Class Number Two for Real Quadratic Fields of Richaud-Degert Type*

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Abstract

This paper contains proofs of prove conjectures made in [16] on class number 2 and what this author has dubbed the *Euler-Rabinowitsch polynomial* for real quadratic fields. As well, we complete the list of Richaud-Degert types given in [16] and show how the behaviour of the Euler-Rabinowitsch polynomials and certain continued fraction expansions come into play in the complete determination of the class number 2 problem for such types. For some values the determination is unconditional, and for others, the wide Richaud-Degert types, the determination is conditional on the generalized Riemann hypothesis (GRH).

1 Introduction

Over the past couple of decades there has been much work done on real quadratic fields of Richaud-Degert type, namely those $\mathbb{Q}(\sqrt{D})$ with radicand $D = m^2+r$ where $r \mid 4m$. Some of this work by this author and H.C. Williams in the 1980s made a list of Richaud-Degert types of various class numbers, and showed that the list is complete with one possible exception that is ruled out if the GRH is assumed—see [12] for background. Recently some of these lists

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have been unconditionally verified. In particular conjectures made by this author and others have been affirmatively resolved. For instance, Chowla's conjecture and Yokoi's conjectures were both settled in 2003 by Biro in [1]–[2]. Chowla's conjecture says that if $p = m^2 + 1$ is prime and $m > 26$, then the class number of $\mathbb{Q}(\sqrt{p})$, $h_p > 1$. Yokoi's conjecture says that if $d = m^2 + 4$ is squarefree and $m > 17$, then $h_D > 1$. Mollin's conjecture says that if $d = m^2 - 4$ is squarefree and $m > 21$, then $h_D > 1$. The latter was affirmatively verified in 2007 by Byeon, Kim, and Lee in [3]. This essentially determines all *narrow* Richaud-Degert types of class number one, where a narrow Richaud-Degert type is one for which the aforementioned $r \in \{\pm 1, \pm 4\}$. Also, in 2008, Byeon and Lee unconditionally determined all narrow Richaud-Degert types with radicand $D = n^2 + 1$ with n odd having class number 2. This allows the unconditional verification of conjectures made by this author and H.C. Williams in [16].

In this paper, we look at the class number 2 problem for Richaud-Degert types. We provide several theorems that delineate the relationship between class number 2 for all Richaud-Degert types with necessary and sufficient conditions in terms of the behaviour of the simple continued fraction expansions of certain specified quadratic irrationals as well as the prime-producing behaviour of the Euler-Rabinowitsch polynomial. This completely describes the class number 2 radicands, some unconditionally, and some with one GRH-ruled-out exception. Moreover, we include values, three of them, missed in the list in [16], which includes one missed in [13] of type $D = m^2 \pm 2$. Hence, this paper completes the list of all Richaud-Degert types of class number 2 and related continued fraction and polynomial performance criteria, heretofore not presented in the literature. What remains, and this is not likely to be resolved soon, is an *unconditional* verification that the list of 107 values is indeed complete.

2 Notation and Preliminaries

In what follows, a field discriminant Δ, with associated (squarefree) radicand D, is defined by

$$\Delta = \begin{cases} D & \text{if } D \equiv 1 \pmod 4, \\ 4D & \text{if } D \equiv 2, 3 \pmod 4. \end{cases} \tag{2.1}$$

Herein, we will be concerned with the simple continued fraction expansions of quadratic irrationals $\alpha = (P + \sqrt{D})/Q$, namely $P^2 \equiv D \pmod Q$

with $P, Q \in \mathbb{Z}$ (the integers) and $Q \neq 0$. We denote this expansion by,

$$\alpha = \langle q_0; \overline{q_1, q_2, \ldots, q_{\ell-1}, q_\ell} \rangle,$$

where $\ell = \ell(\alpha)$ is the period length, $q_0 = \lfloor \alpha \rfloor$ (the *floor* of α). The *norm* of α is given by $N(\alpha) = (P^2 - D)/Q^2$.

The *complete quotients* are given by $(P_j + \sqrt{D})/Q_j$, where $P_0 = P$, $Q_0 = Q$, and for $j \geq 1$,

$$P_{j+1} = q_j Q_j - P_j, \tag{2.2}$$

$$q_j = \left\lfloor \frac{P_j + \sqrt{D}}{Q_j} \right\rfloor, \tag{2.3}$$

and

$$D = P_{j+1}^2 + Q_j Q_{j+1}. \tag{2.4}$$

We will need the following facts concerning period length.

If $\ell(\alpha) = \ell$ is even, then

$$P_{\ell/2} = P_{\ell/2+1}, \tag{2.5}$$

and if ℓ is odd, then

$$Q_{(\ell+1)/2} = Q_{(\ell-1)/2}. \tag{2.6}$$

The *principal surd* for the discriminant Δ of a real quadratic field $\mathbb{Q}(\sqrt{D})$ having radicand D, is given by

$$\omega_\Delta = \begin{cases} (1+\sqrt{\Delta})/2 & \text{if } D \equiv 1 \pmod 4, \\ \sqrt{\Delta}/2 & \text{if } D \not\equiv 1 \pmod 4. \end{cases}$$

We will need the following which determines the generators of the ideal class group $\mathcal{C}_\Delta$ of $\mathbb{Q}(\sqrt{\Delta})$ having discriminant Δ.

Theorem 2.1 *If Δ is the discriminant of a real quadratic field, then every class of $\mathcal{C}_\Delta$ contains a primitive ideal I with $N(I) \leq \sqrt{\Delta}/2$. Furthermore, $\mathcal{C}_\Delta$ is generated by the non-inert prime $\mathcal{O}_\Delta$-ideals $\mathcal{P}$ with $N(\mathcal{P}) < \sqrt{\Delta}/2$.*

Proof. See [12, Theorem 1.3.1, p. 15]. □

We also need the following, called the *infrastructure theorem* for real quadratic fields.

Theorem 2.2 *Let $I = I_1 = [Q_0/2, (P_0 + \sqrt{\Delta})/2]$ be an $\mathcal{O}_\Delta$-ideal corresponding to the quadratic irrational $\alpha = \alpha_0 = (P_0 + \sqrt{\Delta})/2$, and let P_j, Q_j be as given above. If $I_j = [Q_{j-1}/2, (P_{j-1} + \sqrt{\Delta})/2]$, then $I_1 \sim I_j$ for all $j \geq 1$. Moreover, there exists a least value $m \in \mathbb{N}$ such that I_{m+i} is reduced for all $i \geq 0$.*

Proof. See [12, Theorem 2.1.2, p. 44]. □

Corollary 2.1 A reduced ideal $I = [Q/2, (P + \sqrt{\Delta})/2]$ of $\mathcal{O}_\Delta$ is principal if and only if $Q = Q_j$ for some positive integer $j \leq \ell(\omega_\Delta)$ in the continued fraction expansion of ω_Δ.

Proof. See [9]. □

3 Class Number Two

In what follows, Δ is a discriminant with squarefree radicand D, and $q \in \mathbb{N}$ a squarefree divisor of Δ, with $\alpha_\Delta = 1$ if $4q \mid \Delta$ and $\alpha_\Delta = 2$ otherwise. The polynomial

$$F_{\Delta,q}(x) = qx^2 + (\alpha_\Delta - 1)qx + \frac{(\alpha_\Delta - 1)q^2 - \Delta}{4q} \tag{3.7}$$

is called the *Euler-Rabinowitsch polynomial*, which was introduced by this author in [12] to discuss prime-producing quadratic polynomials as an analogue of the renowned Rabinowitsch result for class number one of complex quadratic fields—see [12, Theorem 4.1.2, p. 108].

The following affirmatively verifies, without the use of GRH, the values for class number 2 given in [16] for the case where $\Delta = 4(m^2 \pm 1)$ and provides both the continued fraction expansion data and behaviour of the Euler-Rabinowitsch polynomial.

Theorem 3.1 *Let $\Delta = 4(m^2 \pm 1) = 4D$ for $m \in \mathbb{N}$. Then following are equivalent.*

(a) $h_\Delta = 2$.

(b) For each split prime $r < \sqrt{\Delta}/2$, there exists a natural number $q \mid \Delta$ such that $|F_{\Delta,q}(x)| = r$ for some integer $x \geq 0$.

(c) Δ is one of the following values:

$$\{10, 15, 26, 35, 122, 143, 362\}. \tag{3.8}$$

(d) For each x with $1 \le x \le m$, $|F_{\Delta,1}(x)|$ equals 1, a prime, twice a prime, or m^2 where m is prime.

(e) One of the following holds:

(i) $D = p^2 + 2p = (p+1)^2 - 1$ for some prime p and there is no split prime $p < \sqrt{D}$.

(ii) $D = p^2 - (p-2)p = 2p = q^2 + 1$ for some prime p where q is prime and the only split prime less than $\sqrt{D}$ is q.

Proof. The equivalence of (a) and (b) is [11, Theorem 3.1, p. 357]. The equivalence of (a) and (c) is [8] and [4, Theorem 1.2, p. 866]. The equivalence of (a) and (d) is [10]. Now assume that (e) holds. If (i) holds, then by [12, Theorem 3.2.1, p. 78], since $D = p^2 + 2p$, then $\ell(\sqrt{D}) = 2$ and $Q_1 = 2p$. Thus, by Corollary 2.1, $h_\Delta > 1$ since $p \ne Q_j$ for any j in the principal cycle ω_Δ. Also, $\ell(\sqrt{D}/p) = 2$ and this is not the principal cycle with $Q_1 = 2$ and $Q_0 = p = Q_2$. Since there are no split primes less than $\sqrt{\Delta}/2$, then by Theorem 2.1, $h_\Delta = 2$. If (ii) holds, namely $D = 2p = m^2 + 1$, then by [12, Theorem 3.2.1, p. 78], $\ell(\sqrt{D}) = 1$ so by Corollary 2.1, $h_\Delta > 1$. Also, $\ell(\sqrt{D}/2) = 3$ with $Q_0 = Q_3 = 2$ and $Q_1 = Q_2 = q$. Since there are no other split or ramified primes less than $\sqrt{D}$ then, by Theorem 2.1, $h_\Delta = 2$.

We have shown that part (e) implies part (a). Since (a) is equivalent to (c) and (c) is easily checked to imply (e), then the logical circle is complete. □

Example 3.1 Our largest example for $D = 2p = q^2 + 1$ is $D = 362 = 2 \cdot 181 = 19^2 + 1$. Here $\ell(\sqrt{362}) = 1$ and $\ell(\sqrt{D}/2) = 3$ with $Q_3 = Q_0 = 2$ and $Q_2 = Q_3 = 19$, where 19 is the only split prime less than $\sqrt{362}$. Indeed, $\lfloor\sqrt{362}\rfloor = 19$.

The largest value of $D = p^2 + 2p$ is $D = 143 = 11^2 + 2 \cdot 11$ where $\ell(\sqrt{143}) = 2$ with $Q_1 = 22$ and $\ell(\sqrt{143})/11 = 2$ with $Q_0 = Q_2 = 11$, $Q_1 = 2$, and there are no split primes less than $\sqrt{143}$.

Remark 3.1 In [8] it was shown that if $D = m^2 + 1 \not\equiv 5 \pmod 8$ is squarefree, then $h_\Delta = 2$ if and only if the list in (3.8) also contains $D = 65$. Thus, our restriction to $D \not\equiv 1 \pmod 4$ eliminates only this one value. Indeed, note that in [13, Theorem 3.3, p. 569], it was *unconditionally* shown that if $D = \Delta = m^2+1 \equiv 1 \pmod 8$, then $h_\Delta = 2$ if and only if $\Delta = 65$ or $\Delta = 105$. See also [10]. Now we look at $D = \Delta \equiv 5 \pmod 8$. Also, we found the values in part (e) (i) in [15] by different methods.

Theorem 3.2 If $\Delta = D = 4m^2+1$, with m odd, is a squarefree discriminant, then the following are equivalent.

(a) $h_\Delta = 2$.

(b) For each split prime $r < \sqrt{\Delta}/2$, there exists a natural number $q \mid \Delta$ such that $|F_{\Delta,q}(x)| = r$ for some integer $x \geq 0$.

(c) Δ is one of the following values, with one GRH-ruled-out exception:

$$\{485, 1157, 2117, 3365\}. \tag{3.9}$$

(d) m is prime, $\Delta = 4 \cdot p^2 + q^2$, where $p < \sqrt{\Delta}/2$ is a split prime and if $\alpha = (q + \sqrt{\Delta})/p$, then $\ell(\alpha) = 3$ or $\ell(\alpha) = 5$, and in the latter case $Q_3 = r$ a split prime with either $r > \sqrt{\Delta}/2$ or else, $r < \sqrt{\Delta}/2$ and p, m, and r are the only split primes less than $\sqrt{\Delta}/2$.

Proof. The equivalence of (a) and (b) is [11, Theorem 3.1, p. 357]. The equivalence of (a) and (c) is [16, Theorem 3.2, p. 99]. Now assume that (d) holds. By [12, Theorem 3.2.1, p. 78], $\ell((1 + \sqrt{\Delta})/2) = 3$ and by (2.6), $Q_1 = Q_2 = 2m$, so by Corollary 2.1, $h_\Delta > 1$. By the hypothesis, the continued fraction expansion of $(q + \sqrt{\Delta})/p$ has the norms in Q_j of all split primes less than $\sqrt{\Delta}/2$, so by Theorem 2.1, $h_\Delta = 2$. We have shown that (d) implies (a). Since (a) is equivalent to (c) and (c) is easily checked to imply (d), then the proof is secured. $\square$

Remark 3.2 Allowing m to be even in Theorem 3.2 adds only the value $\Delta = 65$ via [16, Theorem 3.2, p. 99], and we have dealt with this already as mentioned in Remark 3.1. Also, note that $\Delta = 2117$ is the only value from the list (3.9) that has three split primes less than $\sqrt{\Delta}/2$, namely $p = 17$, $m = 23$ and $r = 11$.

Now that we have secured all of the class number 2 values for narrow Richaud-Degert types, we turn our attention to the wide values.

Theorem 3.3 Let $\Delta = 4(m^2 + q)$ with $q = 2p$ where the prime $p \mid m$. Then the following are equivalent.

(a) $h_\Delta = 2$.

(b) $|F_{\Delta,2p}(x)|$ is 1 or prime for all non-negative integers $x < (\sqrt{D} - 1)/2$.

(c) Δ is one of the following values, with one GRH-ruled-out exception:

$$\{42, 87, 110, 395, 447, 635\}. \tag{3.10}$$

(d) There are no split primes less than $\sqrt{D}$ and in the simple continued fraction expansion of $\alpha = \sqrt{D}/p$, $\ell(\alpha) = 2$, with $Q_0 = Q_2 = p$ and $Q_1 = 2$.

Proof. That (b) implies (a) is [12, Theorem 4.2.5 (a), p. 134]. The equivalence of (a) and (c) is [16, Theorem 3.2, p. 99]. Assuming that (d) holds, by [12, Theorem 3.2.1, p. 78], $\ell(\sqrt{D}) = 2$ and $Q_1 = 2p$. Since the continued fraction expansion of α contains all the non-inert primes less than $\sqrt{D}$, then by Theorem 2.1, $h_\Delta = 2$. Thus, (d) implies (a) which is equivalent to (c) and is easily checked to imply (d), as well as (b). □

In exactly the same fashion as in Theorem 3.3, we achieve the following Theorems 3.4–3.5, which we therefore state without proof.

Theorem 3.4 Let $\Delta = 4(m^2 - q)$ with $q = 2p$ where the prime $p \mid m$. Then the following are equivalent.

(a) $h_\Delta = 2$.

(b) $|F_{\Delta,2p}(x)|$ is 1 or prime for all non-negative integers $x < (\sqrt{D} - 1)/2$.

(c) Δ is one of the following values, with one GRH-ruled-out exception:

$$\{138, 182, 215, 318\}. \tag{3.11}$$

(d) There are no split primes $p < \sqrt{D}$ and in the simple continued fraction expansion of $\alpha = \sqrt{D}/p$, $\ell(\alpha) = 4$, with $Q_0 = Q_4 = p$ and $Q_2 = 2$.

Theorem 3.5 Let $\Delta = 4(m^2 \pm p)$ with the prime $p \mid m$. Then the following are equivalent.

(a) $h_\Delta = 2$.

(b) $|F_{\Delta,2p}(x)|$ is 1 or prime for all non-negative integers $x < (\sqrt{D} - 1)/2$.

(c) Δ is one of the following values, with one GRH-ruled-out exception:

$$\{30, 39, 78, 95, 203, 222, 230, 327\}. \tag{3.12}$$

(d) One of the following occurs:

(i) $\Delta = 4((pq)^2 \pm p)$ where $q = 1$ or q is prime, and there is at most one split prime less than $\sqrt{D}$, which when it exists, appears as Q_j for $j \in \{1, 2\}$ in the simple continued fraction expansion of $\alpha = \sqrt{D}/2$, where $\ell(\sqrt{D}/2) = 4$.

(ii) $\Delta = 4(4p^2 + p)$ or $\Delta = 4(4p^4 + p)$ and there is one split prime less than $\sqrt{D}$, which appears as a Q_j for $j \in \{1, 2\}$ in the simple continued fraction expansion of $\alpha = (1 + \sqrt{D})/2$, where $\ell((1 + \sqrt{D})/2) = 6$.

(iii) $\Delta = 4(4p^2 - p)$ and there is one split prime less than $\sqrt{D}$, which appears as a Q_j for $j \in \{1, 2\}$ in the simple continued fraction expansion of $\alpha = (1 + \sqrt{D})/2$, where $\ell((1 + \sqrt{D})/2) = 4$.

Remark 3.3 The list in [16, Theorem 3.2, p. 99] lacks the values 78 and 222, which we include in Theorem 3.5.

Theorem 3.6 Let $\Delta = 8(m^2 + 2) = 4D = 8pq$ for primes $p < q$. Then the following are equivalent.

(a) $h_\Delta = 2$.

(b) For each split prime $r < \sqrt{\Delta}/2$, there exists a natural number $q \mid \Delta$ such that $|F_{\Delta,q}(x)| = r$ for some integer $x \geq 0$.

(c) Δ is one of the following values, with one GRH-ruled-out exception:

$$\{66, 102, 258, 402, 678, 902, 1298\}. \tag{3.13}$$

(d) One of the following occurs:

  (i) There is one split primes less than $\sqrt{D}$, and it appears as a Q_j for $j \in \{1, 2\}$ in the simple continued fraction expansion of $\alpha = \sqrt{D}/p$, where $\ell(\sqrt{D}/p) = 6$.

  (ii) There exactly two split primes less than $\sqrt{D}$, and they appear as a Q_j for $j \in \{1, 2, 3, 4\}$ in the simple continued fraction expansion of $\alpha = \sqrt{D}/p$, where $\ell(\sqrt{D}/p) = 10$.

Proof. The equivalence of (a) and (b) is [11, Theorem 3.1, p. 357]. The equivalence of (a) and (c) is [16, Theorem 3.2, p. 99]. Assuming that (d) holds, by [12, Theorem 3.2.1, p. 78], $\ell(\sqrt{D}) = 2$ and $Q_1 = 2$. Since the continued fraction expansion of α contains all the non-inert primes less than $\sqrt{D}$, then by Theorem 2.1, $h_\Delta = 2$. Thus, (d) implies (a) which is equivalent to (c) which, in turn, is easily checked to imply (d). □

Theorem 3.7 Let $\Delta = 4(m^2 + 2) = 4D = 4pq$ for primes $p < q$. Then the following are equivalent.

(a) $h_\Delta = 2$.

(b) For each split prime $r < \sqrt{\Delta}/2$, there exists a natural number $q \mid \Delta$ such that $|F_{\Delta,q}(x)| = r$ for some integer $x \geq 0$.

(c) Δ is one of the following values, with one GRH-ruled-out exception:

$$\{51, 123, 843\}. \tag{3.14}$$

(d) There is either one or two split primes less than $\sqrt{D}$, and they appear as a Q_j for $j \in \{1, 2\}$ in the simple continued fraction expansion of $\alpha = \sqrt{D}/p$, where $\ell(\sqrt{D}/p) = 6$.

Proof. The equivalence of (a) and (b) is [11, Theorem 3.1, p. 357]. The equivalence of (a) and (c) is [16, Theorem 3.2, p. 99]. Assuming that (d) holds, by [12, Theorem 3.2.1, p. 78], $\ell(\sqrt{D}) = 2$ and $Q_1 = 2$. Since the continued fraction expansion of α contains all the non-inert primes less than $\sqrt{D}$, then by Theorem 2.1, $h_\Delta = 2$. Thus, (d) implies (a) which is equivalent to (c) which, in turns, is easily checked to imply (d). □

The following Theorems 3.8–3.11 are proved in exactly the same fashion as Theorem 3.7.

Theorem 3.8 Let $\Delta = 8(m^2 + 2) = 4D = 8p$ for where p is prime. Then the following are equivalent.

(a) $h_\Delta = 2$.

(b) For each split prime $r < \sqrt{\Delta}/2$, there exists a natural number $q \mid \Delta$ such that $|F_{\Delta,q}(x)| = r$ for some integer $x \geq 0$.

(c) $\Delta = 146$ with one possible GRH-ruled-out exceptional value.

(d) There are two split primes less than $q < r < \sqrt{D}$, and they appear as a Q_j for $j \in \{1, 2, 3\}$ in the simple continued fraction expansion of $\alpha = (1 + \sqrt{D})/q$, where $\ell((1 + \sqrt{D})/q) = 8$.

Theorem 3.9 Let $\Delta = 8(m^2 - 2) = 4D = 8p$ for where p is prime. Then the following are equivalent.

(a) $h_\Delta = 2$.

(b) For each split prime $r < \sqrt{\Delta}/2$, there exists a natural number $q \mid \Delta$ such that $|F_{\Delta,q}(x)| = r$ for some integer $x \geq 0$.

(c) Δ is one of the following values, with one GRH-ruled-out exception:

$$\{34, 194, 482\}. \tag{3.15}$$

(d) One of the following occurs, where $D \equiv (b_q)^2 \pmod{q}$:

(i) There are exactly two split primes less than $\sqrt{D}$, and they appears as a Q_j for $j \in \{1, 2, 3\}$ in the simple continued fraction expansion of $\alpha = (b_q + \sqrt{D})/q$, where $\ell((b_q + \sqrt{D})/q) = 6$.

(ii) There are exactly three split primes less than $\sqrt{D}$, and they appears as a Q_j for $j \in \{1, 2, 3, 4, 5\}$ in the simple continued fraction expansion of $\alpha = (b_q + \sqrt{D})/q$, where $\ell((b_q + \sqrt{D})/q) = 10$.

Remark 3.4 It is worthy of note that the values in Theorems 3.8–3.9 are the only values of type $D = m^2 \pm 2$ where the class group of C_Δ is generated by an ambiguous class of ideals with no ambiguous ideals in it. See [14] and [12, Chapter 6] for more background and details.

Theorem 3.10 Let $\Delta = 8(m^2 - 2) = 4D = 8pq$ for primes $p < q$. Then the following are equivalent.

(a) $h_\Delta = 2$.

(b) For each split prime $r < \sqrt{\Delta}/2$, there exists a natural number $q \mid \Delta$ such that $|F_{\Delta,q}(x)| = r$ for some integer $x \geq 0$.

(c) Δ is one of the following values, with one GRH-ruled-out exception:

$$\{782, 1022\}. \tag{3.16}$$

(d) One of the following occurs:

(i) There are exactly two split primes less than $\sqrt{D}$, and they appears as a Q_j for $j \in \{1, 2, 3\}$ in the simple continued fraction expansion of $\alpha = \sqrt{D}/p$, where $\ell(\sqrt{D}/p) = 8$.

(ii) There is exactly one split prime less than $\sqrt{D}$, and it appears as a Q_j for $j \in \{1, 2, 3, 4\}$ in the simple continued fraction expansion of $\alpha = \sqrt{D}/p$, where $\ell(\sqrt{D}/p) = 10$.

Theorem 3.11 Let $\Delta = 4(m^2 - 2) = 4D = 4pq$ for primes $p < q$. Then the following are equivalent.

(a) $h_\Delta = 2$.

(b) For each split prime $r < \sqrt{\Delta}/2$, there exists a natural number $q \mid \Delta$ such that $|F_{\Delta,q}(x)| = r$ for some integer $x \geq 0$.

(c) Δ is one of the following values, with one GRH-ruled-out exception:

$$\{119, 287, 527, 623\}. \tag{3.17}$$

(d) There are one or two split primes less than $\sqrt{D}$ and they appear as a Q_j for some $j \in \{1, 2, 3\}$in the simple continued fraction expansion of $\alpha = \sqrt{D}/p$ where $\ell(\alpha) = 8$.

Remark 3.5 In [13, Conjecture 3.9, p. 571], where we list all class number two of the form $D = m^2 \pm 2$, the value $D = 51$ was omitted. Now that we included this value in Theorem 3.7, then the details for all such values comprise Theorems 3.6–3.11.

Theorems 3.12–3.13 following are proved in the same fashion as above so we state them without proof.

Theorem 3.12 Let $\Delta = m^2 \pm 4$ with m odd. Then the following are equivalent.

(a) $h_\Delta = 2$.

(b) Δ is one of the following values, with one GRH-ruled-out exception:

$$\{85, 165, 221, 285, 357, 365, 533, 629, 957, 965, 1085, 1517,$$
$$1685, 1853, 2397, 2813\}. \tag{3.18}$$

(d) One of the following occurs:

(i) $\Delta = m^2 + 4 = pq$ where $p < q$ are primes, and all split primes less than $\sqrt{\Delta}/2$ appear as some Q_j for $1 < j < \ell(\alpha)$ in the simple continued fraction expansion of $\alpha = \sqrt{D}/p$.

(ii) $\Delta = m^2 - 4 = pqr$ with $p < q < r$ where p, q, r are primes or $p = 1$ and q, r are primes, and all split primes less than $\sqrt{\Delta}/2$ appear as some Q_j for $1 < j < \ell(\alpha)$ in the simple continued fraction expansion of $\alpha = \sqrt{D}/p$.

Remark 3.6 The equivalences in Theorem 3.1 (a)–(c), Theorem 3.2 (a)–(c), and Theorem 3.12 (a)–(b) are also included in the results on class number two for real quadratic fields: [6]–[7] (when the fundamental unit has norm -1, and [5] (when the fundamental unit has norm 1).

Theorem 3.13 Let $\Delta \equiv 5 \pmod 8$ where either $\Delta = m^2 \pm q$ $q = 4r$ where $r \mid m$ and r is a prime or a product of two primes, or $\Delta = 4m^2 \pm q$ where $q \mid m$ and q is prime or a product of two primes. Then the following are equivalent.

(a) $h_\Delta = 2$.

(b) Δ is one of the following values, with one GRH-ruled-out exception:

$$\{205, 429, 645, 741, 885, 1173, 1205, 1245, 1469, 1533, 1605,$$

$$1965, 2013, 2037, 2045, 2085, 2093, 2301, 2373, 2613, 2717, 3005, 3237,$$

$$3597, 3605, 3813, 4245, 4277, 4773, 4893, 5645, 5757, 5885, 5957, 6573,$$

$$7157, 7733, 8333, 9005, 14405\}.$$

(d) $\Delta = pqr$ with $p < q < r$ where p, q, r are primes or $p = 1$ and q, r are primes, and all split primes less than $\sqrt{\Delta}/2$ appear as some Q_j for $1 < j < \ell(\alpha)$ in the simple continued fraction expansion of $\alpha = \sqrt{D}/p$.

Remark 3.7 The value $D = 885$ was missed in [16, Theorem 3.2, p. 99], and its inclusion in Theorem 3.13 completes the description of all Richaud-Degert types of class number 2 in detail in terms of continued fraction expansions and the prime production of Euler-Rabinowitsch polynomials. Note that if we allow $\Delta \equiv 1 \pmod 4$ in Theorem 3.13, then we only add $\Delta = 105$ to the list. As noted in Remark 3.1, this value is already taken into account. Thus, Theorems 3.1–3.13 and Remark 3.1 comprise all class number 2 Richaud-Degert types.

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