# 44. *Class Number One Criteria for Real Quadratic Fields. II*

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This paper continues the work begun in [6]. Therein we gave criteria for real quadratic fields of narrow Richaud-Degert (R-D) type to have class number one. This was a consequence of more general criteria given for real quadratic fields $Q(\sqrt{n})$ with $n\equiv 1 \pmod 4$.

Herein we will deal with positive square-free integers $n$ of wide (R-D) type; i.e., $n=m^2+r$ where $r$ divides $4m$ and $r\in(-m, m]$ with $|r|\neq 1, 4$. The first result generalizes results in [1], [3], [4], [9] and [11].

**Theorem 1.** *Let $n=l^2+r>7$ be of wide R-D type such that $n\not\equiv 1$ (mod 4). If $h(n)=1$ then:*

(1) $|r|=2$.

(2) *$p$ is inert in $Q(\sqrt{n})$ for all odd primes $p$ dividing $l$.*

(3) *If $r=2$ then $l\equiv 0 \pmod 3$.*

(4) *If $r=-2$ then $l\not\equiv 0 \pmod 3$.*

*Proof.* Since $n\not\equiv 1 \pmod 4$ then 2 is ramified in $Q(\sqrt{n})$. Therefore, there are integers $x$ and $y$ such that $x^2-ny^2=\pm 2$. By [5, Theorem 1.1] $2\geq|r|$; where $|r|=2$ since $|r|\neq 1$ by hypothesis. This secures (1). If $p$ is an odd prime dividing $l$ such that $p$ is not inert in $Q(\sqrt{n})$ then there are integers $u$ and $v$ such that $u^2-nv^2=\pm p$. By [5, Theorem 1.2] $n=7$ and $p=3$ are forced. This secures (2).

If 3 is not inert in $Q(\sqrt{n})$ then $x^2-ny^2=\pm 3$ for some integers $x$ and $y$. Assume that $x>0$ and that $y>0$ is the *least* positive solution. Thus we may invoke [7, Theorem 108–108a, pp. 205–207] to get that if $x^2-ny^2=3$ then; for $x_1=(2l^2+r)/|r|$ and $y_1=2l/|r|$ (see [2] and [8]):

(i) $0\leq y\leq y_1\sqrt{3}/\sqrt{2(x_1+1)}$

and if $x^2-ny^2=-3$ then:

(ii) $0<y\leq y_1\sqrt{3}/\sqrt{2(x_1-1)}$.

A tedious check shows that $y=1$.

Therefore $x^2-n=\pm 3$; i.e., $x^2-l^2=r\pm 3$. An easy check shows that the only possible solutions to the latter equation occur when either $l=r=2$ or $l=3$, and $r=-2$. Thus, if $n>6$ when $r=2$, and $n>7$ when $r=-2$ then 3 is inert in $Q(\sqrt{n})$; whence $n\equiv 2 \pmod 3$. Therefore, $l\equiv 0 \pmod 3$ if $r=2$, and $l\not\equiv 0 \pmod 3$ if $r=-2$. This secures (3), (4) and the theorem. Q.E.D.

**Remark 1.** The converse of Theorem 1 is false. For example, if $n=12^2+2=146$ then Theorem 1 (1)–(3) are satisfied, but $h(n)=2$.

The following Table illustrates Theorem 1.

Table

| $l$ | $r$ | $n$ | $h(n)$ |
|---|---|---|---|
| 2 | 2 | 6 | 1 |
| 3 | 2 | 11 | 1 |
| 6 | 2 | 38 | 1 |
| 9 | 2 | 83 | 1 |
| 12 | 2 | 146 | 2 |
| 15 | 2 | 227 | 1 |
| 18 | 2 | 326 | 3 |
| 315 | 2 | 99227 | 18 |
| 3 | −2 | 7 | 1 |
| 4 | −2 | 14 | 1 |
| 5 | −2 | 23 | 1 |
| 7 | −2 | 47 | 1 |
| 8 | −2 | 62 | 1 |
| 11 | −2 | 119 | 2 |
| 13 | −2 | 194 | 2 |
| 20 | −2 | 398 | 1 |
| 316 | −2 | 99854 | 21 |

All class numbers are taken from [10].

**Theorem 2.** *Let* $n=l^2+r$ *be of R-D type with* $r\,|\,2l$, *and* $n\equiv 1 \pmod 4$. *If* $h(n)=1$ *then*:

(1) *If* $n\equiv 1 \pmod 8$ *then* $n=33$.

(2) *If* $n\equiv 5 \pmod 8$ *then* $r<0$, $-r$ *is a prime and* $p$ *is inert in* $Q(\sqrt{n})$ *for all primes* $p<|r|/4$.

*Proof.* If $n\equiv 1 \pmod 8$ then 2 splits in $Q(\sqrt{n})$. Thus there are integers $a$ and $b$ such that $a^2-nb^2=\pm 8$.

By [5, Theorem 1.1] $|r|\leq 8$. Also, using [7, Theorems 108–108a, pp. 205–207] we may achieve that $b=1$ by the same reasoning as in the proof of Theorem 1. Hence $a^2-l^2=r\pm 8$ where $|r|\leq 8$. However, $n\equiv 1 \pmod 8$ and $|r|\neq 1, 4$. Therefore, $r\in\{-7, -3, 5\}$. An easy check of $a^2-l^2=r\pm 8$ for these values of $r$ yields that the only solution is $l=6$ and $r=-3$; i.e., $n=33$.

Suppose that $n\equiv 5 \pmod 8$. If $|r|$ is not prime then there exists a prime $p$ dividing $|r|$ such that $2<p<|r|$ and $p$ is ramified in $Q(\sqrt{n})$. Therefore, there are integers $c$ and $d$ with $c^2-nd^2=\pm 4p$; whence $4p\geq |r|$ by [5, Theorem 1.1]. Hence, $|r|=2p$, $3p$ or $4p$. Either even case contradicts that $n\equiv 5 \pmod 8$. For the $|r|=3p$ case we note that it is well-known that if $h(n)=1$ then $n=s$ or $pq$ where $s$, $p$ and $q$ are primes such that either $p=2$ and $q\equiv 3 \pmod 4$ or $p\equiv q\equiv 3 \pmod 4$, (e.g., see [5]). Thus $|r|=3p$ implies that $n$ is a product of more than two primes. Hence $|r|$ is a prime. Moreover $|r|\equiv 3 \pmod 4$ and $(l^2+r)/|r|\equiv 3 \pmod 4$ is prime. If $r>0$ then $l^2\equiv 2r$

(mod 4) forcing $r$ to be even, a contradiction. Thus $r<0$.

If $p<|r|/4$ is a prime which is not inert in $Q(\sqrt{n})$, then there are integers $e$ and $f$ such that $e^2-nf^2=\pm 4p$ with $|r|>4p$. This contradicts [5, Theorem 1.1]. This secures the theorem. Q.E.D.

Two examples which illustrate Theorem 2 (2) are $n=141=12^2-3$ and $n=1757=42^2-7$ for which $h(n)=1$.

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