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POLYNOMIALS OF PELLIAN TYPE AND CONTINUED FRACTIONS

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Abstract. We investigate infinite families of integral quadratic polynomials $\{f_k(X)\}_{k\in\mathbb{N}}$ and show that, for a fixed $k \in \mathbb{N}$ and arbitrary $X \in \mathbb{N}$, the period length of the simple continued fraction expansion of $\sqrt{f_k(X)}$ is constant. Furthermore, we show that the period lengths of $\sqrt{f_k(X)}$ go to infinity with k. For each member of the families involved, we show how to determine, in an easy fashion, the fundamental unit of the underlying quadratic field. We also demonstrate how the simple continued fraction expansion of $\sqrt{f_k(X)}$ is related to that of $\sqrt{C}$, where $f_k(X) = a_kX^2+b_kX+C$. This continues work in [1]–[4].

1. Introduction. In [4], we described the background to the study of continued fraction expansions of $\sqrt{f(X)}$ where $f(X)$ is a polynomial, especially quadratic, which we study herein. This includes the pioneering efforts of Stern

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[9] in the late nineteenth century, as well as the seminal work of Schinzel [5]–[6], and work on the determination of fundamental units by Stender [8] in the late twentieth century. The basic idea, for quadratic polynomials $f(X)$ is to look at, on the one hand, the period length $\ell(\sqrt{f(X)})$ of the simple continued fraction expansion expansion of $\sqrt{f(X)}$ as $X \to \infty$, and to provide families of such polynomials $\{f_k(X)\}_{k\in\mathbb{N}}$ such that $\ell\left(\sqrt{f_k(X)}\right)$ is some constant $N \in \mathbb{N}$ for all $X \in \mathbb{N}$; and on the other hand, to demonstrate that $\lim_{k\to\infty} \ell(\sqrt{f_k(X)}) = \infty$. Moreover, we are able to obtain, in an explicit fashion, the fundamental unit ε_f of the quadratic order $\mathcal{O}_f = \mathbb{Z}\left[\sqrt{f(X)}\right]$ for all $k, X \in \mathbb{N}$. For these families ε_{f_k} are particularly small, since $\ell(\sqrt{f_k(X)})$ is independent of X, and this means that, for reasons discussed in [4], the class number h_{f_k} of $\mathbb{Z}\left[\sqrt{f_k(X)}\right]$ is particularly large. With numerous examples, we illustrate that $\lim_{k\to\infty} h_{f_k} = \infty$.

The means by which we achieve the above is somewhat ironic given that we employ the well-studied theory behind *Extended Richaud-Degert type* (ERD)-radicands, those of the form $D = a^2 + r > 0$ where $r \mid 4a$ for $a, r \in \mathbb{Z}$ (see [2]). The irony comes from the fact, explained in [4], that polynomials $f(X)$ of ERD-type satisfy $\ell(\sqrt{f(X)}) \leq 12$, so are ostensibly of little interest in the aforementioned scenario. However, in [4], we observed that, for the quadratic polynomials $f_k(X) = A_k^2X^2 + 2B_kX + C$ investigated therein, $A_k^2 f_k(X) = (A_k^2X + B_k)^2 - 1$, which is of ERD-type. From the well-studied ERD-theory, we know that $\varepsilon_{4A_k^2 f_k(X)} = A_k^2X + B_k + \sqrt{A_k^2 f_k(X)}$. In [4], we proved that $\varepsilon_{4A_k^2 f_k(X)} = \varepsilon_{4f_k(X)}$, and in the process developed for so doing, were able to explicitly determine the simple continued fraction expansion of $\sqrt{f_k(X)}$ in terms of that for $\sqrt{C}$.

In this paper, we continue down the path established in [4] with a much more sweeping set of results (see Theorems 4.1–4.2 below) that employ the above techniques, in a different direction than that elucidated in [4]. As a consequence, we are able to get extremely simple proofs of some classical results (see Corollaries 4.1–4.2 below), and we provide numerous informative examples to illustrate the theory.

2. Notation and preliminaries. The background for the following together with proofs and details may be found in [2]. Let $\Delta = d^2D_0$ ($d \in \mathbb{N}$, $D_0 > 1$ squarefree) be the discriminant of a real quadratic order $\mathcal{O}_\Delta = \mathbb{Z} + \mathbb{Z}[\sqrt{\Delta}] = [1, \sqrt{\Delta}]$ in $\mathbb{Q}(\sqrt{\Delta})$, U_Δ the group of units of $\mathcal{O}_\Delta$, and ε_Δ the fundamental unit of $\mathcal{O}_\Delta$.

Now we introduce the notation for continued fractions. Let $\alpha \in \mathcal{O}_\Delta$. We denote the simple continued fraction expansion of α (in terms of its *partial* *quotients*) by:

$$\alpha = \langle q_0; q_1, \ldots, q_n, \ldots \rangle .$$

If α is *periodic*, we use the notation:

$$\alpha = \langle q_0; q_1 . q_2 . \ldots, q_{k-1}; \overline{q_k, q_{k+1}, \ldots, q_{\ell+k-1}} \rangle ,$$

to denote the fact that $q_n = q_{n+\ell}$ for all $n \geq k$. The smallest such $\ell = \ell(\alpha) \in \mathbb{N}$ is called the *period length* of α and $q_0, q_1, \ldots, q_{k-1}$ is called the *pre-period* of α. If $k = 0$ is the least such nonnegative value, then α is *purely periodic*, namely,

$$\alpha = \langle \overline{q_0; q_1, \ldots, q_{\ell-1}} \rangle .$$

The *convergents* (for $n \geq 0$) of α are denoted by

$$\frac{x_n}{y_n} = \langle q_0; q_1, \ldots, q_n \rangle = \frac{q_n x_{n-1} + x_{n-2}}{q_n y_{n-1} + y_{n-2}}. \tag{2.1}$$

We will need the following facts, the proofs of which can be found in most standard undergraduate number theory texts (for example see [3], and see [2] for a more advanced exposition).

$$x_j = q_j x_{j-1} + x_{j-2} \qquad \text{(for } j \geq 0 \text{ with } x_{-2} = 0\text{, and } x_{-1} = 1\text{)}, \tag{2.2}$$

$$y_j = q_j y_{j-1} + y_{j-2} \qquad \text{(for } j \geq 0 \text{ with } y_{-2} = 1\text{, and } y_{-1} = 0\text{)}, \tag{2.3}$$

$$x_j y_{j-1} - x_{j-1} y_j = (-1)^{j-1} \qquad (j \in \mathbb{N}), \tag{2.4}$$

$$\langle q_j, q_{j-1}, \ldots, q_1 \rangle = y_j / y_{j-1} \qquad (j \in \mathbb{N}), \tag{2.5}$$

$$\langle q_j, q_{j-1}, \ldots, q_1, q_0 \rangle = x_j / x_{j-1} \qquad (j \in \mathbb{N}), \tag{2.6}$$

In particular, we will be dealing with $\alpha = \sqrt{D}$ where D is a radicand. In this case, the *complete quotients* are given by $(P_j + \sqrt{D})/Q_j$ where the P_j and Q_j are given by the recursive formulae as follows for any $j \geq 0$ (with $P_0 = 0$ and $Q_0 = 1$):

$$q_j = \left\lfloor \frac{P_j + \sqrt{D}}{Q_j} \right\rfloor , \tag{2.7}$$

$$P_{j+1} = q_j Q_j - P_j , \tag{2.8}$$

and

$$D = P_{j+1}^2 + Q_j Q_{j+1}. \tag{2.9}$$

Thus, we may write:

$$\sqrt{D} = \left\langle q_0; q_1, \dots, q_n, (P_{n+1} + \sqrt{D})/Q_{n+1} \right\rangle. \tag{2.10}$$

We will also need he following facts for $\alpha = \sqrt{D}$. For any integer $j \geq 0$, and $\ell = \ell(\sqrt{D})$:

$$\sqrt{D} = \langle q_0; \overline{q_1, \dots, q_{\ell-1}, 2q_0} \rangle, \tag{2.11}$$

$$\text{where } q_j = q_{\ell-j} \text{ for } j = 1, 2, \dots, \ell - 1, \text{ and } q_0 = \lfloor \sqrt{D} \rfloor \tag{2.12}$$

$$x_{j\ell-1} = q_0 y_{j\ell-1} + y_{j\ell-2}, \tag{2.13}$$

$$D y_{j\ell-1} = q_0 x_{j\ell-1} + x_{j\ell-2}. \tag{2.14}$$

Also, for any $j \in \mathbb{N}$

$$x_j x_{j-1} - D y_j y_{j-1} = (-1)^j P_{j+1}, \tag{2.15}$$

$$P_1 = P_{j\ell} = q_0 \qquad \text{and} \qquad Q_0 = Q_{j\ell} = 1, \tag{2.16}$$

$$x_{j-1}^2 - y_{j-1}^2 D = (-1)^j Q_j, \tag{2.17}$$

$$D y_{j\ell-1} = q_0 x_{k\ell-1} + x_{j\ell-2}, \tag{2.18}$$

$$q_j < 2q_0. \tag{2.19}$$

When ℓ is even,

$$P_{\ell/2} = P_{\ell/2+1} = P_{(2j-1)\ell/2+1} = P_{(2j-1)\ell/2} \text{ and } Q_{\ell/2} = Q_{(2j-1)\ell/2}, \tag{2.20}$$

whereas when ℓ is odd,

$$Q_{(\ell-1)/2} = Q_{(\ell+1)/2}. \tag{2.21}$$

Lastly, the following result on Pell's equation will be quite useful in establishing results in the next section. In fact, this gives more detail to the fact exhibited in (2.17) in the case where the index is a multiple of ℓ.

Theorem 2.1. *For $j \geq 0$, let x_j and y_j be as above in the simple continued fraction expansion of $\sqrt{D}$ for a nonsquare $D > 0$ and let $\ell = \ell(\sqrt{D})$. If ℓ is even, then all positive integer solutions of*

$$x^2 - Dy^2 = 1 \tag{2.22}$$

are given by $(x, y) = (x_{j\ell-1}, y_{j\ell-1})$ for $j \geq 1$, whereas there are no integer solutions of

$$x^2 - Dy^2 = -1. \tag{2.23}$$

If ℓ is odd, then all positive solutions of (2.22) are given by

$$(x, y) = (x_{2j\ell-1}, y_{2j\ell-1})$$

for $j \geq 1$, whereas all positive solutions of (2.23) are given by

$$(x, y) = (x_{(2j-1)\ell-1}, y_{(2j-1)\ell-1}).$$

Proof. See [3, Corollary 5.3.3, p. 249]. $\square$

3. Preliminary lemmas. In this section, we prove a sequence of lemmas which are needed for the main results in the next section. *For the balance of the paper, we make the following assumptions.*

We let $A, B, C, k, X \in \mathbb{N}$ with C not a perfect square. Suppose that $(x, y) = (B, A)$ is the smallest positive solution of $x^2 - Cy^2 = 1$ and define, for each $k \in \mathbb{N}$,

$$B_k + A_k\sqrt{C} = (B + A\sqrt{C})^k.$$

Also, for $n \geq 0$, set

$$\sqrt{C} = \left\langle c_0; \overline{c_1, \ldots, c_n, 2c_0} \right\rangle,$$

where this is understood to mean $\sqrt{C} = \left\langle c_0; \overline{2c_0} \right\rangle$ in the case where $n = 0$.

Lemma 3.1. *For $n \in \mathbb{N}$ odd and $k = (2m+1)$ where m is a nonnegative integer:*

$$B_k = x_{(2m+1)(n+1)/2}y_{(2m+1)(n+1)/2-1} + x_{(2m+1)(n+1)/2-1}y_{(2m+1)(n+1)/2-2}$$

and

$$A_k = y_{(2m+1)(n+1)/2-1}(y_{(2m+1)(n+1)/2} + y_{(2m+1)(n+1)/2-2}).$$

Proof. For convenience, set $N = (2m+1)(n+1)$. By (2.1), (2.5), and (2.11)–(2.12):

$$\frac{x_{N-1}}{y_{N-1}} = \langle c_0; c_1, \ldots, c_{N-1} \rangle = \left\langle c_0; c_1, \ldots, c_{N/2-1}, \frac{y_{N/2}}{y_{N/2-1}} \right\rangle =$$

$$\frac{\left(\frac{y_{N/2}}{y_{N/2-1}} x_{N/2-1} + x_{N/2-2}\right)}{\left(\frac{y_{N/2}}{y_{N/2-1}} y_{N/2-1} + y_{N/2-2}\right)} = \frac{y_{N/2}x_{N/2-1} + x_{N/2-2}y_{N/2-1}}{y_{N/2-1}(y_{N/2} + y_{N/2-2})}.$$

Thus, $y_{N-1} = y_{N/2-1}(y_{N/2} + y_{N/2-2})$ and $x_{N-1} = y_{N/2}x_{N/2-1} + x_{N/2-2}y_{N/2-1}$, which by (2.4) equals,

$$x_{N/2}y_{N/2-1} + (-1)^{N/2} + x_{N/2-1}y_{N/2-2} + (-1)^{N/2-1} = x_{N/2}y_{N/2-1} + x_{N/2-1}y_{N/2-2}.$$

Since n is odd, then $B_k = x_{N-1}$ and $A_k = y_{N-1}$ by Theorem 2.1, so we have secured the result. □

Lemma 3.2. *If n is odd and $k = 2m+1$ with m a nonnegative integer, then*

$$2x_{(2m+1)(n+1)/2-1} = Q_{(n+1)/2}(y_{(2m+1)(n+1)/2} + y_{(2m+1)(n+1)/2-1}).$$

Proof. Let N be as in the proof of Lemma 3.1. Then by (2.17) and (2.20),

$$x^2_{N/2-1} - Cy^2_{N/2-1} = (-1)^{(n+1)/2}Q_{(n+1)/2}, \tag{3.24}$$

and by (2.8), (2.15) and (2.20),

$$x_{N/2}x_{N/2-1} - Cy_{N/2}y_{N/2-1} = (-1)^{(n+1)/2}P_{(n+3)/2} = c_{(n+1)/2}Q_{(n+1)/2}/2. \tag{3.25}$$

Therefore, by (3.24)–(3.25),

$$C = \frac{x^2_{N/2-1} + (-1)^{(n+3)/2}Q_{(n+1)/2}}{y^2_{N/2-1}} = \frac{2x_{N/2}x_{N/2-1} + (-1)^{(n+3)/2}c_{(n+1)/2}Q_{(n+1)/2}}{2y_{N/2}y_{N/2-1}}$$

from which it follows that

$$2x^2_{N/2-1}y_{N/2} + 2(-1)^{(n+3)/2}Q_{(n+1)/2}y_{N/2} =$$

$$2x_{N/2}x_{N/2-1}y_{N/2-1} + (-1)^{(n+3)/2}c_{(n+1)/2}Q_{(n+1)/2}y_{N/2-1} =$$

$$2x_{N/2-1}(x_{N/2-1}y_{N/2} + (-1)^{(n+3)/2}) + (-1)^{(n+3)/2}c_{(n+1)/2}Q_{(n+1)/2}y_{N/2-1},$$

where the last equality follows from (2.4). Thus,

$$2(-1)^{(n+3)/2}Q_{(n+1)/2}y_{N/2} = 2x_{N/2-1}(-1)^{(n+3)/2} + (-1)^{(n+3)/2}c_{(n+1)/2}Q_{(n+1)/2}y_{N/2-1},$$

so

$$Q_{(n+1)/2}y_{N/2} + Q_{(n+1)/2}(c_{(n+1)/2}y_{N/2-1} + y_{N/2-2}) = 2x_{N/2-1} + c_{(n+1)/2}Q_{(n+1)/2}y_{N/2-1},$$

by (2.3) and (2.20). Hence,

$$2x_{N/2-1} = Q_{(n+1)/2}(y_{N/2} + y_{N/2-2}),$$

as required. □

Lemma 3.3. *For $n \in \mathbb{N}$ odd, $k = 2m+1$, and $m \geq 0$,*

$$Q_{(n+1)/2}B_k = x^2_{(2m+1)(n+1)/2-1} + y^2_{(2m+1)(n+1)/2-1}C$$

and

$$Q_{(n+1)/2}A_k = 2x_{(2m+1)(n+1)/2-1}y_{(2m+1)(n+1)/2-1}.$$

Proof. Let N be as the proof of Lemma 3.1. Then by that lemma,

$$Q_{(n+1)/2}B_k = Q_{(n+1)/2}(x_{N/2}y_{N/2-1} + x_{N/2-1}y_{N/2-2}) =$$

$$Q_{(n+1)/2}(x_{N/2-1}y_{N/2} + (-1)^{N/2-1} + x_{N/2-1}y_{N/2-2})$$

where the last equality follows from (2.4), and by Lemma 3.2, this equals,

$$x_{N/2-1}(Q_{(n+1)/2}(y_{N/2} + y_{N/2-2})) + (-1)^{N/2-1}Q_{(n+1)/2} =$$

$$2x^2_{N/2-1} + (-1)^{N/2-1}Q_{(n+1)/2} = x^2_{N/2-1} + Cy^2_{N/2-1}$$

where the last equality follows from (2.17). This yields the first result. For the second result, we invoke Lemmas 3.1–3.2 which tell us that

$$Q_{(n+1)/2}A_k = Q_{(n+1)/2}y_{N/2-1}(y_{N/2} + y_{N/2-2}) = 2x_{N/2-1}y_{N/2-1},$$

and the entire result is proved. □

Lemma 3.4. *For $n \in \mathbb{N}$ odd, $k = 2m+1$, and $m \geq 0$,*

$$A_kx_{(2m+1)(n+1)/2-1} - (B_k \pm 1)y_{(2m+1)(n+1)/2-1} = y_{(2m+1)(n+1)/2-1}((-1)^{(n+1)/2} \mp 1).$$

P r o o f. Let N be as in the proof of Lemma 3.1. Then by Lemma 3.3,

$$A_k x_{N/2-1} - (B_k \pm 1) y_{N/2-1} =$$

$$\frac{2x^2_{N/2-1} y_{N/2-1} - (x^2_{N/2-1} + C y^2_{N/2-1} \pm Q_{(n+1)/2}) y_{N/2-1}}{Q_{(n+1)/2}} =$$

$$\frac{x^2_{N/2-1} y_{N/2-1} - C y^3_{N/2-1} \mp Q_{(n+1)/2} y_{N/2-1}}{Q_{(n+1)/2}} =$$

$$\frac{y_{N/2-1}\left[(x^2_{N/2-1} - C y^2_{N/2-1}) \mp Q_{(n+1)/2}\right]}{Q_{(n+1)/2}} =$$

$$\frac{y_{N/2-1}\left[(-1)^{(n+1)/2} Q_{(n+1)/2} \mp Q_{(n+1)/2}\right]}{Q_{(n+1)/2}} = y_{N/2-1}((-1)^{(n+1)/2} \mp 1)$$

where the penultimate equality comes from (2.17). □

Lemma 3.5. *If $n \in \mathbb{N}$ is odd, and $m \geq 0$, then*

$$B_k + (-1)^{(n+3)/2} = y_{(2m+1)(n+1)/2-1}(x_{(2m+1)(n+1)/2} + x_{(2m+1)(n+1)/2-2})$$

P r o o f. Let N be as in the proof of Lemma 3.1. By that lemma and (2.4) we have, $y_{N/2-1} x_{N/2} + x_{N/2-2} y_{N/2-1} = y_{N/2-1} x_{N/2} + x_{N/2-1} y_{N/2-2} - x_{N/2-1} y_{N/2-2} + x_{N/2-2} y_{N/2-1} = B_k + (-1)^{(n+3)/2}$. □

4. Main results. Assuming $m \geq 0$, in the following, we set:

$$w_m = c_1, \dots, c_n, 2c_0, c_1, \dots, c_n, 2c_0, \dots, c_1, \dots, c_n,$$

which is m iterations of $c_1, \dots, c_n, 2c_0$ followed by one iteration of $c_1, \dots, c_n$. In the case where $n = 0$, w_m is just m iterations of $2c_0$, and when $m = n = 0$, w_m is the empty string. Also, for odd $n > 1$,

$$\vec{v_n} = c_1, c_2, \dots, c_n, 2c_0, c_1, c_2, \dots, c_n, 2c_0, \dots, c_1, \dots, 2c_0, c_1, c_2, \dots, c_{(n-1)/2},$$

which means m iterations of $c_1, c_2, \dots, c_n, 2c_0$ followed by one iteration of

$c_1, c_2, \ldots, c_{(n-1)/2}$, and the reverse of this is denoted by

$$\overleftarrow{v}_m = c_{(n-1)/2}, \ldots, c_2, c_1, 2c_0, c_n, \ldots, c_1, \ldots, 2c_0, c_n, \ldots, c_1,$$

one iteration of $c_{(n-1)/2}, c_n, \ldots, c_1$ followed by m iterations of $2c_0, c_n, \ldots, c_1$. Lastly, the symbols Q_j, defined in formulas (2.7)–(2.9), refer to the continued fraction expansion of $\sqrt{C}$.

Theorem 4.1. *Let*

$$D_k(X) = (B_k - 1)^2 A_k^2 X^2 + 2(B_k - 1)^2 X + C.$$

Then the fundamental solution of

$$x^2 - D_k(X) y^2 = 1$$

is

$$(x, y) = ((B_k - 1)(A_k^2 X + 1)^2 + 1, A_k^3 X + A_k),$$

and, for

$$q_0 = (B_k - 1) A_k X + c_0 :$$

(a) *If both $n, k \in \mathbb{N}$ are even, then $C \mid (B_k - 1)$ and:*

$$\sqrt{D_k(X)} = \left\langle q_0; \overline{w_{k-1}, c_0, 2(B_k - 1) A_k X / C, c_0, w_{k-1}, 2q_0} \right\rangle,$$

with $\ell\left(\sqrt{D_k(X)}\right) = 2k(n+1) + 2.$

(b) *If $n \geq 0$ is even and k is odd, then*

$$\sqrt{D_k(X)} = \langle q_0; \overline{w_{k-1}, 2q_0} \rangle$$

and $\ell\left(\sqrt{D_k(X)}\right) = k(n+1).$

(c) *If n is odd, then one of the following holds.*

(i) *If $k = 2m + 1$, $m \geq 0$, and $(n+1)/2 > 1$ is odd, then $Q_{(n+1)/2}$ divides $(B_k - 1)$ and:*

$$\sqrt{D_k(X)} = \left\langle q_0; \overline{\overrightarrow{v}_m, 2(B_k - 1) A_k X / Q_{(n+1)/2} + c_{(n+1)/2}, \overleftarrow{v}_m, 2q_0} \right\rangle$$

with $\ell\left(\sqrt{D_k(X)}\right) = k(n+1).$

(ii) *If* $k = 2m+1$, $m \ge 0$, $(n+1)/2$ *is even, and* $c_{(n+1)/2}$ *is even, then* C *divides* $Q_{(n+1)/2}(B_k - 1)$ *and:* $\sqrt{D_k(X)} =$

$$\left\langle q_0; \overline{\overrightarrow{v}_m, c_{(n+1)/2}/2, 2Q_{(n+1)/2}(B_k-1)A_kX/C, c_{(n+1)/2}/2, \overleftarrow{v}_m, 2q_0}\right\rangle$$

with $\ell\left(\sqrt{D_k(X)}\right) = k(n+1)+2$.

(iii) *If* $k = 2m$, $m \in \mathbb{N}$, *then* $C \mid (B_k - 1)$ *and:*

$$\sqrt{D_k(X)} = \left\langle q_0; \overline{w_{m-1}, c_0, 2(B_k-1)A_kX/C, c_0, w_{m-1}, 2q_0,}\right\rangle$$

with $\ell(\sqrt{D_k(X)}) = k(n+1)+2$.

Proof. First, we observe that

$$A_k^2 D_k(X) = (B_k - 1)^2(A_k^2X+1)^2 + 2(B_k - 1). \tag{4.26}$$

Since

$$((B_k-1)(A_k^2X+1)^2+1)^2 - (A_k^3X + A_k)^2 D_k(X) = 1,$$

$D_k(X)$ is not a perfect square. Thus, by [2, Theorem 3.2.1, p. 78],

$$\varepsilon_{4A_k^2D_k(X)} = (B_k-1)(A_k^2X+1)^2 + 1 + (A_k^2X+1)\sqrt{A_k^2D_k(X)}.$$

Let $\ell = \ell(\sqrt{D_k(X)})$ and X_i/Y_i be the i-th convergent of $\sqrt{D_k(X)}$. Then by Theorem 2.1, there is a $j \in \mathbb{N}$ such that

$$(B_k-1)(A_k^2X+1)^2 + 1 = X_{j\ell-1} \text{ and } A_k^3X + A_k = Y_{j\ell-1}.$$

First, we prove part (a) for which we now show that $j = 1$. Thus, we show via Theorem 2.1 that $\varepsilon_{4A_k^2D_k(X)} = \varepsilon_{4D_k(X)}$, since ℓ will be shown to be even. In the process of doing this, the continued fraction expansion in (a) will be shown to hold.

If x_i/y_i is the i-th convergent of $\sqrt{C}$, then by (2.1),

$$\langle q_0, w_{k-1}, c_0, 2(B_k-1)A_kX/C, c_0, w_{k-1}\rangle =$$

$$\langle q_0, w_{k-1}, c_0, 2(B_k-1)A_kX/C + y_{k(n+1)-1}/x_{k(n+1)-1}\rangle =$$

$$\left\langle q_0, w_{k-1}, \frac{2c_0(B_k-1)A_kXx_{k(n+1)-1} + Cc_0y_{k(n+1)-1} + Cx_{k(n+1)-1}}{2(B_k-1)A_kXx_{k(n+1)-1} + Cy_{k(n+1)-1}}\right\rangle. \tag{4.27}$$

If we set,

$$M = \frac{2c_0(B_k - 1)A_k X x_{k(n+1)-1} + Cc_0 y_{k(n+1)-1} + Cx_{k(n+1)-1}}{2(B_k - 1)A_k X x_{k(n+1)-1} + Cy_{k(n+1)-1}},$$

then (4.27) equals,

$$(B_k - 1)A_k X + \frac{Mx_{k(n+1)-1} + x_{k(n+1)-2}}{My_{k(n+1)-1} + y_{k(n+1)-2}},$$

which may be manipulated using (2.4), (2.13), and (2.17)–(2.18) to equal:

$$(B_k - 1)A_k X +$$

$$\frac{2x_{k(n+1)-1}y_{k(n+1)-1}C(B_k - 1)A_k X + C(2x^2_{k(n+1)-1} + (-1)^{k(n+1)-1})}{2Cy_{k(n+1)-1}x_{k(n+1)-1} + 2(B_k - 1)A_k X x^2_{k(n+1)-1}}. \tag{4.28}$$

However, since $n+1$ is odd, we must have that $B_k = x_{2k(n+1)-1}$ and $A_k = y_{2k(n+1)-1}$ by Theorem 2.1. Also since

$$B_k + A_k\sqrt{C} = (x_n + y_n\sqrt{C})^{2k} = (x_{k(n+1)-1} + y_{k(n+1)-1}\sqrt{C})^2 =$$

$$x^2_{k(n+1)-1} + y^2_{k(n+1)-1}C + 2x_{k(n+1)-1}y_{k(n+1)-1}\sqrt{C},$$

and in turn, by (2.17), this equals,

$$2x^2_{k(n+1)-1} - 1 + 2x_{k(n+1)-1}y_{k(n+1)-1}\sqrt{C},$$

so,

$$x_{2k(n+1)-1} = 2x^2_{k(n+1)-1} - 1 = B_k \quad \text{and} \quad y_{2k(n+1)-1} = 2x_{k(n+1)-1}y_{k(n+1)-1} = A_k. \tag{4.29}$$

Thus, (4.28) equals

$$(B_k - 1)A_k X + \frac{C(B_k - 1)A_k^2 X + CB_k}{CA_k + (B_k^2 - 1)A_k X} =$$

given that $(n+1)k$ is even by hypothesis, and since $B_k^2 - 1 = CA_k^2$ this equals

$$(B_k - 1)A_k X + \frac{C(B_k - 1)A_k^2 X + CB_k}{CA_k(A_k^2 X + 1)} =$$

$$\frac{(B_k-1)(A_k^2X+1)^2+1}{A_k(A_k^2X+1)}.$$

Since $2((B_k-1)A_kX+c_0)\neq c_j$ for any $0<j<\ell$ by (2.19), and Theorem 2.1 tells us that a convergent $X_{j\ell-1}/Y_{j\ell-1}$ can only occur at the end of the j-th period, then $j=1$. We have shown that

$$\langle (B_k-1)A_kX+c_0; w_{k-1}, c_0, 2(B_k-1)A_kX/C, c_0, w_{k-1}\rangle = \frac{X_{\ell-1}}{Y_{\ell-1}}.$$

Since this is the $(\ell-1)$-th convergent, then $\sqrt{D_k(X)}$ is as given in (a) and $\ell(\sqrt{D_k(X)})=2k(n+1)+2$.

One final note is in order. By (4.29) and Theorem 2.1,

$$B_k-1=2(x^2_{k(n+1)-1}-1)=2Cy^2_{k(n+1)-1},$$

so $C\mid(B_k-1)$, ensuring that $2(B_k-1)A_kX/C\in\mathbb{Z}$ in the simple continued fraction expansion of $\sqrt{D_k(X)}$, thereby establishing part (a).

For part (b), we show that $j=2$. We have,

$$\langle q_0, w_{k-1}\rangle=(B_k-1)A_kX+\frac{x_{k(n+1)-1}}{y_{k(n+1)-1}}=(B_k-1)A_kX+\frac{B_k-1}{A_k},$$

where the second equality follows from (4.29), and this equals,

$$\frac{(B_k-1)(A_k^2X+1)}{A_k}=\frac{X_{\ell-1}}{Y_{\ell-1}},$$

where

$$X_{\ell-1}=\frac{(B_k-1)(A_k^2X+1)}{2x_{k(n+1)-1}} \text{ and } Y_{\ell-1}=\frac{A_k}{2x_{k(n+1)-1}},$$

since

$$\gcd(B_k-1,A_k)=\gcd(2x^2_{k(n+1)-1},2x_{k(n+1)-1}y_{k(n+1)-1})=2x_{k(n+1)-1}$$

by (2.17). Thus, $\ell=k(n+1)$, which is odd since n is even and k is odd. Therefore, the fundamental solution of $X^2-D_k(X)Y^2=1$ comes from $(X_{\ell-1}-Y_{\ell-1}\sqrt{D_k(X)})^2=X_{2\ell-1}+Y_{2\ell-1}\sqrt{D_k(X)}=$

$$\frac{2(B_k-1)((B_k-1)(A_k^2X+1)^2+1)+2(B_k-1)(A_k^3X+A_k)\sqrt{D_k(X)}}{4x^2_{k(n+1)-1}}=$$

$$(B_k-1)(A_k^2X+1)^2+1+(A_k^3X+A_k)\sqrt{D_k(X)},$$

since $4x^2_{k(n+1)-1} = 2(B_k - 1)$. Hence, we have established part (b).

For the proof of case (c), we present only the salient features of the proofs that distinguish them from cases (a)–(b). The reader may fill in the details using the methodology presented in those cases. We now establish case (c), part (i). Let N be as in the proof of Lemma 3.1. Then,

$$\left\langle q_0; \vec{v}_m, 2(B_k - 1)A_k X/Q_{(n+1)/2} + c_{(n+1)/2}, \overleftarrow{v}_m \right\rangle =$$

$$\left\langle q_0; \vec{v}_m, 2(B_k - 1)A_k X/Q_{(n+1)/2} + y_{N/2}/y_{N/2-1} \right\rangle =$$

$$\left\langle q_0; \vec{v}_m, \frac{2(B_k - 1)A_k X y_{N/2-1} + Q_{(n+1)/2} y_{N/2}}{Q_{(n+1)/2} y_{N/2-1}} \right\rangle.$$

If we set

$$M = \frac{2(B_k - 1)A_k X y_{N/2-1} + Q_{(n+1)/2} y_{N/2}}{Q_{(n+1)/2} y_{N/2-1}},$$

then by (2.1) the above equals,

$$(B_k - 1)A_k X + \frac{M x_{N/2-1} + x_{N/2-2}}{M y_{N/2-1} + y_{N/2-2}} = (B_k - 1)A_k X +$$

$$\frac{2(B_k - 1)A_k X y_{N/2-1} x_{N/2-1} + (y_{N/2} x_{N/2-1} + x_{N/2-2} y_{N/2-1}) Q_{(n+1)/2}}{2(B_k - 1)A_k X y^2_{N/2-1} + (y_{N/2} y_{N/2-1} + y_{N/2-2} y_{N/2-1}) Q_{(n+1)/2}},$$

and by Lemmas 3.1–3.4 and (2.4) this equals,

$$(B_k - 1)A_k X + \frac{(B_k - 1)A_k^2 Q_{(n+1)/2} X + (x_{N/2} y_{N/2-1} + y_{N/2-2} x_{N/2-1}) Q_{(n+1)/2}}{2A_k^2 X x_{N/2-1} y_{N/2-1} + A_k Q_{(n+1)/2}},$$

so by Lemmas 3.1 and 3.3, this equals,

$$(B_k - 1)A_k X + \frac{(B_k - 1)A_k^2 Q_{(n+1)/2} X + B_k Q_{(n+1)/2}}{A_k^3 X Q_{(n+1)/2} + A_k Q_{(n+1)/2}} =$$

$$((B_k - 1)A_k^4 X^2 + (B_k - 1)A_k^2 X + (B_k - 1)A_k^2 X + B_k)/(A_k^3 X + A_k) =$$

$$((B_k - 1)(A_k^2 X + 1)^2 + 1)/(A_k^3 X + A_k).$$

Hence, as in cases (a)–(b), the result follows for $D_k(X)$. (Note, as well, that by Lemmas 3.2–3.4, $Q_{(n+1)/2} \mid (B_k - 1)$, so $2(B_k - 1)A_k X/Q_{(n+1)/2} \in \mathbb{N}$.)

We now establish c, part (ii) for $D_k(X)$. We have,

$$\left\langle q_0; \vec{v}_m, \frac{c_{(n+1)/2}}{2}, \frac{2(B_k-1)A_kXQ_{(n+1)/2}}{C}, \frac{c_{(n+1)/2}}{2}, \overleftarrow{v}_m \right\rangle =$$

$$\left\langle q_0; \vec{v}_m, \frac{c_{(n+1)/2}}{2}, \frac{2(B_k-1)A_kXQ_{(n+1)/2}}{C}, -\frac{c_{(n+1)/2}}{2} + \frac{y_{N/2}}{y_{N/2-1}} \right\rangle =$$

$$\left\langle q_0; \vec{v}_m, \frac{c_{(n+1)/2}}{2}, \frac{2(B_k-1)A_kXQ_{(n+1)/2}}{C}, \frac{2y_{N/2} - c_{(n+1)/2}y_{N/2-1}}{2y_{N/2-1}}) \right\rangle.$$

By Lemma 3.1, (2.3) and (2.20), this equals,

$$\left\langle q_0; \vec{v}_m, \frac{c_{(n+1)/2}}{2}, \frac{2(B_k-1)A_kXQ_{(n+1)/2}}{C} + \frac{2y_{N/2-1}^2}{A_k} \right\rangle =$$

$$\left\langle q_0; \vec{v}_m, \frac{(B_k-1)A_k^2XQ_{(n+1)/2}c_{(n+1)/2} + Cy_{N/2-1}^2c_{(n+1)/2} + 2A_kC}{2(B_k-1)A_k^2XQ_{(n+1)/2} + 2Cy_{N/2-1}^2} \right\rangle =$$

$$\left\langle q_0; \vec{v}_m, M \right\rangle,$$

where M is the last term in the preceding continued fraction, and by (2.1), this equals,

$$(B_k-1)A_kX + \frac{Mx_{N/2-1} + x_{N/2-2}}{My_{N/2-1} + y_{N/2-2}}. \tag{4.30}$$

The denominator of (4.30) equals,

$$(B_k-1)A_k^2XQ_{(n+1)/2}(c_{(n+1)/2}y_{N/2-1} + 2y_{N/2-2}) + A_kCy_{N/2-1} +$$

$$Cy_{N/2-1}^2(c_{(n+1)/2}y_{N/2-1} + 2y_{N/2-2}),$$

so by (2.3) and (2.20), this equals,

$$(B_k-1)A_k^2XQ_{(n+1)/2}(y_{N/2} + y_{N/2-2}) + A_kCy_{N/2-1} + Cy_{N/2-1}^2(y_{N/2} + y_{N/2-2}),$$

and by Lemma 3.1, this equals,

$$(B_k-1)A_k^3XQ_{(n+1)/2}/y_{N/2-1} + A_kCy_{N/2-1} + Cy_{N/2-1}A_k =$$

$$\frac{A_k C}{y_{N/2-1}}((B_k-1)A_k^2 X Q_{(n+1)/2}/C + 2y_{N/2-1}^2). \tag{4.31}$$

However, we may employ Lemmas 3.3–3.4, (2.17), and (2.20) to verify that $(B_k-1)Q_{(n+1)/2}/C = 2y_{N/2-1}^2$, so (4.31) equals,

$$2A_k C y_{N/2-1}(A_k^2 X + 1). \tag{4.32}$$

Given the calculated denominator (4.32) of (4.30), we may now use it to calculate the numerator, which is,

$$2A_k^4(B_k-1)X^2 C y_{N/2-1}+$$

$$(B_k-1)A_k^2 X(2Cy_{N/2-1} + Q_{(n+1)/2}(c_{(n+1)/2}x_{N/2-1} + 2x_{N/2-2}))+$$

$$A_k C x_{N/2-1} + Cy_{N/2-1}^2(c_{(n+1)/2}x_{N/2-1} + 2x_{N/2-2}),$$

and by using Lemma 3.5, (2.2), and (2.20), this equals,

$$2A_k^4(B_k-1)X^2 C y_{N/2-1}+$$

$$(B_k-1)A_k^2 X(2Cy_{N/2-1} + Q_{(n+1)/2}(x_{N/2} + x_{N/2-2}))+$$

$$A_k C x_{N/2-1} + Cy_{N/2-1}^2(x_{N/2} + x_{N/2-2})) =$$

$$2A_k^4(B_k-1)X^2 C y_{N/2-1}+$$

$$(B_k-1)A_k^2 X(2Cy_{N/2-1} + Q_{(n+1)/2}(B_k-1)/y_{N/2-1})+$$

$$A_k C x_{N/2-1} + C(B_k-1)y_{N/2-1}.$$

However, it may be verified using Lemma 3.3 and (2.17) that

$$2Cy_{N/2-1} = Q_{(n+1)/2}(B_k-1)/y_{N/2-1}$$

so the above equals

$$2A_k^4(B_k-1)X^2 C y_{N/2-1} + 4(B_k-1)A_k^2 X C y_{N/2-1} + A_k C x_{N/2-1} + C(B_k-1)y_{N/2-1}$$

and since Lemma 3.4 tells us that $A_k x_{N/2-1} = (B_k-1)y_{N/2-1} + 2y_{N/2-1}$, then the above equals,

$$2A_k^4(B_k-1)X^2 C y_{N/2-1} + 4(B_k-1)A_k^2 X C y_{N/2-1} + 2C(B_k-1)y_{N/2-1} + 2Cy_{N/2-1} =$$

$$2(B_k-1)Cy_{N/2-1}(A_k^4X^2+2A_k^2X+1)+2Cy_{N/2-1}=$$

$$2Cy_{N/2-1}((B_k-1)(A_k^2X+1)^2+1).$$

Hence, we have shown that (4.30) equals

$$\frac{2Cy_{N/2-1}((B_k-1)(A_k^2X+1)^2+1)}{2A_kCy_{N/2-1}(A_k^2X+1)}=\frac{(B_k-1)(A_k^2X+1)^2+1}{A_k(A_k^2X+1)}.$$

The balance of the proof now follows as in cases (a)–(b). One final note is in order for this case. Since Lemmas 3.3–3.4 and (2.17) allow us to deduce that when $(n+1)/2$ is even, we must have, $Q_{(n+1)/2}(B_k-1)=2Cy_{N/2-1}^2$, then $C \mid Q_{(n+1)/2}(B_k-1)$, so $2(B_k-1)A_kQ_{(n+1)/2}X/C \in \mathbb{N}$ in the simple continued fraction expansion of $\sqrt{D_k(X)}$.

It remains to verify case (c), part (iii). By (2.1) and (2.5) we have,

$$\left\langle q_0; w_{m-1}, c_0, \frac{2(B_k-1)A_kX}{C}, c_0, w_{m-1}\right\rangle =$$

$$\left\langle q_0; w_{m-1}, c_0, \frac{2(B_k-1)A_kX}{C}+\frac{y_{m(n+1)-1}}{x_{m(n+1)-1}}\right\rangle =$$

$$\left\langle q_0; w_{m-1}, \frac{2(B_k-1)A_kXc_0x_{m(n+1)-1}+Cc_0y_{m(n+1)-1}+Cx_{m(n+1)-1}}{2(B_k-1)A_kXx_{m(n+1)-1}+y_{m(n+1)-1}C}\right\rangle =$$

$$\langle q_0; w_{m-1}, M\rangle = (B_k-1)A_kX+\frac{Mx_{m(n+1)-1}+x_{m(n+1)-2}}{My_{m(n+1)-1}+y_{m(n+1)-2}}, \tag{4.33}$$

where M is the last term in the above continued fraction expansion. Now we calculate the denominator of (4.33). It is

$$2(B_k-1)A_kXx_{m(n+1)-1}(c_0y_{m(n+1)-1}+y_{m(n+1)-2})+$$

$$Cy_{m(n+1)-1}(c_0y_{m(n+1)-1}+y_{m(n+1)-2})+Cx_{m(n+1)-1}y_{m(n+1)-1},$$

and by (2.13), this equals

$$2(B_k-1)A_kXx_{m(n+1)-1}^2+2Cx_{m(n+1)-1}y_{m(n+1)-1}. \tag{4.34}$$

However, by Theorem 2.1,

$$B_k+A_k\sqrt{C}=(x_n+y_n\sqrt{C})^k=(x_n+y_n\sqrt{C})^{2m}=(x_{m(n+1)-1}+y_{m(n+1)-1}\sqrt{C})^2=$$

$$x^2_{m(n+1)-1} + y^2_{m(n+1)-1}C + 2x_{m(n+1)-1}y_{m(n+1)-1}\sqrt{C}.$$

Hence,

$$B_k = x^2_{m(n+1)-1} + y^2_{m(n+1)-1}C \text{ and } A_k = 2x_{m(n+1)-1}y_{m(n+1)-1}. \tag{4.35}$$

Thus, using (4.35) in conjunction with (2.17), we get that (4.34) equals,

$$4A_kXx^2_{m(n+1)-1}y^2_{m(n+1)-1}C + A_kC = C(A_k^3X + A_k),$$

which is the denominator of (4.33), so we can now calculate its numerator:

$$(B_k - 1)A_kXC(A_k^3X + A_k) + 2(B_k - 1)A_kXx_{m(n+1)-1}(c_0x_{m(n+1)-1} + x_{m(n+1)-2})+$$

$$C(c_0x_{m(n+1)-1} + x_{m(n+1)-2})y_{m(n+1)-1} + Cx^2_{m(n+1)-1}$$

which (2.14) tells us is equal to,

$$(B_k - 1)A_k^2XC(A_k^2X + 1)+$$

$$2(B_k - 1)A_kXCx_{m(n+1)-1}y_{m(n+1)-1} + C^2y^2_{m(n+1)-1} + Cx^2_{m(n+1)-1}$$

and (4.35) allows us to rewrite this as

$$(B_k - 1)A_k^2XC(A_k^2X + 1) + CB_k = C[(B_k - 1)(A_k^2X + 1)^2 + 1]$$

Hence, we have shown that (4.33) equals,

$$\frac{C[(B_k - 1)(A_k^2X + 1)^2 + 1]}{C(A_k^3X + A_k)} = \frac{(B_k - 1)(A_k^2X + 1)^2 + 1}{A_k^3X + A_k},$$

and the balance of the proof follows as in cases (a)–(b). (Note, as well, that from (2.17), it follows that $C \mid (B_k - 1)$, so $2(B_k - 1)A_kX/C \in \mathbb{N}$.) □

Remark 4.1. Theorem 4.1 provides a paradigm for construction of infinite families of polynomials $f_k(X)$ such that $\ell(\sqrt{f_k(X)})$ is independent of the variable X, $\lim_{k\to\infty}(\ell(\sqrt{f_k(X)})) = \infty$, and we are able to explicitly give the fundamental unit of the order $\mathbb{Z}[\sqrt{f_k(X)}]$. Furthermore, it is known that if D is a fundamental radicand with discriminant Δ, and $f \in \mathbb{N}$ is the least value such that $f^2D = a^2 + r$ with $f \mid 4a$, an ERD-type, then

$$\varepsilon_\Delta = \begin{cases} a + f\sqrt{D} & \text{if } |r| = 1,\ (D, f) \neq (5, 1), \\ (a + f\sqrt{D})/2 & \text{if } |r| = 4, \\ (2a^2 + r + 2af\sqrt{D})/|r| & \text{if } |r| \notin \{, 4\}, \end{cases}$$

(see [2, Exercise 3.2.7, p. 85], for example). In Theorem 4.1, one of the things we demonstrate is that for $D_k(X)$, $f = A_k$ ensures that $f^2D_k(X)$ is of ERD-type. However, we cannot ensure that A_k is the *least* such value. For instance, if $C = 3$, $B_3 = 26$, $A_3 = 15$, $D_k(X) = D_3(3) = 1269378$ and $9D_k(X) = 3380^2 + 2$ (see Example 4.2 below). Nevertheless, we have verified in general for the cases given in Theorem 4.1 that $\varepsilon_{4A_k^2 D_k(X)}$ is either $\varepsilon_{4D_k(X)}$ or $\varepsilon^2_{4D_k(X)}$.

We provide an illustration of all cases in Theorem 4.1 and of subsequent results below.

Example 4.1. *Let $C = 85$ for which*

$$\sqrt{C} = \left\langle 9; \overline{4,1,1,4,18} \right\rangle = \left\langle c_0; \overline{c_1, c_2, c_3, c_4, 2c_0} \right\rangle,$$

so $n = 4$ and we can illustrate Theorem 4.1 (a)–(b) *as follows. We have that $B_1 = 285769$ and $A_1 = 30996$. Thus,*

$$\sqrt{D_k(X)} = \sqrt{D_1(2)} = \sqrt{313832912233538380117} =$$

$$\left\langle 17715329865; \overline{4,1,1,4,35430659730} \right\rangle =$$

$$\left\langle (B_1-1)A_1X + c_0; \overline{w_0, 2((B_1-1)A_1X + c_0)} \right\rangle,$$

and $\ell(\sqrt{D_1(2)}) = 5 = k(n+1)$. Also, the least positive solution of

$$X^2 - D_1(2)Y^2 = 1$$

is given by $(X, Y) = ((B_1 - 1)(2A_1^2 + 1)^2 + 1, 2A_1^3 + A_1) =$

$$(1055106250929156033953353, 59558939006868),$$

which illustrates part (b)*. Note that this is not the same as the fundamental unit of $\mathbb{Z}[\sqrt{D_1(2)}]$ which is*

$$\varepsilon_{4D_1(2)} = 726328524474 + 41\sqrt{313832912233538380117}.$$

In fact, the (X, Y) above is achieved by squaring the fundamental unit whose norm is necessarily -1 since $\ell(\sqrt{D_1(2)})$ is odd (*see Theorem* 2.1).

Also, since $B_2 = 163327842721$ and $A_2 = 17715391848$, then for $k = 2$ and $X = 1$,

$$\sqrt{D_2(1)} = \sqrt{8371860393543383432522015026708959273830485} =$$

$$\langle 2893416733473314146569;$$

$$\overline{4, 1, 1, 4, 18, 4, 1, 1, 4, 9, 68080393728783862272,}$$

$$\overline{9, 4, 1, 1, 4, 18, 4, 1, 1, 4, 57868334669466282931 38}\rangle =$$

$$\left\langle (B_2 - 1)A_2X + c_0; \overline{w_1, c_0, 2(B_2 - 1)A_2X/C, c_0, w_1, 2((B_2 - 1)A_2X + c_0)} \right\rangle,$$

and $\ell(\sqrt{D_2(1)}) = 22 = 2k(n+1)+2$, *which illustrates part* (a). *Note as well that the least positive solution of* $X^2 - D_2(1)Y^2 = 1$ *is given by,*

$$X = 1608656350173170298294936670688048858287633864198800 1 =$$

$$(B_2 - 1)(A_2^2 + 1)^2 + 1,$$

and

$$Y = 5559711919693322890764178184040 = A_2^3 + A_2,$$

and $\varepsilon_{4D_2(1)} = X + Y\sqrt{D_2(1)}$

Remark 4.2. In Theorem 4.1, we have that $\ell(\sqrt{D_k(X)}) \to \infty$ as $k \to \infty$, so we have numerous parametric families for which the fundamental unit is explicitly known even though the period length gets arbitrarily large. As explained in [4], the first such sequence discovered with this property was the Shanks sequence, $S_n = (2^n + 3)^2 - 8$. For such families, the regulator $R = \log(\varepsilon_{4D_k(X)})$ is "small" compared to $\log(\sqrt{4D_k(X)})$ so we necessarily have large class number $h_{4D_k(X)}$. The following provide illustrations.

Example 4.2. *Let* $C = 21$ *for which we have* $B_1 = 55$, $A_1 = 12$, *and* $\sqrt{C} = \langle 4; \overline{1, 1, 2, 1, 1, 8}\rangle$, *so* $n = 5$. *For* $k = X = 1$, $\ell(\sqrt{D_1(1)}) = 6 = k(n+1)$, $Q_3 = 3$, *where*

$$\sqrt{D_1(1)} = \sqrt{425757} = \langle 652; \overline{1, 1, 434, 1, 1, 1304}\rangle =$$

$$\left\langle q_0; \overline{\overrightarrow{v}_0, 2(B_1 - 1)A_1X/Q_3 + c_3, \overleftarrow{v}_0, 2q_0} \right\rangle.$$

Also, $\varepsilon_{4D_1(1)} =$

$$1135351 + 1740\sqrt{D_1(1)} = (B_k - 1)(A_k^2X + 1)^2 + 1 + (A_k^3X + A_k)\sqrt{D_1(1)}.$$

The above illustrates part (c)-(i). *Note as well, with respect to Remark* 4.2, *that* $h_{4D_1(1)} = 30$ *while* $\log(\varepsilon_{4D_1(1)}) = 14.6355\ldots$, *and* $\log(\sqrt{4D_1(1)}) = 7.17395\ldots$.

Example 4.3. *Take* $C = 3$ *for which* $\sqrt{3} = \langle 1; \overline{1, 2} \rangle$ *so* $n = 1$. *Since* $B_4 = 97$ *and* $A_4 = 56$, *we have for* $k = 4$, *and* $X = 1$, $\sqrt{D_k(X)} =$

$$\sqrt{D_4(1)} = \sqrt{28919811} = \langle 5377; \overline{1, 2, 1, 1, 3584, 1, 1, 2, 1, 10754} \rangle =$$

$$\left\langle (B_4 - 1)A_4X + c_0; \overline{w_1, c_0, \frac{2(B_4 - 1)A_4X}{3}, c_0, w_1, 2((B_4 - 1)A_4X + c_0)} \right\rangle,$$

and $\ell(\sqrt{D_4(1)}) = 10 = k(n+1) + 2$, *which illustrates case* (c)-(iii). *Also, we may illustrate Remark* 4.2 *with the facts:*

$$h_{4D_4(1)} = 348, \log(\varepsilon_{4D_4(1)}) = 21.35953\ldots.$$

and

$$\log\left(\sqrt{4D_4(1)}\right) = 9.97631\ldots.$$

Example 4.4. *Let* $C = 46$ *for which we have*

$$\sqrt{C} = \langle 6; \overline{1, 3, 1, 1, 2, 6, 2, 1, 1, 3, 1, 12} \rangle,$$

so $n = 11$ *and* $(n + 1)/2$ *is even. Since* $B_3 = 57643991108495$ *and* $A_3 = 8499142809612$, *we have for* $k = 3$, $X = 1$, $\sqrt{D_k(X)} = \sqrt{D_3(1)} =$

$$\sqrt{2400260279945084907570092244370817462970207372908677 02} =$$

$$\langle 48992451254709484147804433 4;$$

$$\overline{1, 3, 1, 1, 2, 6, 2, 1, 1, 3, 1, 12, 1, 3, 1, 1, 2, 3, 426021315258343340415690 72,}$$

$$\overline{3, 2, 1, 1, 3, 1, 12, 1, 3, 1, 1, 2, 6, 2, 1, 1, 3, 1, 979849025094189682956088668} \rangle =$$

$$\left\langle q_0; \overline{\vec{v}_1, \frac{c_6}{2}, \frac{2Q_{(n+1)/2}(B_k - 1)A_kX}{C}, \frac{c_6}{2}, \overleftarrow{v}_1, 2q_0} \right\rangle,$$

where $q_0 = (B_3 - 1)A_3X + c_0$, *which illustrates case* (c)-(ii).

The following provides an analogue of Theorem 4.1 for the case where the minus sign therein is replaced by a plus sign in the representation of $D_k(X)$, and this result illustrates the comments made in the last statement of Remark 4.1. We do not provide a proof of he following since the reader may use the techniques in the proof of Theorem 4.1 *mutatis mutandis*.

Theorem 4.2. *Let*

$$D_k(X) = (B_k + 1)^2 A_k^2 X^2 + 2(B_k + 1)^2 X + C,$$

and let (x, y) *denote the fundamental solution of* $x^2 - D_k(X)y^2 = 1$. *Then for*

$$q_0 = (B_k + 1)A_k X + c_0 :$$

(a) *If both* $n, k \in \mathbb{N}$ *are even, then,*

$$\sqrt{D_k(X)} = \langle q_0; \overline{w_{k-1}, 2q_0} \rangle ,$$

with $\ell\left(\sqrt{D_k(X)}\right) = k(n+1)$, *and*

$$(x, y) = \left(\frac{(B_k + 1)(A_k^2 + 1)}{2x_{k(n+1)-1}}, \frac{A_k}{2x_{k(n+1)-1}} \right).$$

(b) *If* n *is even and* k *is odd, then* $C \mid (B_k + 1)$,

$$\sqrt{D_k(X)} = \left\langle q_0; \overline{w_{k-1}, c_0, 2(B_k + 1)A_k X/C, c_0, w_{k-1}, 2q_0} \right\rangle ,$$

with $\ell\left(\sqrt{D_k(X)}\right) = 2k(n+1) + 2$, *and*

$$(x, y) = ((B_k + 1)(A_k^2 X + 1)^2 - 1, A_k^3 X + A_k).$$

(c) *If* n *is odd, then one of the following holds.*

(i) *If* $k = 2m + 1$, $m \geq 0$, $(n + 1)/2 > 1$ *is odd, and* $c_{(n+1)/2}$ *is even, then* $C \mid Q_{(n+1)/2}(B_k + 1)$, $\sqrt{D_k(X)} =$

$$\left\langle q_0; \overline{\overrightarrow{v}_m, c_{(n+1)/2}/2, 2Q_{(n+1)/2}(B_k + 1)A_k X/C, c_{(n+1)/2}/2, \overleftarrow{v}_m, 2q_0} \right\rangle$$

with $\ell\left(\sqrt{D_k(X)}\right) = k(n+1) + 2$, *and*

$$(x, y) = ((B_k + 1)(A_k^2 X + 1)^2 - 1, A_k^3 X + A_k).$$

(ii) *If* $k = 2m+1$, $m \geq 0$, *and* $(n+1)/2$ *is even, then* $Q_{(n+1)/2}$ *divides* $2(B_k+1)$,

$$\sqrt{D_k(X)} = \left\langle q_0; \overline{\overrightarrow{v}_m, 2(B_k+1)A_kX/Q_{(n+1)/2} + c_{(n+1)/2}, \overleftarrow{v}_m, 2q_0} \right\rangle$$

with $\ell\left(\sqrt{D_k(X)}\right) = k(n+1)$, *and*

$$(x,y) = ((B_k+1)(A_k^2X+1)^2 - 1, A_k^3X + A_k).$$

(iii) *If* $k = 2m$, $m \in \mathbb{N}$, *then*

$$\sqrt{D_k(X)} = \langle q_0; \overline{w_{m-1}, 2q_0} \rangle$$

with $\ell\left(\sqrt{D_k(X)}\right) = m(n+1)$, *and*

$$(x,y) = \left(\frac{(B_k+1)(A_k^2+1)}{2x_{m(n+1)-1}}, \frac{A_k}{2x_{m(n+1)-1}} \right).$$

Now we illustrate how to use the above results to obtain quite simple proofs of some classical results in the literature. The first may be found, for instance, in [7, Theorem 5, p. 324].

Corollary 4.1. *Given* $c_0, c_1, \ldots, c_n, n \in \mathbb{N}$ *fixed and*

$$\sqrt{C} = \langle c_0; \overline{c_1, \ldots, c_n, 2c_0} \rangle,$$

there exist infinitely many $D, d_0 \in \mathbb{N}$ *such that*

$$\sqrt{D} = \langle d_0; \overline{c_1, \ldots, c_n, 2d_0} \rangle.$$

Proof. If n is even, let $k = 1$ in Theorem 4.1 part (b). Then

$$\sqrt{D_1(X)} = \left\langle (B_1-1)A_1X + c_0; \overline{c_1, \ldots, c_n, 2[(B_1-1)A_1X + c_0]} \right\rangle$$

for all $X \in \mathbb{N}$. If n is odd, take $m = 1$ in Theorem 4.2, part (c)–(iii). Then

$$\sqrt{D_1(X)} = \left\langle (B_1+1)A_1X + c_0; \overline{c_1, \ldots, c_n, 2[(B_1+1)A_1X + c_0]} \right\rangle$$

for all $X \in \mathbb{N}$. □

The following provides an alternative proof of another classic result (see [7, Theorem 6, p. 325], for instance).

Corollary 4.2. *For any $k \in \mathbb{N}$ there are infinitely many $D \in \mathbb{N}$ such that $\ell(\sqrt{D}) = k$.*

Proof. If $k = 2m$ for any $m \in \mathbb{N}$, set $C = 3$, for which $n = 1$, and invoke Theorem 4.2, part (c)-(iii) to get that for any $X \in \mathbb{N}$, and any fixed $m \in \mathbb{N}$, $\ell(\sqrt{D_k(X)}) = 2m$. If $k = 2m+1$, then take $C = 2$ for which $n = 0$ and invoke Theorem 4.1 part (b) to get that for any $X \in \mathbb{N}$ and any fixed $m \geq 0$, $\ell(\sqrt{D_k(X)}) = 2m+1$. $\square$

To see how Corollary 4.2 works, for instance in the case where $k = 2m+1$ is arbitrary but fixed, note that $C = 2$, $B_k + A_k\sqrt{2} = (3+2\sqrt{2})^k$, so for any $X \in \mathbb{N}$,

$$\sqrt{D_{2m+1}(X)} = \left\langle (B_k - 1)A_kX + 1; \overline{\underbrace{2, 2, \ldots, 2}_{2m \text{ copies}}, 2[(B_k - 1)A_kX + 1]} \right\rangle$$

provides infinitely many values with period length $2m+1$.

In the case where $k = 2m$ is arbitrary but fixed, choose $C = 3$, $X \in \mathbb{N}$ arbitrary, $B_k + A_k\sqrt{3} = (2+\sqrt{3})^k$, and $n = 1$ in Theorem 4.2, part (c)-(iii). Then

$$\sqrt{D_{2m}(X)} = \left\langle (B_k + 1)A_kX + 1; \overline{\underbrace{1, 2, 1, 2, \ldots, 1, 2}_{m-1 \text{ copies of } 1,2}, 1, 2[(B_k + 1)A_kX + 1]} \right\rangle$$

provides infinitely many values with period length $2m$. We conclude with some illustrations of Theorem 4.2.

Example 4.5. *We revisit the value $C = 85$ discussed in Example 4.1. For $n = 4$, $k = 1 = X$, we have,*

$$D_1(1) = (B_1 + 1)^2A_1^2 + 2(B_1 + 1)^2 + C = 78459326352621672285,$$

and $\sqrt{D_1(1)} =$

$$\langle 8857726929 : \overline{4, 1, 1, 4, 9, 208417104, 9, 4, 1, 1, 4, 17715453858} \rangle =$$

$$\left\langle (B_k + 1)A_kX + c_0; \overline{w_0, c_0, 2(B_1 + 1)A_1/C, c_0, w_0, 2q_0} \right\rangle,$$

with $\ell(\sqrt{D_1(1)}) = 12 = 2k(n+1) + 2$, which illustrates part (b) *of Theorem* 4.2. *Notice that the fundamental solution of $x^2 - D_1(1)^2y^2 = 1$ is*

$$(x, y) = ((B_k + 1)(A_k^2X + 1)^2 - 1, A_k^3 + A_k) =$$

$$(26377840909571752994752\,9, 29779469518932),$$

and $x + y\sqrt{D_1(1)}$ *is also the fundamental unit of the order* $\mathbb{Z}\left[\sqrt{D_1(1)}\right]$.

Now consider $\sqrt{D_2(1)} = \langle q_0; \overline{w_1, 2q_0}\rangle =$

$$\langle 289341673350874493026\,5; \overline{4,1,1,4,18,4,1,1,4,57868334670174898605\,30}\rangle$$

with $\ell(\sqrt{D_2(1)}) = 10 = k(n+1)$. *The fundamental unit of* $\mathbb{Z}[\sqrt{D_2(1)}]$ *is*

$$(x, y) = (896843450718370578585007\,45, 30996) =$$

$$\left(\frac{(B_2+1)(A_2^2+1)}{2x_9}, \frac{A_2}{2x_9}\right),$$

which is the fundamental solution of $x^2 - D_2(1)y^2 = 1$. *This illustrates Theorem* 4.2, *part* (a).

Example 4.6. *Now we return to the value* $C = 21$ *considered in Example* 4.2. *For* $k = 1 = X$ *and* $n = 5$, *we have*

$$\sqrt{D_1(1)} = \sqrt{457877} = \langle 676; \overline{1,1,1,192,1,1,1,1352}\rangle =$$

$$\langle q_0; \overline{\overrightarrow{v_0}, c_3/2, 2Q_3(B_1+1)A_1/C, c_3/2, \overleftarrow{v_0}, 2q_0}\rangle,$$

and $\ell(\sqrt{D_1(1)}) = 8 = k(n+1) + 2$. *Also,*

$$\varepsilon_{4D_1(1)} = 1177399 + 1740\sqrt{D_1(1)}$$

which is the fundamental solution of $x^2 - D_1(1)y^2 = 1$. *This illustrates Theorem* 4.2, *part* (c)–(i).

Example 4.7. *Let* $C = 46$ *as in Example* 4.4. *Then* $(n+1)/2 = 6$, *and for* $k = 1 = X$, $B_1 = 24335$ *and* $A_1 = 3588$,

$$\sqrt{D_1(1)} = \sqrt{7624358865916462} =$$

$$\langle 87317574; \overline{1,3,1,1,2,87317574,2,1,1,3,1,174635148}\rangle =$$

$$\langle q_0; \overline{\overrightarrow{v_0}, 2(B_1+1)A_1X/Q_6 + c_6, \overleftarrow{v_0}, 2q_0}\rangle,$$

and $\ell(\sqrt{D_1(1)}) = 12 = k(n+1)$. *Also,*

$$\varepsilon_{4D_1(1)} = 4033285840069808399 + 46190997060\sqrt{D_1(1)}$$

which is also the fundamental solution of $x^2 - D_1(1)y^2 = 1$. *This depicts Theorem* 4.2, *part* (c)–(ii).

Example 4.8. *Let* $C = 3$ *as in Example* 4.3, *for which* $n = 1$. *Also, for* $X = 1$, *and* $k = 4$,

$$D_4(1) = (B_4+1)^2A_4^2 + 2(B_4+1)^2 + C = 30137355,$$

for which

$$\sqrt{D_4(1)} = \langle 5489; \overline{1, 2, 1, 10978}\rangle = \langle q_0; \overline{w_1, 2q_0}\rangle$$

$\ell(\sqrt{D_4(1)}) = 4 = m(n+1)$, *and*

$$\varepsilon_{4D_4(1)} = 21959 + 4\sqrt{D_4(1)} = \frac{(B_4+1)(A_4^2+1)}{2x_3} + \frac{A_4}{2x_3}\sqrt{D_4(1)}$$

which illustrates Theorem 4.2, *part* (c)–(iii).

Remark 4.3. In later work, we will show how to obtain such families, as above, for the simple continued fraction expansions of $(1+\sqrt{D})/2$ where $D \equiv 1 \pmod 4$ is a nonsquare natural number. Moreover, we will show that *any palindrome* of natural numbers may be represented by the symmetric part of the simple continued fraction expansion of either $\sqrt{D}$ or $(1+\sqrt{D})/2$ in infinitely many ways. Finally, we will demonstrate how to find, explicitly and easily for a given palindrome, the fundamental unit of the underlying quadratic order. This has applications for the class numbers of those orders.

REFERENCES

[1] R. A. Mollin. Polynomial solutions for Pell's equation revisited. *Indian J. Pure Appl. Math.* **28** (1997), 429–438.

[2] R. A. Mollin. Quadratics. CRC Press, Boca Raton, New York, London, Tokyo, 1996.

[3] R. A. Mollin. Fundamental Number Theory with Applications. CRC Press, Boca Raton, New York, London, Tokyo, 1998.

[4] R. A. Mollin, K. Cheng, B. Goddard. Pellian Polynomials and Period Lengths of Continued Fractions (to appear).

[5] A. Schinzel. On some problems of the arithmetical theory of continued fractions. *Acta Arith.* **6** (1961), 393–413.

[6] A. Schinzel. On some problems of the arithmetical theory of continued fractions II. *Acta Arith.* **7** (1962), 287–298; (Corrigendum, ibid. **47** (1986), 295).

[7] W. Sierpiński. Elementary Theory of Numbers. North-Holland, Amsterdam, New York, Oxford, 1988.

[8] H.-J. Stender. Über die Grundeinheit der reell-quadratischen Zahlkörper $\mathbb{Q}(\sqrt{A^2N^2+BN+C})$. *J. Reine Angew. Math.* **311/312** (1979), 302–306.

[9] M. A. Stern. Zur Theorie der periodischen Kettenbrüche. *J. Riene Angew. Math.* **8** (1857), 1–102.

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