -df - ed - fc$ is a solution of $\phi_3(x) \equiv 0 \pmod{n}$.

EXAMPLE. Let $n = 4339$, a prime. Then $n = 64^2 + 3 \cdot 9^2 = 55^2 + 18 \cdot 55 + 18^2$. Thus $a = 64$, $b = 9$, $e = 55$, $f = 18$, and $c = -1$ while $d = -3$. Therefore $k = 237$; that is, $\phi_3(237) \equiv 0 \pmod{4339}$.

3. THE pq -TH CYCLOTOMIC POLYNOMIAL

$\phi_{pq}(x)$ has been studied by Ivanov [4], Carlitz [2], Beiter [1], and Zeitlin [8] among others. We provide herein an explicit determination of $\phi_{pq}(x)$ from which we obtain much of the above as well as some generalizations thereof.

¹ Note that in [5, Theorem 2.8] m should be $(-1 + 3bc + ad)/2$. Also, in the subsequent example, m should be $-((p^2 - p + 14)/4)$.

THEOREM 3.1. *Let $q < p$ be primes, and let l_i be the quadratic residue of $-pi$ modulo q , where $0 \leq l_i < q$ for $i = 1, 2, \dots, q - 1$. Then $\phi_{pq}(x) = 1 + \sum_{j=1}^{q-1} \sum_{i=1}^{k_j} (x^{(q-j)p-qi+1} - x^{(q-j)p-qi})$, where $k_j = ((q-j)p - l_j)/q$.*

Proof. Let $f(x)$ denote the above polynomial for the moment so as not to prejudice the situation. We now verify that $\phi_1(x) \phi_p(x) \phi_q(x) f(x) = x^{pq} - 1$. We have $\phi_1(x) \phi_p(x) \phi_q(x) = x^{p+q-1} + x^{p+q-2} + \dots + x^p - x^{q-1} - x^{q-2} \dots - 1$. Now $x^{p+q-i} f(x)$ has all of its positive terms, except x^{p+q-i} , cancel with all of the negative terms of $x^{p+q-i+1} f(x)$ for $i = 2, 3, \dots, q$. Moreover all of the positive terms of $x^{p+q-1} f(x)$, except $x^{pq} + x^{pq-p} + \dots + x^{2p} + x^{p+q-1}$, cancel with all of the negative terms of $x^p f(x)$, except $-x^{p+l_1} - x^{p+l_2} - \dots - x^{p+l_{q-1}}$. Thus $(x^{p+q-1} + x^{p+q-2} + \dots + x^p) f(x) = x^{p+q-2} + x^{p+q-3} + \dots + x^p + x^{pq} + x^{pq-p} + \dots + x^{2p} + x^{p+q-1} - x^{p+l_1} - x^{p+l_2} - \dots - x^{p+l_{q-1}}$. Furthermore $-x^{q-i} f(x)$ has all of its negative terms except $-x^{q-i}$ cancel with all of the positive terms of $-x^{q-i+1} f(x)$ for $i = 2, 3, \dots, q$. Moreover $-x^{q-1} f(x)$ has all of its negative terms, except $-x^{pq-p} - x^{pq-2p} - \dots - x^p - x^{q-1}$, cancel with all of the positive terms of $-f(x)$, except $x^{l_1} + \dots + x^{l_{q-1}}$. Thus $-(x^{q-1} + x^{q-2} + \dots + 1) f(x) = -x^{q-2} - x^{q-3} - \dots - 1 - x^{pq-p} - x^{pq-2p} - \dots - x^p - x^{q-1} + x^{l_1} + x^{l_2} + \dots + x^{l_{q-1}}$. Hence we have in total that

$$\begin{aligned} &\phi_1(x) \phi_p(x) \phi_q(x) f(x) \\ &= x^{p+q-2} + x^{p+q-3} + \dots + x^{p+1} + x^p \\ &\quad + x^{pq} + x^{pq-p} + \dots + x^{2p} + x^{p+q-1} - x^{p+l_1} - x^{p+l_2} \\ &\quad - \dots - x^{p+l_{q-1}} - x^{q-2} - x^{q-3} - \dots - 1 - x^{pq-p} - x^{pq-2p} \\ &\quad - \dots - x^p - x^{q-1} + x^{l_1} + x^{l_2} + \dots + x^{l_{q-1}}, \\ &= x^{p+q-1} + x^{p+q-2} + \dots + x^{p+1} + x^{pq} - x^{q-1} - x^{q-2} \\ &\quad - \dots - x - 1 + x^{l_1} + x^{l_2} + \dots + x^{l_{q-1}} - x^{p+l_1} - x^{p+l_2} \\ &\quad - \dots - x^{p+l_{q-1}}, \\ &= (x^{pq} - 1) + (x^p - 1)(x^{q-1} + x^{q-2} + \dots + x) \\ &\quad - (x^p - 1)(x^{l_1} + x^{l_2} + \dots + x^{l_{q-1}}), \\ &= x^{pq} - 1 \quad \text{since } \phi_q(x) = x^{l_1} + x^{l_2} + \dots + x^{l_{q-1}}. \end{aligned} \qquad \text{Q.E.D.}$$

APPLICATIONS OF THEOREM 3.1

(i) For arbitrary $p > 2 = q$ we have that $\phi_{2p}(x) = x^{p-1} - x^{p-2} + \dots - x + 1$. Also we have as examples

$$\phi_{21}(x) = x^{12} - x^{11} + x^9 - x^8 + x^6 + x^3 - x^4 - x + 1,$$

and

$$\begin{aligned} \phi_{35}(x) &= x^{24} - x^{23} + x^{19} - x^{18} + x^{17} - x^{16} + x^{14} - x^{13} \\ &\quad + x^{12} - x^{11} + x^{10} - x^8 + x^7 - x^6 + x^5 - x + 1. \end{aligned}$$

(ii) For $0 < j, h < q$ and $0 < i \leq k_j; 0 < m \leq k_h$ we have $(q-j)p - qi = (q-h)p - qm$ if and only if $h=j$ and $i=m$. Thus the only coefficients of $\phi_{pq}(x)$ are $0, \pm 1$. This was first obtained by V. Ivanov [4].

(iii) The following result was first obtained by Carlitz in [2, Theorem, p. 980]. We now show how the result may be obtained from Theorem 3.1.

THEOREM 3.2. *Let p and q be arbitrary primes with $q < p$, and let u be defined by $pu = -1 + qt$ with $0 < u < q$. If $\theta_0(pq)$ denotes the number of terms with positive coefficient in $\phi_{pq}(x)$, then $\theta_0(pq) = (q-u)(pu+1)/q$.*

Proof. For $i = 1, 2, \dots, q-1$ define j_i by $(q-i)p = l_i + qj_i$, where l_i is as in Theorem 3.1. If we have cancellation of terms in the form of $\phi_{pq}(x)$ as given in Theorem 3.1, then for some set of integers i, k, l, m with $0 < i, k < q$ and $1 \leq l \leq j_k; 1 \leq m \leq j_i$ we have $(q-i)p - qm = (q-k)p - ql + 1$; that is, $q(l-m) - p(i-k) = 1$. There are only two possible sets of solutions for this equation.

Case 1. $m = l - t$ and $k = i - u$ in which case $i = u + 1, \dots, q - 1$ which implies $k = 1, 2, \dots, q - u - 1$ and $l = t + 1, \dots, t + j_i$ which implies $m = 1, \dots, j_i$. Thus for Case 1 there are $\sum_{i=u+1}^{q-1} j_i$ cancellations.

Case 2. For $u \geq 2$ set $k = i - (u - q)$ and $j = l - (t - p) = l + j_u$ in which case $i = 1, \dots, u - 1$ which implies $k = q - u + 1, \dots, q - 1$ and $l = j_1 - j_u = j_{q-u+1}, \dots, j_{q-u+i}$ which implies $m = j_1, \dots, j_i$. Thus there are $\sum_{c=q-u+1}^{q-1} j_c$ cancellations in Case 2.

Hence the number of terms in $\phi_{pq}(x)$ with positive coefficient is

$$\begin{aligned} \theta_0(pq) &= 1 + \sum_{b=1}^{q-1} j_b - \sum_{c=u+1}^{q-1} j_c - \sum_{d=q-u+1}^{q-1} j_d \\ &= 1 + \sum_{b=1}^u j_b - \sum_{c=q-u+1}^{q-1} j_c. \end{aligned}$$

We now show that this equals $(q - u)(pu + 1)/q$.

$$\begin{aligned} & 1 + \sum_1^u j_b - \sum_{q-u+1}^{q-u} j_b \\ &= 1 + pu - \frac{pu(u+1)}{2q} - \sum_{b=1}^u \frac{l_b}{q} - \sum_{b=1}^{u-1} \frac{(q - (q - u + b))p - l_b}{q} \\ &= \frac{(q - u)(pu + 1) + u + (\sum_{b=1}^{u-1} l_{q-u+b} - \sum_{b=1}^u l_b)}{q}. \end{aligned}$$

It now suffices to show that $\sum_1^{u-1} l_{q-u+b} - \sum_1^u l_b = -u$. First we prove the following:

Claim. $l_{q-u+b} = l_b - 1$ and $j_{q-u+b} - j_b - j_u$. We have $(l_b - l_u) + q(j_b - j_u) = (q - b)p - (q - u)p = (q - (q - u + b)) = l_{q-u+b} + qj_{q-u+b}$. But since $l_u + j_u q = (q - u)p = qp - pu = qp + 1 - qt = 1 + q(p - t)$, then $l_u = 1$ and so $l_b - 1 = l_{q-u+b}$ and $j_{q-u+b} = j_b - j_u$. This completes the proof of the claim.

Therefore $\sum_1^{u-1} l_{q-u+b} - \sum_1^u l_b = \sum_1^u (l_b - 1) - \sum_1^u l_b = \sum_1^{u-1} l_b - (u - 1) - \sum_1^u l_b = -(u - 1) - l_u = -(u - 1) - 1 = -u$. Q.E.D.

It is interesting to note that Carlitz [2, p. 981] mentions that the value of $\theta_0(pq)$ depends strongly on the residue of p modulo q . We can see from the above why this is the case.

Now let $d = \phi(pq) = (p - 1)(q - 1)$. As the above proof indicates, cancellations occur in a symmetric fashion about $x^{d/2}$. Thus we have the following which is [8, Lemma 4, p. 978]:

COROLLARY 3.3. *If $\theta_0(pq)$ is odd, then the coefficient of $x^{d/2}$ in $\phi_{pq}(x)$ is 1 and it is -1 otherwise.*

Furthermore we have the following result which generalizes [8, Lemma 5, p. 978]:

COROLLARY 3.4. *Let $p = a + q^k$ for primes p and q , and let $q \equiv b \pmod{a}$ for $0 < b < a$, with $r =$ the order of b modulo a . Then if $b^{r-1}q \equiv 1 \pmod{2a}$, then the coefficient of $x^{d/2}$ in $\phi_{pq}(x)$ is 1. Otherwise the coefficient is -1 .*

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