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Matrices and Continued Fractions *

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Abstract

In this presentation, we establish the fact that any palindrome of natural numbers can be realized as the symmetric part of either the simple continued fraction expansion of $\sqrt{D}$ or $(1+\sqrt{D})/2$ for a suitable nonsquare natural number D, the polishing of a result known to Perron via the use of matrices. This result allows us to generalize our notion of a *continued fraction beeper*, introduced in earlier work, to what what we call *Mega-beepers*, which have arbitrarily long period lengths in their simple continued fraction expansions and the symmetric part is the repetition, for any numbers of times required, of a palindrome that is itself the repetition of a given simple continued fraction expansion. We display several examples including examples related to Fibonacci numbers. We are also able to explicitly determine the fundamental unit of the underlying quadratic order.

1 Introduction

In [8]–[9], Schinzel studied integral polynomials f from the perspective of the period lengths $\ell(\sqrt{f(X)})$ of the simple continued fraction expansion of $\sqrt{f(X)}$. The case that he covered, in which we are most interested, is the quadratic case. In this case, Schinzel showed that $\overline{\lim}_{x\to\infty}\ell(\sqrt{f(X)}) < \infty$ if and only if $f(X) = A^2X^2 + BX + C$, with $A \in \mathbb{N}$, $B^2 - 4A^2C \neq 0$, and $(B^2 - 4A^2C) \mid 4\gcd(2A^2, B)^2$. It is easy to verify that without loss of generality, we look at $f(X) = A^2X^2 + 2BX + C$ where A is even. Moreover, if $B^2 - A^2C = 1$, for nonsquare C, then Schinzel's condition is always satisfied. In [6], we

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exhibited infinitely many such families $f_k(X)$ with $\lim_{k\to\infty} \ell(\sqrt{f_k(X)}) = \infty$ for any fixed $X \in \mathbb{N}$, whereas for fixed $k \in \mathbb{N}$, $\ell(\sqrt{f_k(X)}) = \ell(\sqrt{f_k(X+1)})$ for all $X \in \mathbb{N}$, and we were able to easily exhibit the explicit fundamental unit for each order $\mathbb{Z}[\sqrt{f(X)}]$. We continued this work in [3]–[5].

In this current endeavor, we observe that Perron knew the fact that any palindrome of natural numbers is realizable as the simple continued fraction expansion of either $\sqrt{D}$ or $(1+\sqrt{D})/2$ for some nonsquare natural number D. This appears not to be *widely* known and we exhibit generalizations, using this result, of some recent results in the literature. Moreover, we are able to extend our notion, introduced in [3], of a *continued fraction beeper*, which allows us to display simple continued fraction expansions of arbitrary length and complexity building upon recent results of Madden [1]. Of course, we are indebted to the pioneering work, not only of Schinzel, but also of Dan Shanks [10]–[11] and others who followed over the last 20 years in developing families of quadratic surds with unbounded period lengths (see [6] for an overview). However, as we will see below, we perhaps owe the greatest debt to Perron [7].

The following sets some notation. We denote the simple continued fraction expansion of α (in terms of its *partial quotients*) by:

$$\alpha = \langle q_0; q_1, \ldots, q_n, \ldots \rangle ,$$

and if α is *periodic*, such as $\sqrt{D}$ for nonsquare integers $D > 0$ for instance, we denote the simple continued fraction expansion by

$$\sqrt{D} = \langle q_0; \overline{q_1, \ldots, q_{\ell-1}, 2q_0} \rangle ,$$

with period length $\ell(\sqrt{D}) = \ell$. The *convergents* (for $n \geq 0$) of a given $\alpha = \langle q_0; q_1, \ldots \rangle$ are denoted by

$$\frac{x_n}{y_n} = \langle q_0; q_1, \ldots, q_n \rangle = \frac{q_n x_{n-1} + x_{n-2}}{q_n y_{n-1} + y_{n-2}}. \tag{1}$$

We also will need the following for the convergents of any quadratic irrational $\alpha = (P + \sqrt{D})/Q$, with period length $\ell = \ell(\alpha)$.

$$x_{\ell-1} = q_0 y_{\ell-1} + y_{\ell-2}, \tag{2}$$

$$x_j y_{j-1} - x_{j-1} y_j = (-1)^{j-1} \qquad (j \in \mathbb{N}), \tag{3}$$

and if we set, for any $j \geq -2$

$$g_j = Qx_j - Py_j, \tag{4}$$

then the following important result holds.

Theorem 1 *If $D > 0$ is a nonsquare integer and we set $\sigma = 1$ if $D \not\equiv 1 \pmod 4$, and $\sigma = 2$, otherwise then the fundamental unit of the real quadratic order $\mathbb{Z}[(\sigma - 1 + \sqrt{D})/\sigma]$ is*

$$\frac{g_{\ell-1} + y_{\ell-1}\sqrt{D}}{\sigma},$$

where $\ell = \ell((\sigma - 1 + \sqrt{D})/\sigma)$.

Proof. See [2, Theorems 5.5.2–5.5.3, pp. 261–269]. $\square$

2 Results

A beautiful result by Perron, (so-called by Perron himself–see [7, bottom page 88]), is the following which should be well-known but appears not to be as we shall see.

Theorem 2 *Given a palindrome $q_1, \ldots, q_{\ell-1}$ of natural numbers for $\ell \geq 2$, there exist integers $u, v, w \in \mathbb{Z}$ such that the following matrix equation holds:*

$$\prod_{j=1}^{\ell-1} \begin{pmatrix} q_j & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} u & v \\ v & w \end{pmatrix}. \tag{5}$$

If we set

$$\sigma = \begin{cases} 1 & \text{if } u \equiv vw \equiv 0 \pmod 2, \\ 2 & \text{if } u \equiv vw + 1 \equiv 0 \pmod 2, \end{cases}$$

and either choice of $\sigma = 1$ or $\sigma = 2$ is allowed if u is odd, then there exists a nonsquare $D \in \mathbb{N}$ such that

$$\frac{\sigma - 1 + \sqrt{D}}{\sigma} = \left\langle q_0; \overline{q_1, \ldots, q_{\ell-1}, 2q_0 - \sigma + 1} \right\rangle, \tag{6}$$

where

$$q_0 = (\sigma - 1 + ux - (-1)^{\ell} vw)/2 \tag{7}$$

for some $x \in \mathbb{Z}$. *Moreover, when this holds, and* x_j/y_j *is the* j^{th} *convergent of* $(\sigma - 1 + \sqrt{D})/\sigma$, *then*

$$u = y_{\ell-1}, \quad v = y_{\ell-2}, \text{ and } \quad w = x_{\ell-2} - q_0 y_{\ell-2}, \tag{8}$$

and

$$D = (\sigma q_0 - \sigma + 1)^2 + \sigma^2 x v - \sigma^2 (-1)^\ell w^2 = \tag{9}$$
$$\left(\frac{\sigma}{2}\right)^2 u^2 x^2 + \left(\sigma^2 v - \frac{(-1)^\ell}{2} uvw\right) x + \left(\frac{\sigma}{2}\right)^2 v^2 w^2 - (-1)^\ell \sigma^2 w^2.$$

Proof. See [7]. □

Notice that any choice of x which makes $\sigma - 1 + ux - (-1)^\ell vw$ even and positive will suffice. Thus for a suitable choice of

$$x \geq \begin{cases} \lfloor vw/u \rfloor + 1 & \text{if } \ell \text{ is even,} \\ -\lfloor vw/u \rfloor & \text{if } \ell \text{ is odd,} \end{cases}$$

we may find infinitely many such $(\sigma - 1 + \sqrt{D})/\sigma$ with the palindrome as symmetric part.

It is also quite worth observing another matrix sequence of values. we present this here with proof since we are unaware of any place in the literature where the result appears for *arbitrary* quadratic orders, and the result is certainly worthy of display as a result of interest in its own right.

Theorem 3 (Fundamental Unit Theorem for Quadratic Orders) *Suppose that* (6) *holds. Then*

$$\prod_{j=0}^{\ell-1} \begin{pmatrix} q_j & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{(\sigma-1)\sigma t + (\sigma-1)s + Ds}{\sigma^2} & \frac{(\sigma-1)s+t}{\sigma} \\ \frac{(\sigma-1)s+t}{\sigma} & s \end{pmatrix}, \tag{10}$$

where

$$t^2 - s^2 D = \pm \sigma^2,$$

and $(t + s\sqrt{D})/\sigma$ *is the fundamental unit of the order* $\mathbb{Z}[(\sigma - 1 + \sqrt{D})/\sigma]$.

Proof. Using (5), we get:

$$\prod_{j=0}^{\ell-1} \begin{pmatrix} q_j & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} q_0^2 u + 2q_0 v + w & q_0 u + v \\ q_0 u + v & u \end{pmatrix}. \tag{11}$$

Now let

$$s = u \quad \text{and} \quad t = \sigma(q_0 u + v) - (\sigma - 1)u. \tag{12}$$

Therefore, $((\sigma - 1)s + t)/\sigma = q_0 u + v$. We now show that upper left entries in the matrices (10)–(11) agree. By substituting the values for s, t, and D given in (9) and (12), we have,

$$\frac{(\sigma - 1)\sigma t + (\sigma - 1)s + Ds}{\sigma^2} = q_0^2 u + uvx - (-1)^\ell w^2 u + (\sigma - 1)v.$$

Thus, we need only show that $2q_0 v + w = uvx - (-1)^\ell uw^2 + (\sigma - 1)v$. However, from (7), we deduce that we only need to verify that $v^2 - uw = (-1)^\ell$, which follows from (2)–(3) and (8).

It remains to show that $(t + s\sqrt{D})/\sigma$ is indeed the fundamental unit. However, by (2), (4), (8), and (12),

$$\frac{t + s\sqrt{D}}{\sigma} = \frac{g_{\ell-1} + y_{\ell-1}\sqrt{D}}{\sigma},$$

and by Theorem 1, this is the fundamental unit. □

Example 1 *Given* $D = 245$,

$$\sqrt{245} = \langle 15; \overline{1, 1, 1, 7, 6, 7, 1, 1, 1, 30} \rangle = \langle q_0; \overline{q_1, \ldots, q_{\ell-1}, 2q_0} \rangle,$$

so

$$\prod_{j=0}^{\ell-1} \begin{pmatrix} q_j & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 15 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 811440 & 51841 \\ 51841 & 3312 \end{pmatrix} = \begin{pmatrix} sD & t \\ t & s \end{pmatrix},$$

with $t^2 - s^2 D = 51841^2 - 3312^2 \cdot 245 = 1$, *where* $51841 + 3312\sqrt{245}$ *is the fundamental unit of* $\mathbb{Z}[\sqrt{245}]$.

Example 2 *For* $D = 45$, $(1 + \sqrt{45})/2 = \langle 3; \overline{1, 5} \rangle$, *and*

$$\prod_{j=0}^{1} \begin{pmatrix} q_j & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 15 & 4 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} (2t + s + Ds)/4 & (s + t)/2 \\ (s + t)/2 & s \end{pmatrix},$$

so $s = 1$, $t = 7$, *and* $t^2 - s^2 45 = 4$, *with* $(7 + \sqrt{45})/2$ *being the fundamental unit of* $\mathbb{Z}[(1 + \sqrt{45})/2]$.

Notice that since $q_1 = 1 = q_{\ell-1}$, *then we could also have chosen* $\sigma = 1$ *in Theorem 2. for instance,* $\sqrt{3} = \langle 1; \overline{1, 2}\rangle$ *where*

$$\prod_{j=0}^{1} \begin{pmatrix} q_j & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} sD & t \\ t & s \end{pmatrix},$$

so $(t + s\sqrt{D})/\sigma = 2 + \sqrt{3}$ *is the fundamental unit of* $\mathbb{Z}[\sqrt{3}]$.

As noted in Theorem 2, it is a fact that the existence of a palindrome of natural numbers

$$s = q_1, \dots, q_{\ell-1}$$

necessarily gives rise to a symmetric matrix given by (5). we may now invoke *Lucas-Lehmer theory* (for instance, see [2]). First we notice that the eigenvalues α, β of the matrix

$$M = \begin{pmatrix} u & v \\ v & w \end{pmatrix} \tag{13}$$

are given by

$$\alpha = \frac{u + w + \sqrt{(u-w)^2 + 4v^2}}{2} \qquad \beta = \frac{u + w - \sqrt{(u-w)^2 + 4v^2}}{2},$$

so we may define the Lucas functions for this scenario as

$$U_k = \frac{\alpha^k - \beta^k}{\alpha - \beta}.$$

A simple induction shows us that for any $k \in \mathbb{N}$,

$$M^k = \begin{pmatrix} U_{k+1} - wU_k & vU_k \\ vU_k & U_{k+1} - uU_k \end{pmatrix} \tag{14}$$

using he fact that $U_{k+2} = (u + w)U_{k+1} - \det(M)U_k$, also easily established by induction. Thus, by Theorem 2, if either $U_{k+1} - wU_k$ is odd or

$$U_{k+1} - wU_k \equiv vU_k(U_{k+1} - uU_k) \pmod 2,$$

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$$\sqrt{D} = \langle q_0; \overline{\underbrace{s, s, \ldots, s}_{k \text{ copies}}, 2q_0} \rangle,$$

and otherwise,

$$\frac{1+\sqrt{D}}{2} = \langle q_0; \overline{\underbrace{s, s, \ldots, s}_{k \text{ copies}}, 2q_0 - 1} \rangle.$$

Example 3 *Let* $s = q_1, \ldots, q_4 = 2, 1, 1, 2$. *Then we may calculate that*

$$M = \begin{pmatrix} 13 & 5 \\ 5 & 2 \end{pmatrix},$$

$$\alpha = \frac{15+\sqrt{221}}{2}; \qquad \beta = \frac{15-\sqrt{221}}{2},$$

$$U_k = \frac{(15+\sqrt{221})^k - (15-\sqrt{221})^k}{2^k\sqrt{221}},$$

and

$$M^k = \begin{pmatrix} U_{k+1} - wU_k & vU_k \\ vU_k & U_{k+1} - uU_k \end{pmatrix}.$$

In particular, if $k = 2$, *then*

$$M^2 = \begin{pmatrix} 194 & 75 \\ 75 & 29 \end{pmatrix},$$

so by choosing $x = -\lfloor vw/u \rfloor = -\lfloor 75 \cdot 29/194 \rfloor = -11$, $\sigma = 2$,*and* $\ell = 9$ *we get*

$$q_0 = 21 = \frac{1}{2}(\sigma - 1 + ux - (-1)^\ell vw)$$

and

$$D_2 = 1745 = (\sigma q_0 - \sigma + 1)^2 + \sigma^2 xv - \sigma^2(-1)^\ell w^2.$$

Thus,

$$\frac{1+\sqrt{1745}}{2} = \langle 21; \overline{2, 1, 1, 2, 2, 1, 1, 2, 41} \rangle,$$

containing the two copes of the original palindrome in the symmetric part, and for any choice of $x \geq -11$ *we get infinitely many such* D *to represent these two copies. Similarly, for* $k = 3$,

$$M^3 = \begin{pmatrix} 2897 & 1120 \\ 1120 & 433 \end{pmatrix},$$

so $\sigma = 1$, $\ell = 13$ *and by choosing* $x = -166$, *we get* $q_0 = 2029$ *and* $D_3 = 4118410$ *with*

$$\sqrt{4118410} = \langle 2029; \overline{2, 1, 1, 2, 2, 1, 1, 2, 2, 1, 1, 2, 40587}\rangle,$$

representing the three copies of the original palindrome. Again infinitely many such Ds may be found to so represent these three copies, and so on for any k *such that*

$$\frac{\sigma - 1 + \sqrt{D_k}}{\sigma} = \langle q_0; \overline{\underbrace{s, s, \ldots, s}_{k \text{ copies}}, 2q_0 - 1}\rangle.$$

Note that $D_1 = 29$ *and* $\sqrt{29} = \langle 5; \overline{2, 1, 1, 2, 10}\rangle$.

The process illustrated in Example 3 of repeating a part of a symmetric period in a given continued fraction was studied from another perspective in [3], wherein we introduced he following concept.

Definition 1 *Suppose that* $C \in \mathbb{N}$ *and* $\alpha = (P + \sqrt{D})/Q$ *is a quadratic irrational having simple continued fraction expansion* $\alpha = \langle c_0; \overline{c_1, \ldots, c_n, c_\ell}\rangle$. *Furthermore, let*

$$w_m^{(\alpha)} = c_1, \ldots, c_n, c_\ell, c_1, \ldots, c_n, c_\ell, \ldots, c_1, \ldots, c_n, \tag{15}$$

which is $m \geq 0$ *iterations of*

$$c_1, \ldots, c_n, c_\ell$$

followed by one iteration of

$$c_1, \ldots, c_n,$$

which is the empty string if $n = 0$. *Then a simple continued fraction of the form*

$$\langle q_0; \overline{w_m^{(\alpha)}, q_\ell}\rangle,$$

is called an m-beeper *for* α.

Example 4 *Set* $\alpha = (1+\sqrt{5})/2 = \langle \overline{1} \rangle$ *and for any* $k, X \in \mathbb{N}$, *let*

$$D_k(X) = 4F_{2k}^2 X^2 + (20F_k^2 + 8(-1)^k)X + 5,$$

where F_n *is the* n^{th} *Fibonacci number for any* $n \in \mathbb{N}$. *In* [4], *we showed that*

$$\frac{1+\sqrt{D_k(X)}}{2} = \left\langle F_{2k}X + 1; \overline{w_{2k-1}^{(\alpha)}, 2F_{2k}+1} \right\rangle =$$

$$\left\langle F_{2k}X + 1; \overline{\underbrace{1, 1, \ldots, 1}_{2k-1 \text{ copies}}, 2F_{2k}+1} \right\rangle,$$

a $(2k-1)$*-beeper for* α. *In* [3], *we presented the following as an example, involving Fibonacci numbers, where*

$$\ell(\sqrt{D_k(X)}) = \frac{1}{3}\ell\left(\frac{1+\sqrt{D_k(X)}}{2}\right) = 2t,$$

the minimum possible according to [12]. *If* $k = 3t$ *for* $t \in \mathbb{N}$, *and* $\beta = \sqrt{5} = \langle 2; \overline{4} \rangle$, *then*

$$\sqrt{D_k(X)} = \left\langle 2(F_{6t}X + 1); \overline{w_{2k-1}^{(\beta)}, 4(F_{6t}+1)} \right\rangle =$$

$$\left\langle 2(F_{6t}X + 1); \overline{\underbrace{4, 4, \ldots, 4}_{2t-1 \text{ copies}}, 4(F_{6t}+1)} \right\rangle,$$

a $(2t-1)$*-beeper for* β. *However, we did not prove this, relying instead on referring to the techniques developed in* [4] *as a means for the reader to verify this. However, with Theorem 2 at our disposal, it is an easy task, and a rather informative one, to prove this result here. If we let* $q_j = 4$ *for* $j = 1, 2, \ldots, \ell - 1 = 2t - 1$, *then for* $\ell = 2t$ *and* $k = 3t$:

$$\prod_{j=1}^{\ell-1} \begin{pmatrix} q_j & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} F_{6t}/2 & F_{6t-3}/2 \\ F_{6t-3}/2 & F_{6t-6}/2 \end{pmatrix},$$

and

$$\left\lfloor \frac{F_{6t-3}F_{6t-6}}{2F_{6t}} \right\rfloor = \frac{F_{6t-3}F_{6t-6}+16}{2F_{6t}} - 1.$$

Thus, by choosing

$$x = \left\lfloor \frac{F_{6t-3}F_{6t-6}}{2F_{6t}} \right\rfloor + 8X + 1 = \frac{F_{6t-3}F_{6t-6}+16}{2F_{6t}} + 8X$$

in Theorem 2, we get the desired expansion since

$$q_0 = \frac{1}{2}(ux - vw) = \frac{1}{2}\left(\frac{F_{6t}}{2}\left(\frac{F_{6t-3}F_{6t-6}+16}{2F_{6t}} + 8X\right) - \frac{F_{6t-3}}{2}\frac{F_{6t-6}}{2}\right) =$$

$$2(F_{6t}X + 1).$$

Also, note that $y_{\ell-1} = y_{2t-1} = F_{6t}/2$, $y_{\ell-2} = F_{6t-3}/2$, $x_{\ell-2} - q_0 y_{\ell-2} = F_{6t-6}/2$, *and the fundamental unit of* $\mathbb{Z}[\sqrt{D_{3t}(X)}]$ *is*

$$\varepsilon_{4D_{3t}(X)} = x_{\ell-1} + y_{\ell-1}\sqrt{D_{3t}(X)},$$

given by

$$(x_{\ell-1}, y_{\ell-1}) = (F_{6t}^2 X + 5F_{3t}^2/2 + (-1)^{3t}, F_{6t}/2).$$

Notice as well that in Example 4,

$$\lim_{k\to\infty} \ell(\sqrt{D_k(X)}) = \lim_{k\to\infty} \ell((1+\sqrt{D_k(X)})/2) = \infty,$$

whereas for fixed $k = 3t$, say,

$$\lim_{X\to\infty} \ell(\sqrt{D_k(X)}) = \frac{1}{3}\lim_{X\to\infty} \ell((1+\sqrt{D_k(X)})/2) = 2t.$$

Another instance from [3] that we did not prove is the following, which is also an opportunity to provide an illustration of Theorem 2 and establish more infinite families related to Fibonacci numbers.

Example 5 *Set* $\alpha = (1+\sqrt{45})/2 = \langle 3; \overline{1,5}\rangle$, *and for* $k, X \in \mathbb{N}$, *let*

$$D_k(X) = \left(\frac{2F_{4k}}{3}\right)^2 X^2 + 4\frac{F_{8k}}{F_{4k}}X + 45.$$

In [4], we proved that

$$\frac{1+\sqrt{D_k(X)}}{2} = \left\langle F_{4k}X/3 + 3; \overline{w_{k-1}^{(\alpha)}, 2F_{4k}/3 + 5}\right\rangle =$$

$$\frac{1+\sqrt{D_k(X)}}{2} = \left\langle F_{4k}X/3+3; \overline{\underbrace{1,5,1,5,\ldots,1,5,}_{k-1 \text{copies of } 1,5} 1, 2F_{4k}/3+5} \right\rangle,$$

a $(k-1)$-beeper for α. In [3], we stated without proof that for $\beta = \sqrt{45} = \langle 6; \overline{1,2,2,2,1,12} \rangle$, and $k = 3^t$ with $t \in \mathbb{N}$,

$$\sqrt{D_k(X)} = \left\langle 2F_{4k}X/3+6; \overline{w^{(\beta)}_{k/3-1}, 4F_{4k}X/3+12} \right\rangle =$$

$$\langle 2F_{4k}X/3+6; \overline{\underbrace{1,2,2,2,1,12,\ldots,1,2,2,2,1,12,}_{3^{t-1}-1 \text{copies of } 1,2,2,2,1,12} 1,2,2,2,1,4F_{4k}X/3+12} \rangle,$$

a $(3^{t-1}-1)$-beeper for β, and for any $X, t \in \mathbb{N}$,

$$\ell(\sqrt{D_{3^t}(X)}) = \ell\left(\frac{1+\sqrt{D_{3^t}(X)}}{2}\right) = 2 \cdot 3^t = \ell,$$

We now use Theorem 2 to establish this fact. One may establish that for $\{q_j\}_{j=1}^{(3^{t-1}-1)6+5}$ given by $w^{(\beta)}_{k/3-1}$, we get

$$\prod_{j=1}^{\ell-1} \begin{pmatrix} q_j & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} F_{4\cdot 3^t}/6 & F_{4\cdot 3^t-3}/2 \\ F_{4\cdot 3^t-3}/2 & 3F_{4\cdot 3^t-6}/2 \end{pmatrix},$$

and

$$\left\lfloor \frac{9F_{4\cdot 3^t-3}F_{4\cdot 3^t-6}}{2F_{4\cdot 3^t}} \right\rfloor = \frac{9F_{4\cdot 3^t-3}F_{4\cdot 3^t-6}+144}{2F_{4\cdot 3^t}} - 1.$$

Thus, by choosing $x = \left\lfloor \frac{9F_{4\cdot 3^t-3}F_{4\cdot 3^t-6}}{2F_{4\cdot 3^t}} \right\rfloor + 8X + 1$ in Theorem 2, we get the result since

$$q_0 = \frac{1}{2}(ux - vw) =$$

$$\frac{1}{2}\left(\frac{F_{4\cdot 3^t}}{6}\left(\frac{9F_{4\cdot 3^t-3}F_{4\cdot 3^t-6}+144}{2F_{4\cdot 3^t}} + 8X\right) - \frac{3F_{4\cdot 3^t-3}}{2}\frac{F_{4\cdot 3^t-6}}{2}\right) = \frac{2F_{4\cdot 3^t}X}{3} + 6.$$

Moreover, $y_{\ell-1} = F_{4\cdot 3^t}/6$, $y_{\ell-2} = F_{4\cdot 3^t-3}/2$, $x_{\ell-2} - q_0 y_{\ell-2} = 3F_{4\cdot 3^t-6}/2$, so the fundamental unit of $\mathbb{Z}[\sqrt{D_{3^t}(X)}]$ is given by

$$\varepsilon_{4D_{3^t}(X)} = x_{\ell-1} + y_{\ell-1}\sqrt{D_{3^t}(X)}$$

gien by

$$(x_{\ell-1}, y_{\ell-1}) = (x_{2\cdot 3^t-1}, y_{2\cdot 3^t-1}) = (F_{4\cdot 3^t}X/9 + F_{4\cdot 3^t} + F_{4\cdot 3^t-3}/2).$$

Example 6 *We can also use* $C = 29$ *from Example 3, so* $A_1 = 1$, $B_1 = 5$, $\alpha = (1+\sqrt{29})/2 = \langle 3; \overline{5}\rangle$, $B_k + A_k\sqrt{29} = (5+\sqrt{29})^k/4^{k-1}$, *so if*

$$D_k(X) = A_k^2X^2 + 2B_kX + 29,$$

then

$$\frac{1+\sqrt{D_k(X)}}{2} = \langle q_0; \overline{w_{2k-1}^{(\alpha)}, 2q_0 - 1}\rangle = \langle q_0; \overline{\underbrace{5, 5, \ldots, 5}_{2k-1 \text{ copies}}, 2q_0 - 1}\rangle,$$

since $n = 0$ *in Definition 1 for this case. Also,* $\ell((1+\sqrt{D_k(X)})/2) = 2k$.

Now we are in a position to exploit both Definition 1 and Theorem 2 to create a new infinite family of continued fractions.

Definition 2 *If* C, α, *and* $w_m^{(\alpha)}$ *are given as in Definition 1, and* $c_1, c_2, \ldots, c_\ell$ *is* not *a palindrome, and* $m > 0$, *then a simple continued fraction of the form*

$$\langle q_0; \overline{\underbrace{w_m^{(\alpha)}, w_m^{(\alpha)}, \ldots, w_m^{(\alpha)}}_{k \text{ copies}}, 2q_0 - \sigma + 1}\rangle$$

is called a k*-times* Mega-m-beeper *for* α.

Example 7 *The situation in Example 4 does not apply since* $c_1, c_2, \ldots, c_\ell$ *is a palindrome and so the application of Theorem 2 to it would only lead to a longer beeper. However, Example 5 provides us with an opportunity to illustrate Definition 2. For instance, take* $w_{k/3-1}^{(\beta)}$, *where* $k = 3^t$ *with* $t \in \mathbb{N}$.*we have that*

$$M = \begin{pmatrix} F_{4\cdot 3^t}/6 & F_{4\cdot 3^t-3}/2 \\ F_{4\cdot 3^t-3}/2 & 3F_{4\cdot 3^t-6}/2 \end{pmatrix},$$

so we invoke theorem 2 with $u = F_{4\cdot 3^t}/6$, $v = F_{4\cdot 3^t-3}/2$, $w = 3F_{4\cdot 3^t-6}/2$,

$$\alpha = \frac{F_{4\cdot 3^t}/6 + F_{4\cdot 3^t-3}/2 + \sqrt{(F_{4\cdot 3^t}/6 - F_{4\cdot 3^t-3}/2)^2 + 4(F_{4\cdot 3^t-3}/2)^2}}{2},$$

$$\beta = \frac{F_{4\cdot 3^t}/6 + F_{4\cdot 3^t-3}/2 - \sqrt{(F_{4\cdot 3^t}/6 - F_{4\cdot 3^t-3}/2)^2 + 4(F_{4\cdot 3^t-3}/2)^2}}{2}.$$

Then taking powers of M *will achieve copies of* $w^{(\beta)}_{k/3-1}$, *a* Mega-beeper *for* $\alpha = (1+\sqrt{45})/2$. *For instance, take* $k = 2$, *and* $t = 2$. *Then*

$$M^2 = \begin{pmatrix} 92977572651856 & 6584708258628 \\ 6584708258628 & 4663316283121 \end{pmatrix},$$

so we choose

$$x = -3302578930194 = -\left\lfloor \frac{6584708258628 \cdot 4663316283121}{92977572651856} \right\rfloor,$$

$$q_0 = \frac{9297757265185(-3302578930194) + 6584708258628 \cdot 4663316283121}{2} = 1554438761049,$$

and

$$D = 1554438761049^2 + (-3302578930194)6584708258628 + 4663316283121^2 = 2416279861853751838867210.$$

Then a 2-times Mega 2-beeper for $\gamma = \sqrt{45}$ *is:*

$$\sqrt{D} = \langle 1554438761049; \overline{w_2^{(\gamma)}, w_2^{(\gamma)}, 3108877522098} \rangle,$$

where

$$w_2^{(\gamma)} = 1, 2, 2, 2, 1, 12, 1, 2, 2, 2, 1, 12, 1, 2, 2, 2, 1,$$

where $\ell(\sqrt{D}) = 35$.

In [6], we proved the following.

Theorem 4 *Let* $A, B, C, k, X \in \mathbb{N}$ *with* $2 \mid A$ *and* C *not a perfect square, and*

$$\alpha = \sqrt{C} = \langle c_0; \overline{c_1, \ldots, c_n, 2c_0} \rangle.$$

Suppose that $(x, y) = (B, A)$ *is the smallest positive solution of* $x^2 - Cy^2 = 1$ *with* x *even and define, for each* $k \in \mathbb{N}$,

$$B_k + A_k\sqrt{C} = (B + A\sqrt{C})^k$$

and

$$D_k(X) = A_k^2 X^2 + 2B_k X + C.$$

Then the fundamental solution of $X^2 - D_k(X)Y^2 = 1$ *is*

$$(X, Y) = (A_k^2 X + B_k, A_k)$$

and,

(a) *if* $n \in \mathbb{N}$ *is even,*

$$\sqrt{D_k(X)} = \langle A_k X + c_0, \overline{w_{2k-1}^{(\alpha)}, 2(A_k X + c_0)}\rangle$$

with $\ell(\sqrt{D_k(X)}) = 2k(n+1)$

(b) *and if* n *is odd,*

$$\sqrt{D_k(X)} = \langle A_k X + c_0, \overline{w_{k-1}^{(\alpha)}, 2(A_k X + c_0)}\rangle$$

with $\ell(\sqrt{D_k(X)}) = k(n+1)$.

Before proceeding with the use of the above in conjunction with [1], we first present an illustration of the difference between a k-times Mega m-beeper and a km-times beeper from Example 7.

Example 8 *For* $\gamma = \sqrt{45} = \sqrt{C}$, *we have that* $B_5 = 1730726404001$ *and* $A_5 = 258001459320$, *so*

$$\sqrt{D_5(1)} = 66564753014711067670447 = \langle A_5 + 6; \overline{w_4^{(\gamma)}, 2(A_5 + 6)}\rangle$$

$$\langle 258001459326; \overline{1, 2, 2, 2, 1, 12, 1, 2, 2, 2, 1, 12, 1, 2, 2, 2, 1, 12, 1, 2, 2, 2, 1, 12,}$$
$$\overline{1, 2, 2, 2, 1, 516002918652}\rangle,$$

with $\ell(\sqrt{D_5(1)}) = 30$. *Compare this with Example 7 to see the subtle differences between the above* 4*-times beeper for* γ *and the* 2*-times Mega* 2*-beeper for* γ *given therein.*

We can now get very intricate repetitions from a recent result of Madden [1], by using his explicit descriptions and employing the above as follows.

Example 9 *In* [1], *the following type of examples were created. For* $b, c, n \in \mathbb{N}$,

$$\sqrt{(b(2bn+1)^c+n)^2+2(2bn+1)^c} =$$
$$\langle b(2bn+1)^c+n; \overline{b, 2b(2bn+1)^{c-1}, b(2bn+1), 2b(2bn+1)^{c-2}, b(2bn+1)^2, \ldots,}$$
$$\overline{b(2bn+1)^{c-2}, 2b(2bn+1), b(2bn+1)^{c-1}, 2b, b(2bn+1)^c+n,}$$
$$\overline{2b, b(2bn+1)^{c-1}, 2b(2bn+1), b(2bn+1), b(2bn+1)^{c-2}, 2b(2bn+1)^2, \ldots,}$$
$$\overline{2b(2bn+1)^{c-2}, b(2bn+1), 2b(2bn+1)^{c-1}, b, 2b(2bn+1)^c+2n}\rangle,$$

with $\ell(\sqrt{C}) = 4c+2$, *and the fundamental unit of*

$$\mathbb{Z}[\alpha] = \mathbb{Z}\left[\sqrt{C}\right] = \mathbb{Z}[\sqrt{(b(2bn+1)^c+n)^2+2(2bn+1)^c}]$$

is given by

$$B + A\sqrt{C} = \frac{(1+bn+b^2(2bn+1)^c+b\sqrt{C})^{2c}}{(\frac{1}{2}(-b(2n+1)^c-n+\sqrt{C})^2}.$$

Thus, we may form $B_k + A_k\sqrt{C} = (B+A\sqrt{C})^k$ *when* $B^2 - A^2C = 1$ *and* A *is even, then set*

$$D_k(X) = A_k^2X^2 + 2B_kX + C,$$

which allows us to invoke Theorem 4 to get

$$\sqrt{D_k(X)} = \langle A_kX + b(2bn+1)^c + n, \overline{w_{k-1}^{(\alpha)}, 2(A_kX + b(2bn+1)^c+n)}\rangle$$

with $\ell(\sqrt{D_k(X)}) = k(4c+2)$. *Hence, the intricate patterns for continued fractions formed by Madden in* [1] *can be replicated with period lengths* k *times that of his continued fractions for any* $k \in \mathbb{N}$. *As a specific instance, we look at the example given in* [1, p. 144], *with* $b = 1$, *and* $n = 5$. *Then*

$$\alpha = \sqrt{C} = \sqrt{11^{2c} + 12 \cdot 11^c + 25} =$$
$$\langle 11^c + 5; \overline{1, 2 \cdot 11^{c-1}, 11, 2 \cdot 11^{c-2}, 11^2, 2 \cdot 11^{c-3}, \ldots, 11^{c-2}, 2 \cdot 11, 11^{c-1}, 2, 11^{c+5},}$$
$$\overline{2, 11^{c-1}, 2 \cdot 11, 11^{c-2}, 2 \cdot 11^2, \ldots, 2 \cdot 11^{c-2}, 11, 2 \cdot 11^{c-1}, 1, 2 \cdot 11^c + 10}\rangle,$$

so with B_k *and* A_k *as above, and*

$$D_k = A_k^2X^2 + 2B_kX + C,$$

we have that

$$\sqrt{D_k(X)} = \langle A_kX + 11^c + 5, \overline{w^{(\alpha)}_{k-1}, 2(A_kX + 11^c + 5)}\rangle,$$

a $(k-1)$*-beeper for* α *with* $\ell(\sqrt{D_k(X)}) = k(4c+2)$. *For instance, if* $c = 3$, *and* $k = 2$, *then* $\sqrt{D_k(X)} =$

$$\sqrt{41302790891926679907850997291126488084791718951714899395384166944714008194328} =$$

$$\langle 20323088075370504556143166232013242806 0; \overline{1, 242, 11, 22, 121, 2, 1336,}$$

$$\overline{2, 121, 22, 11, 242, 1, 2672, 1, 242, 11, 22, 121, 2, 1336,}$$

$$\overline{2, 121, 22, 11, 242, 1, 40646176150741009112286332464026485612 0}\rangle,$$

where $\ell(\sqrt{D_k(X)}) = 28$,

$$B_k = B_2 = 27171885153811222467070116286425838851729 7,$$

and

$$A_k = A_2 = 20323088075370504556143166232013242672 4.$$

Hence, any pattern that can be produced in [1] *can be replicated* k *times plus a "tail" consisting of the symmetric part for* $\sqrt{C}$, *and a period length* k *times larger than that of* $\sqrt{C}$ *for any* $k \in \mathbb{N}$. *Moreover, using the techniques of Example 5 we can provide* k*-times* m*-beepers for* $\sqrt{C}$ *for any and all values of* k *and* m *chosen.*

Remark 1 *It is worth observing that each of Madden's expansions developed in* [1] *can be immediately extracted from Theorem 2 on the palindromic part. Hence, we owe much, perhaps unexpressed, debt to Perron for his pioneering work and cataloging in continued fractions.*

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