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Global Versus Local Solvability of Quadratic Diophantine Equations

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Abstract

We present infinite classes of quadratic Diophantine equations of the form $px^2 - cy^2 = \pm 1$, p any prime and c a positive integer, for which there are no solutions (x, y), yet for which there are solutions modulo n for all $n \geq 1$. This generalizes earlier work where only the case $p = 2$ was considered.

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1 Introduction

In [4], we provided reasons, in terms of continued fractions, as to why such Diophantine equations as $2x^2 - 219y^2 = -1$ have no integer solutions (not solvable globally), yet for which $2x^2 - 219y^2 \equiv -1 \pmod{n}$ is solvable for all $n \geq 1$ (solvable everywhere locally). The latter equation is found in many introductory number theory texts (see [2, Exercise 2.5.29, p. 118] for instance), but no explanation is given for why this phenomenon occurs. In [4], we provided an infinite class of quadratic Diophantine equations of the form $2x^2 - cy^2 = \pm 1$, which are not solvable globally, but are solvable locally everywhere. In this note, we expand these ideas to the class of Diophantine equations of the form $px^2 - cy^2 = \pm 1$, p any prime, which are locally solvable everywhere, but not globally. This includes an expansion of the $p = 2$ case from mere Richaud-Degert types, considered in [4], to a class infinitely larger. This involves continued fraction expansions of $\sqrt{pc}$, congruence conditions on the fundamental unit of the order $\mathbb{Z}[\sqrt{pc}]$, and central norms (defined via Equation (4) below).

2 Notation and Preliminaries

We will have occasion to refer to the simple continued fraction expansion of $\sqrt{D}$, where $D = pc$, whose period length we denote by $\ell = \ell(\sqrt{D})$ and whose partial quotients we denote by q_j for $j \geq 0$, where $q_0 = \lfloor\sqrt{D}\rfloor$ (the *floor* of $\sqrt{D}$), and $q_1 q_2, \ldots, q_{\ell-1}$ is a palindrome.

The jth *convergent* of α for $j \geq 0$ are given by,

$$\frac{A_j}{B_j} = \langle q_0; q_1, q_2, \ldots, q_j \rangle,$$

where

$$A_j = q_j A_{j-1} + A_{j-2},$$

$$B_j = q_j B_{j-1} + B_{j-2},$$

with $A_{-2} = 0$, $A_{-1} = 1$, $B_{-2} = 1$, $B_{-1} = 0$. The *complete quotients* are given by, $(P_j + \sqrt{D})/Q_j$, where $P_0 = 0$, $Q_0 = 1$, and for $j \geq 1$,

$$P_{j+1} = q_j Q_j - P_j, \tag{1}$$

$$q_j = \left\lfloor \frac{P_j + \sqrt{D}}{Q_j} \right\rfloor,$$

and

$$D = P_{j+1}^2 + Q_j Q_{j+1}.$$

We will also need the following facts (which can be found in most introductory texts in number theory, such as [2]. Also, see [1] for a more advanced exposition).

$$A_{j-1}^2 - B_{j-1}^2 D = (-1)^j Q_j. \tag{2}$$

In particular,

$$A_{\ell-1}^2 - B_{\ell-1}^2 D = (-1)^\ell, \tag{3}$$

and it follows that $(x_0, y_0) = (A_{\ell-1}, B_{\ell-1})$ is the fundamental solution of the Pell Equation $x^2 - Dy^2 = (-1)^\ell$.

When ℓ is even, $P_{\ell/2} = P_{\ell/2+1}$, so by Equation (1),

$$Q_{\ell/2} \mid 2P_{\ell/2},$$

where $Q_{\ell/2}$ is called the *central norm*, (via Equation (2)), where

$$Q_{\ell/2} \mid 2D. \tag{4}$$

In the following (which we need in the next section), and all subsequent results, the notation for the A_j, B_j, Q_j and so forth apply to the above-developed notation for the continued fraction expansion of $\sqrt{D}$.

The following is crucial in what follows.

Theorem 1 *Let p be a prime dividing $2D$ where D is not a perfect square and $D > p^2$. Then the following are equivalent.*

1. The Diophantine equation
$$x^2 - Dy^2 = \pm p \tag{5}$$
has a solution.

2. $\ell = \ell(\sqrt{D})$ is even and $A_{\ell-1} \equiv (-1)^{\ell/2} \pmod{2D/p}$.

3. ℓ is even, $(-1)^{\ell/2}Q_{\ell/2} = \pm p$, and $(A_{\ell/2-1}, B_{\ell/2-1})$ is the fundamental solution of Equation (5).

Proof. This (with more consequences) is proved in [5]. □

Remark 1 *Underlying Theorem 1 is the celebrated result known as Hilbert's Theorem 90* (*see* [3, Exercise 3.87, p. 167], *for instance*), *which tells us that if $\Delta > 0$ is a discriminant and $\mathbb{Q}(\sqrt{\Delta})$ has fundamental unit of norm 1, then there exists a unique $d > 1$ dividing Δ such that $d < \sqrt{\Delta}$ and there is a principal ideal of norm d. For instance, Theorem 1 characterizes those non-square positive integers D such that certain equations $x^2 - Dy^2 = \pm p$ have integer solutions when $p < \sqrt{D}$ is a prime divisor of $2D$. From this comes the natural question concerning the solvability of equations $px^2 - cy^2 = \pm 1$. Below, we look at the global versus the local solvability of such equations.*

3 Local-Global Considerations

Theorem 2 *Let p be a prime and c a positive integer with $\ell = \ell(\sqrt{pc})$, such that either*

1. $p = 2$, $c \equiv 3 \pmod 8$, and $c > 3$ is divisible only by primes congruent to 1 or 3 modulo 8. Also, in the simple continued fraction expansion of $\sqrt{2c}$, $A_{\ell-1} \not\equiv \pm 1 \pmod c$.

2. $p \equiv 7 \pmod 8$, $c \equiv 1 \pmod p$, c is not divisible by any primes q such that the Legendre symbol equality $(\frac{-p}{q}) = -1$ holds, and in the simple continued fraction expansion of $\sqrt{pc}$, $A_{\ell-1} \not\equiv \pm 1 \pmod{2c}$.

Then the Diophantine equation,

$$px^2 - cy^2 = -1 \tag{6}$$

has no solutions (x, y)*, but*

$$px^2 - cy^2 \equiv -1 \pmod{n} \tag{7}$$

has a solution (x, y) *for all* $n \geq 1$*.*

Proof. Part 1 is established in [4] for the case where $2c$ is a Richaud-Degert type. However, the argument generalizes by replacing the Richaud-Degert type by the conditions in part 1, and invoking Theorem 1 to conclude that ℓ is even, and $Q_{\ell/2} \neq 2$. Then the argument proceeds exactly as in [4].

For part 2, we proceed as follows. Since $p \equiv 3 \pmod{4}$, ℓ must be even by Equation (3). By Theorem 1, Equation (6) has no solutions (x, y) since $A_{\ell-1} \not\equiv \pm 1 \pmod{2c}$. It remains to show that Congruence (7) has solutions for all positive integers n. For $n = 1$, this is trivial. For $n > 1$, by the Chinese Remainder Theorem, we may prove the result by showing that the congruence holds for any prime power. We begin with $n = 2^\alpha$ for $\alpha \geq 1$.

Given that $-p \equiv 1 \pmod{8}$, then it follows from elementary number theory that there is an integer z such that $-p \equiv z^2 \pmod{2^\alpha}$ (see [2, Exercise 3.3.4, p. 158], for instance). By setting $x = z^{-1}$, where z^{-1} is the multiplicative inverse of z modulo 2^α, and $y = 0$, we get $px^2 - cy^2 \equiv -1 \pmod{2^\alpha}$.

If $n = p^\alpha$ for $\alpha \geq 1$, then since $c \equiv 1 \pmod{p}$, there is an integer z such that $c \equiv z^2 \pmod{p^\alpha}$, again by elementary number theory considerations (see [2, Corollary 3.3.1, p. 156], for example). Thus, by setting $x = 0$ and $y = z^{-1}$, where z^{-1} is the multiplicative inverse of z modulo p^α, then

$$px^2 - cy^2 \equiv -1 \pmod{p^\alpha}.$$

It remains to look at prime powers $n = q^\alpha$ for $\alpha \geq 1$, where $q \neq 2, p$. There are two cases.

Case 1 *The Legendre symbol identity* $\left(\frac{-p}{q}\right) = 1$ *holds.*

In this case, there is an integer z such that $-p \equiv z^2 \pmod{q^\alpha}$ by elementary number-theoretic considerations (see [2, Exercise 4.1.6, p. 195] for instance). Let j be the multiplicative inverse of p modulo q^α. Then by setting $x = jz$ and $y = 0$, we get that $px^2 - cy^2 \equiv -1 \pmod{q^\alpha}$.

Case 2 *The Legendre symbol identity* $\left(\frac{-p}{q}\right) = -1$ *holds.*

By hypothesis, q does not divide c. By Jacobsthal's Legendre sum Theorem (see [2, Exercises 4.2.7–4.2.8, p. 202], for example), there exists an integer t such that the Legendre symbol identity $\left(\frac{1-ct^2}{q}\right) = -1$ holds. Therefore, $\left(\frac{pct^2-p}{q}\right) = 1$. Hence, as above, there is an integer $z \in \mathbb{Z}$ such that

$$z^2 \equiv pct^2 - p \pmod{q^\alpha}.$$

If j is the multiplicative inverse of p modulo q^α, then by setting $x = jz$ and $y = t$, we get $px^2 - cy^2 \equiv -1 \pmod{q^\alpha}$. $\square$

Example 1 *Let $p = 7$, $c = 281$, and $\ell = \ell(\sqrt{1967}) = \ell(\sqrt{7 \cdot 281}) = \ell(\sqrt{pc})$. Then $c \equiv 1 \pmod{p}$, $\left(\frac{-7}{281}\right) = 1$, where 281 is prime, and $A_5 = A_{\ell-1} = 2528 \equiv 280 \not\equiv \pm 1 \pmod{2c}$. Thus,*

$$7x^2 - 281y^2 = -1 \tag{8}$$

has no solutions, but

$$7x^2 - 281y^2 \equiv -1 \pmod{n}$$

has solutions for all $n \geq 1$. In this case $Q_{\ell/2} = 14$, so $x^2 - pcy^2 = -14$ has a solution, namely $(x, y) = (A_{\ell/2-1}, B_{\ell/2-1}) = (133, 3)$, so $7X^2 - 281Y^2 = -2$ has a solution $(X, Y) = (19, 3)$, but it is not possible for Equation (8) to have any solutions. By Theorem 1, such solutions would imply that $Q_{\ell/2} = 7$.

To illustrate the proof of Theorem 2 in the construction of solutions to the congruence, we'll take an n at random, say $n = 2^8 \cdot 3^5 \cdot 7^3$. Then via the techniques in that proof, we get,

$$7 \cdot 29^2 - 281 \cdot 0^2 \equiv -1 \pmod{2^8},$$
$$7 \cdot 226^2 - 281 \cdot 74^2 \equiv -1 \pmod{3^5},$$
$$7 \cdot 0^2 - 281 \cdot 237^2 \equiv -1 \pmod{7^3}.$$

We may now put these together via the Chinese remainder Theorem to get a solution modulo n via:

$$\begin{aligned} x = 226 \cdot 2^8 \cdot 7^3 \cdot 2^{-8} \cdot 7^{-3} \pmod{3^5} + 29 \cdot 3^5 \cdot 7^3 \cdot 3^{-5} \cdot 7^{-3} \pmod{2^8} \\ +7^3 \cdot 2^8 \cdot 3^5 \cdot 2^{-8} \cdot 3^{-5} \pmod{7^3} \equiv 226 \cdot 2^8 \cdot 7^3 \cdot 187 \cdot 226 + \\ 29 \cdot 3^5 \cdot 7^3 \cdot 59 \cdot 103 + 7^3 \cdot 2^8 \cdot 3^5 \cdot 205 \cdot 24 \equiv 17269021 \pmod{2^8 \cdot 3^5 \cdot 7^3}, \end{aligned}$$

and

$$\begin{aligned} y = 74 \cdot 2^8 \cdot 7^3 \cdot 2^{-8} \cdot 7^{-3} \pmod{3^5} + 2^8 \cdot 3^5 \cdot 7^3 \cdot 3^{-5} \cdot 7^{-3} \pmod{2^8} \\ +237 \cdot 2^8 \cdot 3^5 \cdot 2^{-8} \cdot 3^{-5} \pmod{7^3} \equiv 74 \cdot 2^8 \cdot 7^3 \cdot 187 \cdot 226 + \\ 2^8 \cdot 3^5 \cdot 7^3 \cdot 59 \cdot 103 + 237 \cdot 2^8 \cdot 3^5 \cdot 205 \cdot 24 \equiv 9452288 \pmod{2^8 \cdot 3^5 \cdot 7^3}. \end{aligned}$$

We find that this yields the solution

$$7 \cdot 7269021^2 - 281 \cdot 9452288^2 \equiv -1 \pmod{2^8 \cdot 3^5 \cdot 7^3}.$$

The following provides an infinite class of equations solvable everywhere locally, but not globally. These are examples of Richaud-Degert types (those of the form $a^2 + r$ where $|r| \mid 4a$), that were the focus of the result in [4].

Corollary 1 *Suppose that p and q are odd primes where $p \equiv 7 \pmod 8$, $2q \equiv 1 \pmod p$ and $pc = p^2q^2 + 2pq = p(pq^2 + 2q)$, with c not divisible by any prime r for which $-p$ is a quadratic nonresidue. Then Equation (6) has no solutions but Equation (7) has a solution for all $n \geq 1$.*

Proof. By [1, Theorem 3.2.1, p. 78], $\ell = \ell(\sqrt{pc}) = 2$ and $Q_{\ell/2} = 2pq$. Thus, by Theorem 1, Equation (6) has no solutions and $A_{\ell-1} \not\equiv \pm 1 \pmod{2c}$. Thus, by Theorem 2, Equation (7) has a solution for all $n \geq 1$. $\square$

Example 2 *Let $pc = 7 \cdot 869 = 7^2 \cdot 11^2 + 2 \cdot 7 \cdot 11 = p^2q^2 + 2pq$. Since $c = 869 = 11 \cdot 79$ with*

$$\left(\frac{-7}{11}\right) = \left(\frac{-7}{79}\right) = 1,$$

and $2q = 22 \equiv 1 \pmod 7$, then the hypothesis of Corollary 1 is satisfied. Hence,

$$7x^2 - 869y^2 \equiv -1 \pmod n \tag{9}$$

has a solution for all $n \geq 1$, but $7x^2 - 869y^2 = -1$ has no solutions. By Theorem 1, if the latter did have solutions, then $Q_{\ell/2} = 7$ would necessarily hold, where $\ell = \ell(\sqrt{pc})$. However, as noted in the proof of Corollary 1, $Q_{\ell/2} = Q_1 = 154$.

By the same technique as illustrated in Example 1, we may take a random n, say $n = 2^5 \cdot 5^3 \cdot 11^2$, and manufacture a solution to Equation (9), as follows,

$$7 \cdot 238893^2 - 869 \cdot 65824^2 \equiv -1 \pmod n.$$

There is a dual result to Theorem 2 which we now present without proof since the verification is entirely analogous to that given above.

Theorem 3 *Let p be a prime and c a positive integer with $\ell = \ell(\sqrt{pc})$, such that either*

1. $p = 2$, $c \equiv 1 \pmod 8$, and c is divisible only by primes congruent to 1 or 7 modulo 8, with at least one prime congruent to 7 modulo 8. Also, in the simple continued fraction expansion of $\sqrt{2c}$, $A_{\ell-1} \not\equiv \pm 1 \pmod c$.

2. $p \equiv 1 \pmod 8$, $c \equiv -1 \pmod p$, $\ell(\sqrt{pc})$ even, and c is not divisible by any primes q such that the Legendre symbol equality $(\frac{p}{q}) = -1$ holds. Also, $A_{\ell-1} \not\equiv \pm 1 \pmod{2c}$ in the simple continued fraction expansion of $\sqrt{pc}$.

Then the Diophantine equation,

$$px^2 - cy^2 = 1 \tag{10}$$

has no solutions (x, y)*, but*

$$px^2 - cy^2 \equiv 1 \pmod{n} \tag{11}$$

has a solution (x, y) *for all* $n \geq 1$.

The analogue of Corollary 1 is given as follows.

Corollary 2 *Suppose that* p *and* q *are odd primes where* $p \equiv 1 \pmod{8}$, $2q \equiv 1 \pmod{p}$ *and* $pc = p^2q^2 - 2pq = p(pq^2 - 2q)$*, with* c *not divisible by any prime* r *for which* p *is a quadratic nonresidue. Then Equation* (10) *has no solutions but Equation* (11) *has a solution for all* $n \geq 1$.

A simple instance of Corollary 2 is the following.

Example 3 *Let* $pc = 17 \cdot 544339 = 17^2 \cdot 179^2 - 2 \cdot 17 \cdot 179 = p^2q^2 - 2pq$*, Then* $Q_{\ell/2} = Q_1 = 6082 = 2 \cdot 17 \cdot 179$, $2q \equiv 1 \pmod{p}$, $\left(\frac{17}{179}\right) = \left(\frac{17}{3041}\right) = 1$*, and* $A_{\ell-1} = A_1 = 6082 \not\equiv \pm 1 \pmod{2c}$*. Thus,* $17x^2 - 544339y^2 = 1$ *has no solutions but* $17x^2 - 544339y^2 \equiv 1 \pmod{n}$ *has solutions for all* $n \geq 1$.

An illustration of Theorem 3 is the following.

Example 4 *Let* $pc = 17 \cdot 628$*, where* $c \equiv -1 \pmod{17}$*, and* $c = 2^2 \cdot 157$ *with* $\left(\frac{17}{157}\right) = 1$*. Also,* $\ell(\sqrt{pc}) = 10$*, and* $A_{\ell-1} \equiv 313 \not\equiv \pm 1 \pmod{2c}$*. Thus, the hypothesis of part 1 of Theorem 3 is satisfied. Hence,* $17x^2 - 628y^2 = 1$ *is not solvable, but* $7x^2 - 628y^2 \equiv 1 \pmod{n}$ *is solvable for all* $n \geq 1$*. Here* $Q_{\ell/2} = 68$*. We now illustrate how to manufacture solutions using Theorem 3, as an analogue to that provided in Example 1 for the Theorem 2 case.*

We select $n = 2^5 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 101$*, at random. Using the analogue of the techniques in the proof of Theorem 2, we get.*

$$17 \cdot 1^2 - 628 \cdot 6^2 \equiv 1 \pmod{2^5},$$

$$17 \cdot 7^2 - 628 \cdot 13^2 \equiv 1 \pmod{3^4},$$

$$17 \cdot 2^2 - 628 \cdot 8^2 \equiv 1 \pmod{5^3},$$

$$17 \cdot 0^2 - 628 \cdot 16^2 \equiv 1 \pmod{7^2},$$

$$17 \cdot 39^2 - 628 \cdot 0^2 \equiv 1 \pmod{101}.$$

Therefore,

$$\begin{aligned} x = {} & 1 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 101 \cdot 3^{-4} \cdot 5^{-3} \cdot 7^{-2} \cdot 101^{-1} \pmod{2^5} \\ & +7 \cdot 2^5 \cdot 5^3 \cdot 7^2 \cdot 101 \cdot 2^{-5} \cdot 5^{-3} \cdot 7^{-2} \cdot 101^{-1} \pmod{3^4} \\ & +2 \cdot 2^5 \cdot 3^4 \cdot 7^2 \cdot 101 \cdot 2^{-5} \cdot 3^{-4} \cdot 7^{-2} \cdot 101^{-1} \pmod{5^3} \\ & +7^2 \cdot 2^5 \cdot 3^4 \cdot 5^3 \cdot 101 \cdot 2^{-5} \cdot 3^{-4} \cdot 5^{-3} \cdot 101^{-1} \pmod{7^2} \\ & +39 \cdot 2^5 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 2^{-5} \cdot 3^{-4} \cdot 5^{-3} \cdot 7^{-2} \pmod{101} \end{aligned}$$

It follows that,

$$\begin{aligned} x \equiv {} & 1 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 101 \cdot 17 \cdot 21 \cdot 17 \cdot 13 + 7 \cdot 2^5 \cdot 5^3 \cdot 7^2 \cdot 101 \cdot 38 \cdot 35 \cdot 43 \cdot 77 \\ & +2 \cdot 2^5 \cdot 3^4 \cdot 7^2 \cdot 101 \cdot 43 \cdot 71 \cdot 74 \cdot 26 + 7^2 \cdot 2^5 \cdot 3^4 \cdot 5^3 \cdot 101 \cdot 23 \cdot 23 \cdot 20 \cdot 33 \\ & +39 \cdot 2^5 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 60 \cdot 5 \cdot 80 \cdot 33 \equiv 774017377 \pmod{2^5 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 101}. \end{aligned}$$

Similarly,

$$\begin{aligned} y \equiv {} & 6 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 101 \cdot 17 \cdot 21 \cdot 17 \cdot 13 + 13 \cdot 2^5 \cdot 5^3 \cdot 7^2 \cdot 101 \cdot 38 \cdot 35 \cdot 43 \cdot 77 \\ & +8 \cdot 2^5 \cdot 3^4 \cdot 7^2 \cdot 101 \cdot 43 \cdot 71 \cdot 74 \cdot 26 + 16 \cdot 2^5 \cdot 3^4 \cdot 5^3 \cdot 101 \cdot 23 \cdot 23 \cdot 20 \cdot 33 \\ & +101 \cdot 2^5 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 60 \cdot 5 \cdot 80 \cdot 33 \equiv 1163414758 \pmod{2^5 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 101}, \end{aligned}$$

This provides the solution

$$17 \cdot 774017377^2 - 628 \cdot 1163414758^2 \equiv 1 \pmod{2^5 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 101}.$$

Remark 2 *Those familiar with Hasse's global-local principle will perhaps be given pause by what is presented herein. However, the two are not related for the following reasons. If one homogenizes, for instance, the equation* $2x^2 - 219y^2 = -1$ *to get* $2x^2 - 219y^2 = -z^2$, *then Hasse's principal says that the homogenized equation* must *have global solutions since it to has local solutions everywhere* (*with* $z = 1$). *However, those global solutions will be for* $z \neq 1$. *Once we insist that we restrict to* $z = 1$, *this takes the problem out of the realm of the Hasse principle into the realm of a Diophantine equation that is not homogeneous. In other words, it is certainly true that if a Diophantine equation* (*not necessarily homogeneous*) *has a global zero then it has local zero everywhere. However, the converse is false, with the classic case being* $2x^2 - 219y^2 = -1$ *not solvable for any integers but solvable for any positive* n *with* $2x^2 - 219y^2 \equiv -1 \pmod{n}$. *The results herein provide an explanation of the local-global phenomenon related to the quadratic Diophantine equations of types* (6) *and* (7) *given in terms of continued fraction expansions, which we have shown to be intimately linked to the role played by the central norm. Thus,* *a ubiquitous Diophantine Equation, cited above, is now fully explained* (*and generalized to an infinite class*), *in relatively simple terms. Also, the techniques presented in Examples 1 and 4 show how to construct the local solutions for all* $n \geq 1$.

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