10. On Class Numbers of Quadratic Extensions of Algebraic Number Fields

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In [14] Nagell showed that there are infinitely many imaginary quadratic extensions of the rational number field $\boldsymbol{Q}$, each of which has class number divisible by a given integer. Subsequently several authors have proved this result (see [1], [4], [5] and [17] as well as the most recent proof by Uehara [16]). In this paper we generalize this well-known result by explicit construction of infinitely many imaginary quadratic extensions of a given number field K (subject only to having a totally ramified rational prime) each with class number divisible by a given integer. The proof and construction given is simpler than that given in previous proofs cited above for the trivial case $K=\boldsymbol{Q}$, and applications are given. The next result is a sufficient condition for an arbitrary quadratic extension of $\boldsymbol{Q}$ to have an element of given order in its class group. Finally for a certain class of real quadratic extensions of $\boldsymbol{Q}$ we give a sufficient condition for its class number to be divisible by a given prime, and we provide applications.

Before presenting the first result some comments on notation and a lemma are required. For a given number field K, $h(K)$ denotes the class number of K, C_K denotes the class group of K, $\mathcal{O}_K$ denotes the ring of integers of K, (α) for $\alpha \in \mathcal{O}_K$ denotes the principal ideal generated by α, and $N(\cdot)$ denotes the norm from K to $\boldsymbol{Q}$.

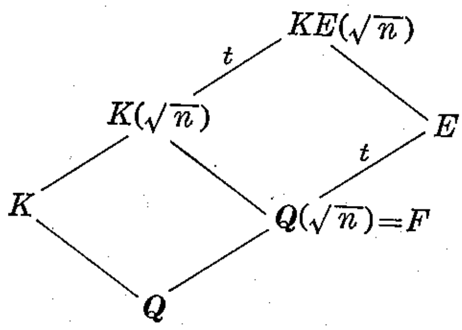
In the proof of Theorem 1 we will need the following result whose proof (mutatis mutandis) is the same as that of [1, Lemma 1, p. 321] of which the following lemma is a generalization.

Lemma 1. *Let ε be any positive real number and let p be any odd prime. Denote by N the number of square-free integers of the form p^g-x^2 where x is an even integer such that $0<x<\varepsilon p^{g/2}$. Then for g sufficiently large, $N\geq c_p\varepsilon p^{g/2}$ where c_p is a positive constant depending only on p.*

Theorem 1. *Let $t>1$ be any integer. If K is any algebraic number field in which there is a totally ramified rational odd prime p, then there are infinitely many imaginary quadratic extensions L of K such that $t \mid h(L)$. Moreover L may be chosen of the form $K(\sqrt{n})$ where n is any square-free rational integer of the form $n=r^2-m^t$ where p does not divide n and r is an even integer subject to $r^2\leq m^{t-1}(m-1)$.*

Proof. Let r be an arbitrarily chosen but fixed even integer. Let n

be an integer of the form $n=r^2-m^t$ where m is any odd integer with $r^2 \leq m^{t-1}(m-1)$ and p does not divide n. By [7, Corollary 2.6] $t \mid h(\boldsymbol{Q}(\sqrt{n}))$. Therefore there exists an abelian unramified extension E of $F=\boldsymbol{Q}(\sqrt{n})$ with $|E:F|=t$. By Abhyankar's Lemma (see [2] or [3]) $KE(\sqrt{n})$ is an unramified extension of $K(\sqrt{n})$. Moreover we claim that $K \cap E=F$. To see this we recall that p does not ramify in F since p fails to divide n. Since p is totally ramified in K and ramification degrees multiply in towers then any $\boldsymbol{Q}(\sqrt{n})$-prime above p is totally ramified in $K(\sqrt{n})$. This proves the claim. Hence from [6, Theorem 7, p. 263] it follows that $KE(\sqrt{n})$ is of degree t over $K(\sqrt{n})$. The following diagram describes the situation:



To conclude the proof of the theorem it remains to show that there are infinitely many square-free integers of the form $n=r^2-m^t$ where r is even, $r^2 \leq m^{t-1}(m-1)$ and p does not divide n.

Let $\varepsilon=[(p-1)/p]^{1/2}$ and let k be sufficiently large such that $g=kt$ satisfies the hypothesis of Lemma 1; that is, the number N of square-free integers of the form m^t-r^2, with $m=p^k$, and $0<r<\varepsilon m^{t/2}$ is greater than $c_p \varepsilon m^{t/2}$. Since ε is fixed and c_p is a positive constant depending only on p then k may be chosen such that N is as large as we want. Q.E.D.

The following application to biquadratic fields is immediate from Theorem 1.

Corollary 1. *Let $K=\boldsymbol{Q}(\sqrt{s})$ where s is any square-free integer, and let $F=\boldsymbol{Q}(\sqrt{n})$ where g.c.d. $(n, 2s)=1$, $n=r^2-m^t$ is square-free, $r^2 \leq m^{t-1}(m-1)$ and r even, then $t \mid h(KF)$. (In fact $t \mid h(F)$.)*

The following is an application to imaginary quadratic extensions of pure fields of prime degree (see Mollin [11, pp. 421–423]).

Corollary 2. *Let $K=\boldsymbol{Q}(\sqrt[p]{p})$ where p is an odd prime, and let n be a square-free integer of the form $n=r^2-m^t$ relatively prime to p and with r even, and $r^2 \leq m^{t-1}(m-1)$; then $t \mid h(K(\sqrt{n}))$.*

The reader may compare the above with Mollin [8, pp. 166–168] where conditions for the divisibility of the class numbers of imaginary quadratic extensions of cyclotomic fields by a power of 2 are given.

We now turn to establishing a sufficient condition for any quadratic field to have an element of order $t>1$ in its class group for a given integer t.

Theorem 2. *Let $K=\boldsymbol{Q}(\sqrt{n})$, where $n=a^2-4b^t$ is a square-free integer where $b>1$ and $t>1$ are integers. If $\pm b^c$ is not the norm of any element of $\mathcal{O}_K$ for all c properly dividing t then t divides the exponent of C_K.*

Proof. Let $b=p_1^{a_1}\cdots p_r^{a_r}$ where the p_i's are distinct rational primes and the a_i's are positive integers. Clearly each p_i splits in K, so $p_i\mathcal{O}_K=\mathcal{P}_iQ_i$ where $\mathcal{P}_i$ and Q_i are $\mathcal{O}_K$-primes for $i=1, 2, \cdots, r$. Let $\alpha=(a+\sqrt{n})/2$ and $\bar{\alpha}=(a-\sqrt{n})/2$, then $(b)^t=(\alpha\bar{\alpha})=\prod_{i=1}^r(\mathcal{P}_iQ_i)^{a_it}$. Since $\alpha+\bar{\alpha}=a$, $(\alpha-\bar{\alpha})^2=n$ and g.c.d. $(a, b)=1$ (whence g.c.d. $(a, n)=1$), then $\mathcal{P}_i$ divides both α and $\bar{\alpha}$ only if 1 is in $\mathcal{P}_i$. Therefore, for an appropriate choice of $\mathcal{R}_i=\mathcal{P}_i$ or Q_i we must have that $(\alpha)=(\prod_{i=1}^r\mathcal{R}_i^{a_i})^t=\mathcal{A}^t$, say. If $\mathcal{A}^c$ is principal for any c properly dividing t then $N(\mathcal{A}^c)=\pm b^c$ violates the hypothesis. Hence $\mathcal{A}$ is an element of order t in C_K; i.e., t divides the exponent of C_K. Q.E.D.

Maintaining the notation of Theorem 2 we have:

Corollary 3 (Mollin [7, Corollary 2.4]). *If $n=a^2-4b^t<0$ and $a^2\leq 4b^{t-1}(b-1)$ then t divides $h(K)$.*

Note that if t divides the exponent of C_K then there is a non-principal ideal $\mathcal{J}$ such that $\mathcal{J}^t=(\alpha)$ for some $\alpha\in\mathcal{O}_K$, but $\mathcal{J}^c$ is not principal for any c properly dividing t. Therefore if $\alpha=(a+s\sqrt{m})/2$ then $a^2-s^2m=4b^t$ where $N(\mathcal{J})=b$; i.e., $K=\boldsymbol{Q}(\sqrt{n})=\boldsymbol{Q}(\sqrt{m})$ for $n=s^2m$. Is the converse of Theorem 2 valid?; i.e., is it true that if t divides the exponent of C_K then $\pm b^c$ is not the norm of any $\beta\in\mathcal{O}_K$ for all c properly dividing t? Note that if such a β exists then $N(\mathcal{J}^c)=N(\beta)$. However this does not necessarily imply that $\mathcal{J}^c$ is principal. Is there some restriction on K such that the condition "$\pm b^c$ is not a norm of an integer in $\mathcal{O}_K$" becomes necessary and sufficient for t to divide the exponent of C_K? Compare the above with Uehara [16, Theorem 2, p. 257].

We now turn to the real quadratic field case.

Proposition 1. *Let $K=\boldsymbol{Q}(\sqrt{n})$ where n is a square-free integer of the form $n=a^2+t^p\not\equiv 1 \pmod 4$ where $a>0$ and $t>1$ are integers and p is a prime. Suppose furthermore that $n=(st)^2+r>7$ where the following conditions are satisfied:*

(i) *$s>1$, t not a square and g.c.d. $(t, r)=1$.*

(ii) *r divides $4s$ with $-2s<r\leq 2s$;*

then p divides $h(K)$.

Proof. By Mollin [9, Theorem 1.2] $x^2-ny^2=\pm t$ is not solvable in integers (x, y), and so by Mollin [10, Theorem 3], p divides $h(K)$. Q.E.D.

The following table provides examples as an application of Proposition 1.

Table I

r	t	s	a	p	n	$h(n)$
1	3	1	1	2	10	2
1	5	1	1	2	26	2
1	9	1	1	2	82	4
1	11	1	1	2	122	2
1	13	1	1	2	170	4
−2	3	5	14	3	223	3
1	15	1	1	2	226	8
−2	3	7	14	5	439	5
4	9	3	4	3	733	3

All class numbers are taken from B. Oriat's "Groupes des Classes des Corps Quadratiques Réels $Q(\sqrt{d})$, $d<10,000$", Faculté des Sciences de Besançon.

Finally we note that Proposition 1 has relevance to the representation of integers as sums of powerful numbers, (see [12] and [13]), a difficult problem in elementary number theory.

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