

## CLASS GROUPS, TOTALLY POSITIVE UNITS, AND SQUARES

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**ABSTRACT.** Given a totally real algebraic number field  $K$ , we investigate when totally positive units,  $U_K^+$ , are squares,  $U_K^2$ . In particular, we prove that the rank of  $U_K^+ / U_K^2$  is bounded above by the minimum of (1) the 2-rank of the narrow class group of  $K$  and (2) the rank of  $U_L^+ / U_L^2$  as  $L$  ranges over all (finite) totally real extension fields of  $K$ . Several applications are also provided.

**1. Notation and preliminaries.** Let  $K$  be an algebraic number field and let  $C_K$  denote the ideal class group in the ordinary or “wide” sense. Let  $C_K^{(+)}$  denote the “narrow” ideal class group of  $K$ . Thus  $|C_K| = h_K$ , the “wide” class number of  $K$ , and  $|C_K^{(+)}| = h_K^{(+)}$ , the “narrow” class number of  $K$ . We denote the Hilbert class field of  $K$  by  $K^{(1)}$ ; i.e.,  $\text{Gal}(K^{(1)}/K) \cong C_K$ , and we denote the “narrow” Hilbert class field by  $K^{(+)}$ ; i.e.,  $\text{Gal}(K^{(+)}/K) \cong C_K^{(+)}$ . Moreover we adopt the “bar” convention to mean “modulo squares”; for example,  $\bar{C}_K = C_K / C_K^2$ .

Let  $U_K$  denote the group of units of the ring of algebraic integers of  $K$ . When  $K$  is totally real, we let  $U_K^+$  denote the subgroup of totally positive units; i.e., those units  $u$  such that  $u^\sigma > 0$  for all embeddings  $\sigma$  of  $K$  into  $\mathbf{R}$ . Finally, for any finite abelian group  $A$  with  $|\bar{A}| = 2^d$ ,  $d$  is called the 2-rank of  $A$ , which we denote by  $\dim_2 A$ .

**2. Results.** We are concerned with the question:

(\*) When is  $U_K^+ = U_K^2$ ?

We begin by observing that  $\dim_2(\bar{U}_K^+) = 0$  if and only if  $K^{(+)} = K^{(1)}$  [6, Theorem 3.1, p. 203]. In particular, when  $K$  is a real finite Galois extension of 2-power degree over  $\mathbf{Q}$ , then  $\dim_2(\bar{U}_K^+) = 0$  if and only if  $N(U_K) = \{\pm 1\}$  [3, Theorem 1, p. 166]. For example, when  $K$  is a real quadratic field, then  $\dim_2(\bar{U}_K^+) = 0$  if and only if the norm of the fundamental unit is  $-1$ . Necessary and sufficient conditions (in terms of the arithmetic of the underlying quadratic field  $K$ ) for the existence of a fundamental unit of norm  $-1$  are unknown (see [8]). This indicates the difficulty of solving (\*) for the simplest even degree case. In this regard one may ask whether (\*) is equivalent to such a norm statement for other fields. In a recent letter to the authors, V. Ennola answered (\*) for cyclic cubic fields  $K$  as follows: Let  $\varepsilon$  be a norm positive unit of  $K$  such that  $-1$  and the conjugates of  $\varepsilon$  generate the unit group. Then  $\dim_2(\bar{U}_K^+) = 0$  if and only if  $\varepsilon$  is *not* totally positive. However, as with

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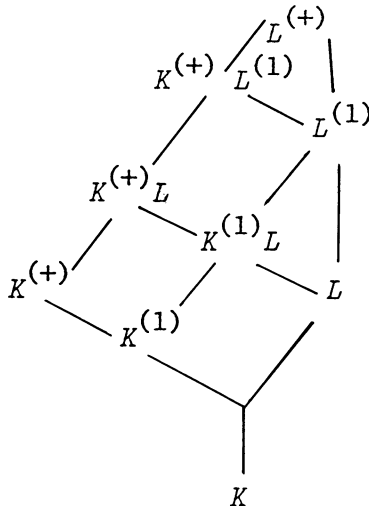
the quadratic field case, the latter does not readily translate into arithmetic conditions on the underlying cyclic cubic field  $K$ . We have not been able to verify that such a norm condition holds for a larger class of fields. For example, it would be interesting to investigate this question for quartic fields. However, we do have the following result which gives an upper bound on  $\bar{U}_K^+$  in terms of  $\bar{U}_L^+$  for a totally real extension field  $L$  of  $K$ . For example, this will allow us to translate the norm criterion from a quadratic field to any of its totally real number field extensions. We note that the following generalizes [3, Theorem 2, p. 168].

**THEOREM 2.1.** *Suppose  $Q \subseteq K \subseteq L$  with  $L$  totally real and finite over  $Q$ . Then  $\dim_2 \bar{U}_K^+ \leq \dim_2 \bar{U}_L^+$ .*

**PROOF.** First we show that  $K^{(1)} \subseteq L^{(1)}$  and  $K^{(+)} \subseteq L^{(+)}$ . Both  $K^{(1)}L/L$  and  $K^{(+)}L/L$  are Galois extensions with abelian Galois groups, since  $\text{Gal}(K^{(1)}L/L) \cong \text{Gal}(K^{(1)}/(K^{(1)} \cap L))$ , which is a subgroup of the class group  $C_K = \text{Gal}(K^{(1)}/K)$ , and  $\text{Gal}(K^{(+)}L/L) \cong \text{Gal}(K^{(+)}/K^{(+)} \cap L)$ , which is a subgroup of the narrow class group  $C_K^{(+)} = \text{Gal}(K^{(+)}/K)$ . By [1] we have that all  $L$ -primes are unramified in  $K^{(1)}L$  and all finite  $L$ -primes are unramified in  $K^{(+)}L$ . It follows that  $K^{(1)} \subseteq K^{(1)}L \subseteq L^{(1)}$  and  $K^{(+)} \subseteq K^{(+)}L \subseteq L^{(+)}$ .

We show next that  $K^{(+)} \cap L^{(1)} = K^{(1)}$ . The inclusion  $K^{(1)} \subseteq K^{(+)} \cap L^{(1)}$  is clear. Since  $K \subseteq K^{(+)} \cap L^{(1)} \subseteq K^{(+)}$ , we see that  $K^{(+)} \cap L^{(1)}$  is an abelian extension of  $K$  and that every finite  $K$ -prime is unramified in  $K^{(+)} \cap L^{(1)}$ . Moreover, since  $L$  is totally real, then  $L^{(1)}$  is totally real, and since  $K^{(+)} \cap L^{(1)} \subseteq L^{(1)}$ , then all infinite  $K^{(+)} \cap L^{(1)}$ -primes are real. Hence  $K^{(+)} \cap L^{(1)} \subseteq K^{(1)}$ , and we have shown that  $K^{(+)} \cap L^{(1)} = K^{(1)}$ .

The following diagram illustrates our situation.



Since  $K^{(+)} \cap L^{(1)} = K^{(1)}$  and  $K^{(+)}/K^{(1)}$  is Galois, the extensions  $K^{(+)}/K^{(1)}$  and  $L^{(1)}/K^{(1)}$  are linearly disjoint, and  $|K^{(+)L^{(1)}}:L^{(1)}| = |K^{(+)}:K^{(1)}|$ . Thus  $2^{\dim_2 \bar{U}_K^+} = |K^{(+)}:K^{(1)}| = |K^{(+)L^{(1)}}:L^{(1)}|$  divides  $|L^{(+)L^{(1)}}:L^{(1)}| = 2^{\dim_2 \bar{U}_L^+}$ , and so  $\dim_2 \bar{U}_K^+ \leq \dim_2 \bar{U}_L^+$ . Q.E.D.

With  $L$  and  $K$  as in Theorem 2.1 we have the following

**COROLLARY 2.2.** *Let  $p$  be a prime and let  $K$  be a subfield of  $L = Q(\zeta_p + \zeta_p^{-1})$ . If the class number of  $Q(\zeta_p)$  is odd, then  $\dim_2 \bar{U}_K^+ = \dim_2 \bar{U}_L^+ = 0$ .*

**PROOF.** A classical result of Kummer (for example see [8, p. 128]) is that  $\dim_2 \bar{U}_L^+ = 0$  whenever the class number of  $Q(\zeta_p)$  is odd. Therefore the result now follows from Theorem 2.1. Q.E.D.

It is worth noting at this juncture that as a result of recent work of Shimura [10] Kummer's classical result cited in the proof of Corollary 2.2 is now extended to  $\dim_2(\bar{U}_L^+) = 0$  if and only if the class number of  $Q(\zeta_p)$  is odd. Here  $L = Q(\zeta_p + \zeta_p^{-1})$  as in Corollary 2.2.

Furthermore, by [3, p. 175], if  $L = Q(\zeta_f + \zeta_f^{-1})$ , where  $f$  is composite, then  $\dim_2 \bar{U}_L^+ > 0$ , and  $N(U_L) = \{1\}$ .

An application of Theorem 2.1 is the following

**EXAMPLE 2.3.** Let  $K = Q(\alpha)$  where the minimal polynomial of  $\alpha$  over  $Q$  is  $x^3 - x^2 - 234x + 729$ . Then  $K$  has conductor 703; i.e., the minimal cyclotomic field containing  $K$  is  $Q(\zeta_{703})$ , so  $K \subseteq L = Q(\zeta_{703} + \zeta_{703}^{-1})$ . It can be shown that  $\dim_2 \bar{U}_K^+ = 2$ . Thus by Theorem 2.1  $\dim_2 \bar{U}_L^+ \geq 2$ . For a list of such examples of cyclic cubic fields the reader may consult [2].

Next we prove

**PROPOSITION 2.4.** *Let  $K$  be a totally real algebraic number field. Then  $\dim_2 \bar{U}_K^+ \leq \dim_2 C_K^{(+)}$ .*

**PROOF.** From [6, Theorem 3.1, p. 203] we have that

$$\dim_2(\bar{U}_K^+) = \dim_2 \text{Gal}(K^{(+)} / K^{(1)}) \leq \dim_2 \text{Gal}(K^{(+)} / K) = \dim_2(C_K^+). \quad \text{Q.E.D.}$$

We note that, in general, it is not correct to claim that  $\dim_2 \bar{U}_K^+ \leq \dim_2 C_K$ . For example, if  $K = Q(\sqrt{3})$ , then  $\dim_2 C_K = 0$ . However,  $K(\sqrt{-1})$  is unramified except at the real primes of  $K$ , and in fact  $K^{(+)} = K(\sqrt{-1})$  [6, Theorem 3.10, p. 210]. Thus  $\dim_2(\bar{U}_K^+) = \dim_2 C_K^{(+)} = 1$ . However, under more restrictive hypotheses it is possible to achieve  $\dim_2 \bar{U}_K^+ \leq \dim_2 C_K$ .

**THEOREM 2.5 (ORLAT [9]).** *Let  $K$  be a finite real Galois extension of  $Q$  with Galois group of odd exponent  $n$  and suppose that  $-1$  is congruent to a power of 2 modulo  $n$ . Then  $\dim_2 \bar{U}_K^+ \leq \dim_2 C_K$ . Moreover  $\dim_2 C_K^{(+)} = \dim_2 C_K$ .*

In particular, when  $h_K$  is odd, we have  $\dim_2 \bar{U}_K^+ = 0$ . (See [5] for an independent proof of this fact, distinct from that of [9]. The authors of [5] were unaware of the existence of [9] at the time of that writing.)

In the presence of Proposition 2.4,  $\dim_2 \bar{U}_K^+ \leq \dim_2 C_K$  is of course an immediate consequence of  $\dim_2 C_K = \dim_2 C_K^{(+)}$ .

We now exhibit a proof of Theorem 2.5 in the case where  $K$  is abelian. This is a simple proof based on the self-duality results established in [11]. Before so doing we need to set the stage with some additional notation and concepts.

Let  $O_K$  denote the ring of integers of an algebraic number field  $K$ . We define  $\alpha \in O_K$  to be *singular* if the ideal  $(\alpha)$  is the square of an ideal. Moreover  $\alpha$  is called *odd* if  $(\alpha)$  is relatively prime to  $(2)$ . Furthermore,  $\alpha$  is called *primary* if it is odd and  $\alpha \equiv \zeta^2 \pmod{4}$  for some  $\zeta \in O_K$ . (For details and further development on these concepts we refer the reader to Hecke [4, pp. 217–237].) We denote by  $O_K^0$  the singular primary numbers and note that by [4, Theorem 120, p. 137] we have  $\mu \in O_K^0$  with  $\mu \notin O_K^2$  if and only if  $K \subsetneq K(\sqrt{\mu}) \subseteq K_2^{(+)}$ . It follows that  $|K_2^{(+)}:K| = |\overline{O}_K^0|$ , where  $K_2^{(+)}$  is the maximal abelian extension of  $K$  unramified at the finite primes such that  $\text{Gal}(K_2^{(+)}/K)$  is a direct sum of copies of  $\mathbf{Z}/2\mathbf{Z}$ . Thus by Kummer theory we have that  $K_2^{(+)}$  is the maximal subfield of  $K^{(+)}$  of the form  $K(\sqrt{\mu_1}, \dots, \sqrt{\mu_r})$  for some  $\mu_i \in K$ ,  $i = 1, 2, \dots, r$ . Similarly, define  $K_2^{(1)}$  as the maximal abelian extension of  $K$  unramified at *all* primes with  $\text{Gal}(K_2^{(1)}/K)$  being a direct sum of copies of  $\mathbf{Z}/2\mathbf{Z}$ . Therefore  $K_2^{(1)}$  is the maximal subfield of  $K^{(1)}$  of the form  $K(\sqrt{\alpha_1}, \dots, \sqrt{\alpha_s})$  for some  $\alpha_i \in K$ ,  $i = 1, 2, \dots, s$ . Furthermore, we note that  $K_2^{(1)} = K_2^{(+)} \cap K^{(1)}$ .

Now if  $O_K^+$  denotes the subgroup of  $O_K$  consisting of totally positive integers, then  $\mu \in O_K^+ \cap O_K^0$  with  $\mu \notin O_K^2$  if and only if  $K \subsetneq K(\sqrt{\mu}) \subseteq K_2^{(1)}$ , from which it follows that  $|K_2^{(1)}:K| = |\overline{O}_K^+|$ .

Now we are in a position to prove Theorem 2.5 under the assumption that  $K$  is a finite real extension of  $\mathbf{Q}$  with abelian Galois group  $G$ .

PROOF. Let  $M$  be a simple  $F_2G$ -module where  $F_2$  is the field of two elements. Then by Schur's lemma  $M \cong F_2Ge$  for some idempotent  $e$ . Now let  $\psi$  be the standard involution of  $F_2G$  given by  $\psi(g) = g^{-1}$  for all  $g \in G$ . Then by the same argument as in the proof of [5, Theorem, p. 615] we have that  $F_2Ge \cong F_2G\psi(e)$ , resulting from  $-1$  being a power of 2 modulo  $n$ . Hence we have shown that all simple  $F_2G$ -modules are self-dual. Therefore from [11, Corollary 1, p. 157] we have that  $\dim_2 \overline{U}_K^+ \leq \dim_2 C_K$ .

By the self-duality established above, we may use exactly the same reasoning as used by Taylor on  $\overline{U}_K^+$  and  $\overline{U}_K^0$  in [11, (\*), p. 157] to establish  $\overline{O}_K^+ \cong \overline{O}_K^0$ . Hence by the discussion preceding the proof, we have  $K_2^{(1)} = K_2^{(+)}$ ; i.e.,  $\dim_2 C_K = \dim_2 C_K^{(+)}$ . Q.E.D.

In what follows, the *signature map* from  $U_K$  to  $F_2G$  is defined by  $\text{sgn}(u) = \sum_{\sigma \in G} s(u^\sigma)$ , where  $s$  is called the *signature* of  $u$  with  $s: K^* \rightarrow F_2$  defined by  $s(k) = 0$  if  $k > 0$  and  $s(k) = 1$  if  $k < 0$ .

It is interesting to note that Lagarias [7] has proved the equivalence of

$$(2.7) \dim_2 C_K = \dim_2 C_K^{(+)}$$

(2.8) All odd singular integers  $\alpha$  have their signature type determined by the congruence class of  $\alpha$  modulo 4.

(2.9) There are  $\alpha$  of all signature types with  $\alpha$  an odd singular integer.

(2.10)  $K_2^{(+)}$  is totally real.

Thus, it would be of interest to investigate those totally real algebraic number fields  $K$  for which the Sylow 2-subgroup of  $\text{Gal}(K^{(+)}/K)$  is elementary abelian. It is not enough to know that the Sylow 2-subgroup of  $\text{Gal}(K^{(1)}/K)$  is elementary abelian. For example in [2] we see that “most” of the Sylow 2-subgroups of  $\text{Gal}(K^{(1)}/K)$  are elementary abelian where  $K$  is a cyclic cubic field. However, there

are no instances known to the authors in the cyclic cubic case where  $\text{Gal}(K^{(+)} / K)$  has elementary abelian Sylow 2-subgroup.

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