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GENERAL CRITERIA FOR DETERMINATION OF CLASS GROUPS OF REAL QUADRATIC ORDERS

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Abstract

The purpose herein is to give necessary and sufficient conditions for the class group of an arbitrary real quadratic order to have its class group completely determined by a given set of ideals in terms of the corresponding continued fraction expansion related to the underlying reduced ideals. Moreover, a class number one criterion is presented as a consequence.

1. Introduction

Several authors have given class number one criterion for real quadratic fields in terms of the continued fraction expansion of the principal class. Among them are, Dubois-Levesque [1], Lu [3], as well as work of this author and others such as [2], [4], and [6]. Linked to this is the search for a class number one, real quadratic field analogue of the well-known Rabinowitsch result [7] for complex quadratic fields. Such criteria can be found in [2] and [6]. This paper produces a completely general result for arbitrary real quadratic orders (not necessarily

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maximal) which links all of the above concepts in one criterion for the class groups of the order to be generated by any given set of primitive, regular ideals. This was first presented in 1994 in [4], but the presentation was flawed and the criterion for class number one that emanated from it turns out to be false. Herein, we provide a correct class number one criterion and a proper elucidation of the general criterion.

2. Notation and Preliminaries

Let $d > 1$ be a square-free positive integer and set

$$r = \begin{cases} 2, & \text{if } d \equiv 1 \pmod 4), \\ 1, & \text{otherwise.} \end{cases}$$

Define $\omega_d = (r - 1 + \sqrt{d})/r$ and $\Delta_0 = (\omega_d - \omega_d')^2 = 4d/r^2$, where ω_d' is the algebraic conjugate of ω_d. Let $\Delta = (\omega_\Delta - \omega_\Delta')^2$, where $\omega_\Delta = f\omega_d + h$ for some $f, h \in \mathbf{Z}$, and set $D = (f/g)^2 d$, where $g = \gcd(f, r)$. Therefore, $\Delta = f^2\Delta_0 = 2D/\sigma^2$, where $\sigma = r/g$. Here, f is called the *conductor* associated with the *discriminant* Δ and d is called the underlying *radicand*, while Δ_0 is known as the *fundamental discriminant*.

Let $[\alpha, \beta] = \alpha\mathbf{Z} \oplus \beta\mathbf{Z}$ and set $O_\Delta = [1, f\omega_d] = [1, \omega_\Delta]$ which is an *order* in $K = \mathbf{Q}(\sqrt{\Delta})$ having conductor f and fundamental discriminant Δ_0. If $f = 1$, then O_Δ is called the *maximal order* or *ring of integers* of K. It is well-known that $I = [a, b + c\omega_\Delta] \not\subseteq Z$ is an *ideal* of O_Δ if and only if $c \mid a$, $c \mid b$ and $ac \mid N(b + c\omega_\Delta)$, where N is the *norm* from $\mathbf{Q}(\sqrt{\Delta})$ to $\mathbf{Q}$, i.e., $N(\alpha) = \alpha\alpha'$. Also, for a given ideal I in O_Δ with $I \not\subseteq \mathbf{Z}$, the integers a and c are unique, and a is, in fact, the *least positive rational integer in* I, denoted by $L(I)$. We denote the value of $cL(I)$ by $N(I)$, which we call *the norm of* I. If $c = 1$, then I is called a *primitive ideal*. Two ideals I and J of O_Δ are *equivalent* (denoted $I \sim J$) if there exist non-zero $\alpha, \beta \in O_\Delta$ such that $(\alpha)I = (\beta)J$ (where (x) denotes the principal ideal generated by x). An ideal I of O_Δ is called *regular* if $O_\Delta = \{\alpha \in K : \alpha I \subseteq I\}$, and all regular ideals are *invertible*.

In particular, if $\gcd(N(I), f) = 1$, then the non-zero ideal I is regular, and any non-zero principal ideal is regular. Let C_Δ denote the equivalence classes of invertible ideals of O_Δ which is a group under ideal multiplication, and in fact $C_\Delta \cong \mathrm{Pic}(O_\Delta)$. The order of this *class group* C_Δ is denoted by h_Δ, the *class number* of O_Δ.

The following is well-known.

Theorem 2.1. *C_Δ is generated by the primitive, regular ideals I with $N(I) < \sqrt{\Delta}/2$.*

A primitive ideal I in O_Δ is said to be reduced if it does not contain any non-zero element α satisfying both $|\alpha| < N(I)$, and $|\alpha'| < N(I)$.

The following results on reduced ideals may be found in [5].

Theorem 2.2. (a) *If I is any ideal of O_Δ, then there exists a reduced O_Δ- ideal $J \sim I$.*

(b) *If I is a reduced ideal of O_Δ, then $N(I) < \sqrt{\Delta}$.*

(c) *If I is a primitive O_Δ- ideal and $N(I) < \sqrt{\Delta}/2$, then I is reduced.*

(d) *If I is a reduced ideal of O_Δ, then there exists an ideal $J \sim I$ such that $N(J) < \sqrt{\Delta}/2$.*

Now we turn to continued fraction expansions as they relate to the above (see [5]).

Let $I = [N(I), b + \omega_\Delta]$ be a primitive ideal in O_Δ and denote the continued fraction expansion of $(b + \omega_\Delta)/N(I)$ by $\langle a, \overline{a_1, a_2, \ldots, a_\ell}\rangle$ with period length $\ell = \ell(I)$, where $(P_0, Q_0) = ((rb + f(r-1) + hr)/g, ar/g)$, and for $i \geq 0$,

$$D = P_{i+1}^2 + Q_i Q_{i+1},$$

$$P_{i+1} = a_i Q_i - P_i,$$

and

$$a_i = \lfloor (P_i + \sqrt{D})/Q_i \rfloor$$

(where $\lfloor x \rfloor$ is the greatest integer less than or equal to x). The continued fraction factoring algorithm (as elucidated in [5]) yields all reduced ideals equivalent to a given reduced ideal $I = [N(I), b + \omega_\Delta]$, i.e., in the continued fraction expansion of $(b + \omega_\Delta)/N(I)$ we have

$$I = I_0 = [Q_0/\sigma, (P_0 + \sqrt{D})/\sigma] \sim$$

$$I_1 = [Q_1/\sigma, (P_1 + \sqrt{D})/\sigma] \sim \cdots$$

$$I_{\ell-1} = [Q_{\ell-1}/\sigma, (P_{\ell-1} + \sqrt{D})/\sigma].$$

Finally, $I_\ell = I_0 = I$ for a complete *cycle of reduced ideals* of length $\ell(I) = \ell$. Therefore, the *complete quotients* of $(b + \omega_\Delta)/N(I)$ are the $(P_i + \sqrt{D})/Q_i$ and the σ_i's are the *partial quotients*. Also the Q_i/σ's represent the *norms of all reduced ideals equivalent to I*. Moreover, the following is well-known (as may be gleaned from [5]).

Theorem 2.3. *Let $I = [N(I), b + \omega_\Delta]$ be a reduced ideal in O_Δ. Thus in the continued fraction expansion of $(b + \omega_\Delta)/N(I)$ we have*

(a) *If J is a reduced ideal of O_Δ and $I \sim J$, then $N(I) = Q_i/\sigma$ for some integer i with $1 \le i \le \ell(I) = \ell$.*

(b) *If J and J' are the only ideals of norm $N(J)$, where J is a reduced ideal in O_Δ and $N(J) = Q_i/\sigma$ for some integer i with $1 \le i \le \ell$, then either $J = I_i$ or $J' = I_i$.*

Thus, from Theorems 2.1 and 2.3, we have

Theorem 2.4. *Let $I_1, I_2, \ldots, I_k$ be primitive, regular ideals. Then $C_\Delta = \{I_i\}_{i=1}^k$ if and only if for each prime $p < \sqrt{\Delta}/2$, such that $(\Delta/p) \neq -1$ there exists an integer i with $1 \le i \le k$ and a reduced ideal $J_i = [a_i, b_i + \omega_\Delta] \sim I_i$ such that in the continued fraction expansion of $(b_i + \omega_\Delta)/a_i$ we have $Q_j/\sigma = p$ for some $1 \le j \le \ell_i = \ell(I_i)$.*

Set

$$f_\Delta(x) = \begin{cases} -x^2 + x + (\Delta - 1)/4, & \textit{if } \Delta \equiv 1 \,(\text{mod } 4), \\ -x^2 + \Delta/4, & \textit{if } \Delta \equiv 0 \,(\text{mod } 4), \end{cases}$$

then we will need the following result which generalizes [2, Lemma 3.1, p. 7].

Lemma 2.1. *If* $p < \sqrt{\Delta}/2$ *is prime, then* $f_\Delta(x) \equiv 0 \,(\text{mod } p)$ *for some integer x with* $1 \le x \le \lfloor \omega_\Delta \rfloor$ *if and only if* $(\Delta/p) \ne -1$, *i.e., all non-inert primes* $p < \sqrt{\Delta}/2$ *divide* $f_\Delta(x)$ *for some integer x with* $1 \le x \le \lfloor \omega_\Delta \rfloor$ *and they are the only such prime divisors less than* $\sqrt{\Delta}/2$.

Proof. If $p < \sqrt{\Delta}/2$ with $(\Delta/p) \ne -1$, then there exists an integer y with $1 \le y \le p$ and $D \equiv 1 \,(\text{mod } p)$. If $p = 2$, then $D \equiv 1 \,(\text{mod } 8)$ and $f_\Delta(x)$ is clearly always even in this case. If $p > 2$, then letting $x = (y + \alpha - 1)/\sigma$ yields the result. The converse is clear.

We will also need the following result in the next section. First, we need a definition.

Definition 2.1. Let $\Delta > 0$ be a discriminant and let $I = [a, (b + \sqrt{\Delta})/2]$ be a reduced ideal in O_Δ. If I is in an ambiguous class (i.e., $I \sim I'$), then in the continued fraction expansion of $(b + \sqrt{\Delta})/2a$, we must have $I' = I_p$ for some integer p with $1 \le p \le \ell(I)$. We call $p = p(I)$ the *palindromic index* of I.

Note that in the rest of the article we will suppress the I and just write p for $p(I)$ and ℓ for $\ell(I)$ whenever no confusion will arise from so doing.

In [5] we proved the following result.

Theorem 2.5. *Let* $\Delta > 0$ *be a discriminant and let* $I = [a, (b + \sqrt{\Delta})/2]$ *be a regular, reduced ideal in an ambiguous class of* C_Δ. *Let* Q_i *be in the* *continued fraction expansion of* $(\sqrt{\Delta} + b)/2a$. *Then* Q_i/σ *is a square-free divisor of* Δ *if and only if one of the following holds*:

(a) $p = \ell$ *and* $i = 0$ *or* ℓ.

(b) p *is even and* $i = p/2$.

(c) p *and* ℓ *have the same parity and* $i = (p + \ell)/2$.

3. Class Number Criteria

The key to the principal result of this paper is the following which opens the door to an understanding of the connection between $f_\Delta(x)$ and class number criteria for real quadratic orders. The following was proved in [5].

Theorem 3.1. *Let* $I = [a, b + \omega_\Delta]$ *be a reduced ideal in* O_Δ. *Then, if in the continued fraction expansion of* $(b + \omega_\Delta)/a$, Q_i/σ *is prime for some positive integer* $i \le \ell$, $f_\Delta(x) \equiv 0 \pmod{Q_i/\sigma}$ *for exactly* $a_i + 1$ *values of x with* $1 \le x \le \lfloor \omega_\Delta \rfloor$ *whenever* Q_i/σ *is not a divisor of* Δ. *If* Q_i/σ *is either* 1 *or a prime divisor of* Δ *for* $0 < i \le \ell$, *then* $i = \ell = p$, $p/2$ *or* $(p + \ell)/2$ *and there are exactly* $\lfloor (a_i + 1)/2 \rfloor$ *such values of x. If* $Q_i/\sigma > 1$ *is not prime, then there are at least that many values.*

Example 3.1. Let $\Delta = 629 = 17 \cdot 37$. Then, if $I = [5, 11 + \omega_\Delta] = [5, (23 + \sqrt{629})/2]$, the continued fraction expansion of $(11 + \omega_\Delta)/5$ is

i	0	1	2	3
P_i	23	17	17	23
Q_i	10	34	10	10
a_i	4	1	4	4

Here $p = p(I) = 2$ and $\ell = \ell(I) = 3$. Also,

X	$f_\Delta(x) = -x^2 + x + 157$
1	157
2	155
3	151
4	145
5	137
6	127
7	115
8	101
9	85
10	67
11	47
12	25
13	1

We observe that $Q_2/\sigma = Q_3/\sigma = 5$ divides $f_\Delta(x)$ exactly 5 times for $1 \le x \le \lfloor \omega_\Delta \rfloor = 13$, and this is exactly $a_2 + 1 = a_3 + 1$ as Theorem 3.1 predicted. Moreover, $f_\Delta(x)$ is divisible (for $1 \le x \le 13$) by $Q_{p/2}/\sigma = Q_2/\sigma = 17$ exactly once which is $\lfloor (a_2 + 1)/2 \rfloor$. These are all the possible values to which Theorem 3.1 applies, as is seen by the fact that, whenever $f_\Delta(x)$ is not divisible by either 5 or 17 for $1 \le x \le 13$, it is a prime bigger than $\sqrt{\Delta}/2$.

For our main result we need the following notation.

Definition 3.1. Let $J_i = [a_i, (b_i + \sqrt{\Delta})/2]$ be reduced, regular ideals for rational integers i with $1 \le i \le k$ and $0 < (\sqrt{\Delta} - b_i)/(2a_i) < 1$, where $\gcd(a_i, f) = 1$. Furthermore, set

$$S_f(p) = \{x : 1 \le x \le \lfloor \omega_\Delta \rfloor \text{ with } p \mid f_\Delta(x)\},$$

and for Q_j appearing in the continued fraction expansion of

$(b_i + \sqrt{\Delta})/(2a_i)$, set

$$T_i(p) = \{j : 1 \le j \le \ell(J_i) = \ell_i \text{ with } Q_j/\sigma = p < \sqrt{\Delta}/2,$$

$$\text{where } p \text{ is a prime counted without multiplicity}\}.$$

Also, let

$$R_i = \{j : 1 \le j \le l \text{ and one of } j = l_i = l(I_i) = p(I_i) = p_i, \text{ or}$$

$$j = p_i/2, \text{ or } j = (p_i + l_i)/2 \text{ holds}\},$$

and

$$a_{(i,j)} = \lfloor (P_j + \sqrt{D})/Q_j \rfloor$$

in the simple continued fraction arising from I_i.

Theorem 3.2. $C_\Delta = \{I_i\}_{i=1}^{k}$ *for primitive, regular ideals* I_i *if and only if for each i with* $1 \le i \le k$ *there exists a* $J_i \sim I_i$ *reduced such that* (*with notation as in Definition* 3.1)

$$\sum_{p<\sqrt{\Delta}/2} |S_f(p)| = \sum_{i=1}^{k} \sum_{j \in T_i - R_i} (a_{(i,j)} + 1), \tag{3.1}$$

where the left-hand sum ranges over all unramified primes, *and*

$$\sum_{p<\sqrt{\Delta}/2} |S_f(p)| = \sum_{i=1}^{k} \sum_{j \in R_i \cap T_i} \lfloor (a_{(i,j)} + 1)/2 \rfloor, \tag{3.2}$$

where the left-hand sum ranges over all ramified primes.

Proof. By Theorem 3.1 the right-hand sides of (3.1)-(3.2), actually represent $\sum |S_f(p)|$, where the sum ranges over only those $p < \sqrt{\Delta}/2$ which appear as some $Q_{(i,j)}/\sigma$. Therefore, if some prime $p < \sqrt{\Delta}/2$ does not appear on the right but does appear on the left, then equality cannot hold. Hence, by Lemma 2.1 and Theorem 2.4 the result follows.

Remark 3.1. R_i in Theorem 3.2 is non-empty if and only if J_i is in an ambiguous class containing an ambiguous ideal. (See [9]-[10] for a complete study of class groups generated by ambiguous ideals.)

Example 3.2. Let $\Delta = 401$. Then in the continued fraction expansion of ω_Δ there are no Q_j/σ which are prime. However, if we look at $I_1 = P_5 = [5, \omega_\Delta]$, then in the continued fraction expansion of $\omega_\Delta/5$ we get $Q_1/2 = 2$, $Q_3/2 = 5$ and $a_{(1,1)} = 9$ with $a_{(1,3)} = 3$. Although P_5^2 is not reduced, $I_2 = P_5^2 \sim P_7 = [7, 1 + \omega_\Delta]$, and in the continued fraction expansion of $(1 + \omega_\Delta)/7$ only $Q_\ell/\sigma = Q_5/\sigma = 7$ is prime and $a_{(2,5)} = 2$. Given that $I_1^5 \sim 1$, then $I_1^3 \sim I_2'$ and $I_1^4 \sim I_1'$, so the above values all occur twice, but by the definition of $T_i(p)$ we do not count multiplicity. These are the only values that yield non-zero $a_{(i,j)}$ since 401 is prime so 2, 5, 7 are the only unramified primes less than $\sqrt{401}/2$ that yield non-zero results. Thus, since we are not in an ambiguous class,

$$\sum_{i=1}^{k} \sum_{j \in T_i - R_i} (a_{(i,j)} + 1) = (a_{(1,1)} + 1) + (a_{(1,3)} + 1) + (a_{(2,5)} + 1)$$

$$= (9 + 1) + (3 + 1) + (2 + 1) = 17,$$

and

$$\sum_{p < \sqrt{\Delta}/2} |S_f(p)| = (|S_f(2)| + |S_f(5)| + |S_f(7)|) = (10 + 4 + 3) = 17.$$

Thus, we see that the hypothesis of Theorem 3.1 is satisfied and so

$$C_\Delta = \{1, P_5, P_5^2, P_5^3, P_5^4\} = \langle P_5 \rangle \text{ with } h_\Delta = 5.$$

Example 3.3. Let $\Delta = \Delta_0 = 4 \cdot 226 = 8 \cdot 113$ with $h_\Delta = 8$. Here 2, 3, 5 and 7 are all the non-inert primes less than $\sqrt{226} = \sqrt{\Delta}/2$. If $I_1 = P_3 = [3, 1 + \sqrt{226}]$, then we observe that in the continued fraction expansion of $(1 + \sqrt{226})/3$ the only Q_j for $j > 0$, which is prime, is $Q_\ell = Q_3 = 3$ and $a_{(1,3)} = 9$. In the continued fraction expansion of $(1 + \sqrt{226})/9$ for

$I_2 = P_3^2 = [9, 1 + \sqrt{226}]$, we see that only $Q_2 = 7$ is prime and $a_{(2,2)} = 3$. If $P_3^3 = [27, 8 + \sqrt{226}]$, then P_3^3 is not reduced. However, $I_3 = P_5 = [5, 14 + \sqrt{226}] \sim P_3^3$ and in the continued fraction expansion of P_5 only $Q_3 = Q_\ell = 5$ is prime and $a_{(3,3)} = 5$. $I_4 = P_3^4 = [81, 8 + \sqrt{226}]$ is also not reduced. However, $P_2 = [2, 226] \sim P_3^4$ and in the continued fraction expansion of $\sqrt{226}/2$ only $Q_\ell = Q_3 = 2$ is prime and $a_{(4,3)} = 14$. Since P_2 is ramified, P_3^4 is ambiguous so $P_3^4 \sim (P_3')^4$, and $P_3^8 \sim 1$. Thus, the above values from P_i for $i = 1, 2, 3$ each occur twice, whereas from P_3^4, $\lfloor (a_{(4,3)} + 1)/2 \rfloor = 7$ occurs once. Hence

$$\sum_{i=1}^{k} \sum_{j \in T_i - R_i} (a_{(i,j)} + 1) = (a_{(1,3)} + 1) + (a_{(2,2)} + 1) + (a_{(3,3)} + 1)$$

$$= (9 + 1) + (3 + 1) + (5 + 1) = 20,$$

and

$$\sum_{i=1}^{8} \sum_{j \in -R_i \cap T_i} \lfloor (a_{(i,j)} + 1)/2 \rfloor = \lfloor (a_{(4,3)} + 1)/2 \rfloor = 7.$$

Also, for ramified primes,

$$\sum_{p < \sqrt{\Delta}/2} | S_f(p) | = | S_f(2) | = 7,$$

and for unramified primes,

$$\sum_{p < \sqrt{\Delta}/2} | S_f(p) | = S_f(3) + S_f(5) + S_f(7) = 10 + 6 + 4 = 20,$$

as predicted by Theorem 3.2.

Thus, by Theorem 3.1, $C_\Delta = \{1, P_3, P_3^2, P_3^3, P_3^4, P_3^5, P_3^6, P_3^7\} = \langle P_3 \rangle$ with $h_\Delta = 8$.

The following class number one criterion follows immediately from Theorem 3.2 and is a simpler criterion than that given by Lu in [3]. Moreover, it corrects [4, Corollary 2.1, p. 440].

Corollary 3.1. *If* $h_\Delta = 1$, *then* $\sum_{p<\sqrt{\Delta}/2} d(p, F) \leq \sum_{i=1}^{\lfloor \ell/2 \rfloor} \beta_i$, *where the left-hand sum ranges over primes* $p < \sqrt{\Delta}/2$, $\ell = \ell(w_\Delta)$, *and*

$$\beta_i = \begin{cases} a_i + 1, & \text{if } i \neq \ell/2 \text{ and } Q_i/2 \text{ is prime}, \\ \lfloor (a_i + 1)/2 \rfloor, & \text{if } i = \ell/2 \text{ and } Q_i/2 \text{ is prime}, \\ 0, & \text{otherwise} \end{cases}$$

with $w_\Delta = \langle a; \overline{a_1, a_2, \ldots, a_\ell} \rangle$. *If equality holds, then* $h_\Delta = 1$, *and in this case, the primes appearing as* Q_j/σ *in the simple continued fraction expansion of* ω *are precisely those less than the Minkowski bound* $\sqrt{\Delta}/2$.

Remark 3.2. By Theorems 3.1-3.2, Corollary 3.1 implies that $h_\Delta = 1$ if and only if every prime $p < \sqrt{\Delta}/2$, which is not inert, appears as some Q_j/σ in the continued fraction expansion of ω_Δ, and no other primes appear as any Q_j/σ. This is a point missed in [4].

Example 3.4. Let $\Delta = 1757 = 7 \cdot 251$ and note that

i	0	1	2	3	4
P_i	1	41	35	35	41
Q_i	2	38	14	38	2
a_i	21	2	5	2	41

so $(a_1 + 1) + \lfloor (a_2 + 1)/2 \rfloor = 6$, and from $f_\Delta(x) = -x^2 + x + 439$,

$$\sum_{p<\sqrt{\Delta}/2} |S_f(p)| = |S_f(7)| + |S_f(19)| = 6.$$

Example 3.5. Note that the inequality in Corollary 3.1 cannot be strengthened (as was another error in [4]). For instance, $\Delta = 4 \cdot 94$ has $h_\Delta = 1$, but $\sum_{p<\sqrt{\Delta}/2} |S_f(p)| = |S_f(2)| + |S_f(3)| + |S_f(5)| = 14$, whereas

$$\sum_{i=1}^{\lfloor \ell/2 \rfloor} \beta_i = \sum_{i=1}^{8} \beta_i = 2 + 4 + 6 + 4 = 16.$$

The reason for the strict inequality here is that $Q_1 = 13$ in the simple continued fraction expansion of ω_Δ and $13 > \sqrt{\Delta}/2$, which illustrates Remark 3.2 which may be seen as a succinct reinterpretation of the class number one criterion.

In conclusion then, Theorem 3.2 is the most general criteria for the complete determination of the class group of an arbitrary real quadratic order in terms of the real quadratic version of the Euler-Rabinowitsch polynomial and continued fractions via the key result (Theorem 3.1). Our comparison with known results in the literature such as [1], [2] and [3] which dealt only with maximal orders, shows the canonical niche into which Theorem 3.2 fits, and our examples depict the ease of use of the criterion. Furthermore, the class number one criterion that is embodied in Corollary 3.1 and encapsulated in Remark 3.2 yields an elegant tool for use in algebraic number theory.

References

[1] E. Dubois and C. Levesque, On determining certain real quadratic fields with class number one and relating this property to continued fractions and primality properties, Nagoya Math. J. 124 (1991), 157-180.

[2] S. Louboutin, R. A. Mollin and H. C. Williams, Class numbers of real quadratic fields, continued fractions, reduced ideals, prime producing quadratic polynomials and quadratic residue covers, Can. J. Math. 44 (1992), 1-19.

[3] H. Lu, On the class number of real quadratic fields, Sci. Sinica (special issue (II)) (1979), 118-130.

[4] R. A. Mollin, Use of the Rabinowitsch polynomial to determine the class groups of a real quadratic field, Bull. Austral. Math. Soc. 50 (1994), 435-443.

[5] R. A. Mollin, Quadratics, CRC Press, Boca Raton, New York, London, Tokyo, 1996.

[6] R. A. Mollin and H. C. Williams, Class number one for real quadratic fields, continued fractions and reduced ideals, Number Theory and Applications, Nato ASI, R. Mollin, ed., C265, Kluwer Academic Publishers, 1989, pp. 481-496.

[7] G. Rabinowitsch, Eindeutigkeit der Zerlegung in Primzahlfaktoren in quadratischen Zahlkörpern, J. Reine Angew. Math 142 (1913), 153-164.

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