A Note on the Diophantine Equation $D_1x^2 + D_2 = ak^n$.*

R.A. Mollin

Abstract

We consider the Diophantine equation of the title which was recently solved, in terms of the number of solutions to it for $a \in \{1, 2, 4\}$, in [1]. However, a counterexample was provided in [6]. We provide another counterexample and show that both the example herein and the one in [6] are results of Ljunggren [7] from the early 1940s. Given that these are all the omissions from [1], this secures the study of the equation in the title.

For relatively prime $D_1, D_2 \in \mathbb{N}$ (the natural numbers), $D = D_1D_2$, $k \in \mathbb{N}$, prime to D, $\lambda \in \{1, \sqrt{2}, 2\}$, with $\lambda = 2$ if k is even, the Diophantine equation

$$D_1x^2 + D_2 = \lambda^2 k^n; \qquad x, n \in \mathbb{N} \tag{1}$$

has a long and distinguished history as outlined in [1], for instance. Therein, the authors present results on the number of solutions $(x, n) \in \mathbb{N}^2$ of (1), which is denoted by $\mathcal{N}(\lambda, D_1, D_2, k)$. A counterexample to one case was presented in [6]. In this note, we provide another counterexample to a different case, but show that both of these examples are consequences of work of Ljunggren, who completely solved the Diophantine equation

$$x^2 = \frac{y^n - 1}{y - 1} \text{ for } x, y > 1, \text{ and } n \geq 3. \tag{2}$$

We begin by observing that for

$$D = k^n(k - 1) = ((k - 1)x)^2 + k - 1 \tag{3}$$

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where $k, x, n \in \mathbb{N}$ with $k > 2$, the period length of the simple continued fraction expansion of $\sqrt{D}$, $\ell = \ell(\sqrt{D}) = 2$ (which is well-known since D is a so-called *Richaud-Degert type*, see [9, Theorem 3.2.1, p. 78], for instance). However, by work of Ljunggren [7] in the 1940s, if D is of the form in (3), then $\ell = 2$ if and only if

$$(n, k, x) \in \{(1, 3, 1), (2, 3, 2), (5, 3, 11), (1, 7, 1), (4, 7, 20)\}. \tag{4}$$

The case $k = 3$ is the result presented in [6] (wherein algebraic number theory in quadratic fields and binary linear recurrences were used in the proof), is merely a footnote to the work of Ljunggren, since $2x^2 + 1 = 3^n$ can be rewritten as $x^2 = (3^n - 1)/(3 - 1)$, which, in turn, is of the form (2). This is a counterexample to the results in [1] since therein they claim that $\mathcal{N}(1, 2, 1, 3) \leq 2$, but by the above, $\mathcal{N}(1, 2, 1, 3) = 3$. The authors of [1] cited Le [5] for the claim that $\mathcal{N}(1, 2, 1, 3) \leq 2$, but no proof exists therein. Instead Le references [3]–[4] as the place to find the proof. Yet, as the authors of [6] note, they could find no such proof, and indeed the claim is incorrect, as the above shows.

The second case $k = 7$ is another counterexample to the results in [1], since the authors claim that $\mathcal{N}(1, 6, 1, 7) \leq 1$. However by the above, $\mathcal{N}(1, 6, 1, 7) = 2$. Observe that $6x^2 + 1 = 7^n$ can be rewritten as $x^2 = (7^n - 1)/(7 - 1)$, again the form in (2).

A similar remark can be made for other Diophantine equations considered in [1], such as the following, which holds since these D are of Richaud-Degert type as well. If D is of the form

$$D = 4k^n(4k - 1) = (4k - 1)^2x^2 + 4k - 1, \text{ with } k, n, x \in \mathbb{N}, \tag{5}$$

then $\ell = \ell(\sqrt{D}) = 2$. Thus, by [1], for D given in (5), with $n > 1$,

$$\ell = 2 \text{ if and only if } (k, n, x) = (2, 4, 3), \text{ namely for } D = 448. \tag{6}$$

Note that an application of the above is the following. It is clear that $\mathbb{Q}(\sqrt{4k^n - 1}) = \mathbb{Q}(\sqrt{4k - 1})$ if and only if there is an $x \in \mathbb{Q}$ such that

$$(4k^n - 1)/(4k - 1) = x^2. \tag{7}$$

For instance for $(k, n, x) = (2, 4, 3)$, we get $\mathbb{Q}(\sqrt{63}) = \mathbb{Q}(\sqrt{7})$. For $n > 1$, this is the only possible integral x by (6). Moreover, if n is odd, then by the following result, there are no solutions to Equation (7) for any $x \in \mathbb{Q}$.

Theorem 1 *If $a, k, m, r \in \mathbb{N}$ with $k > 1$, then there exists an $x \in \mathbb{Q}$ such that*

$$\frac{ak^{r+2m}-1}{ak^r-1} = x^2 \tag{8}$$

if and only if m is even, $a = \frac{1}{4}(3^{m-1}+1)$, $k = 3$, $r = 1$, and $x = 3^m + 2$.

Proof. This is [2, Proposition 14]. □

Theorem 1 allows us to look at more general results than that given above, again by looking at Richaud-Degert types, as follows.

Suppose that $a, k, x \in \mathbb{N}$ with $k > 1$ and D is not a perfect square, where

$$D = ak^n(ak-1) = x^2(ak-1)^2 + ak - 1. \tag{9}$$

Then $\ell = \ell(\sqrt{D}) = 2$. Therefore, if D is given as in (9), then $\ell = 2$ if and only if $k = 3$ and there exists and $s \in \mathbb{N}$ such that $n = 4s+1$, $a = (3^{2s-1}+1)/4$, and $x = 3^{2s} + 2$.

Other than the two counterexamples cited herein to [1], there are no others. Moreover, the power of the results in [1] may be illustrated by the following generalization of some results in [8], as follows.

Theorem 2 *Suppose that $k, m, n, x \in \mathbb{N}$, with k a non-trivial prime power, $x > 1$, $m > n \geq 0$, and $(m, n) \neq (1, 0)$. Then the Diophantine equation*

$$x^2 = 4k^m - 4k^n + 1$$

has solutions if and only if

$$\begin{aligned}(k,m,n,x) \in \{&(7,3,0,37),(2,3,1,5),(2,5,1,11),\\ &(2,13,1,181),(3,5,1,31),(5,7,1,559)\}\end{aligned}$$

Proof. This follows from the results in [1] since $4k^n - 1 = D_1 = 4k^m - x^2$. □

Remark 1 *For the benefit of the reader, we employ this remark to point out where the errors lie in* [1], *and where they do not, so that those working in the area may be aware of what is valid for use. Fortunately, as noted above, it appears that only the two aforementioned counterexamples apply. In other words, the reference to Le's works* [3]–[5], *and the attendant incorrect claim that $\mathcal{N}(1,2,1,3) \leq 2$, when in fact $\mathcal{N}(1,2,1,3) = 3$, is the first mistake. The* *second is of the authors' own making as follows. With respect to* $\mathcal{N}(1,6,1,7)$*, the error occurs on page 71, line -6 of* [1]*, wherein the authors state: "The case* $t_6 = 4$ *implies that* $\lambda = 2$ *and* $(D_1, D_2, k) = (7,1,2)$*." In fact, there is another (overlooked) case, namely* $\lambda = 1$ *and* $(D_1, D_2, k) = (6,1,7)$*, which is our counterexample. In the authors' notation from* [1]*, for the nontrivial case (namely* $(n,k,x) = (4,7,20)$*, the trivial case being* $(n,k,x) = (1,7,1)$*):*

$$\lambda = \lambda' = {\lambda_1}' = \lambda_2' = x_1 = X_1 = Y_1 = Z_1 = t_6' = n_1 = 1;$$

$$\lambda_1 = \sqrt{-1},\ \ \lambda_2 = -1,\ \ t = n = t_6 = 4,\ \ k = 7,\ \ x = 20;$$
$$\varepsilon_1 = \sqrt{6} + \sqrt{-1}; \qquad \overline{\varepsilon}_1 = \sqrt{6} - \sqrt{-1};$$

and

$$20\sqrt{6} + \sqrt{-1} = \sqrt{-1} \cdot \overline{\varepsilon}_1;$$

where ε_1 *is the minimal solution in its class and* $20\sqrt{6} + \sqrt{-1}$ *is the other solution of* $6x^2 + 1 = 7^n$*, namely for* $(x,n) = (20,4)$*. Moreover, the solutions are not in any of the classes* $\mathcal{F}$*,* $\mathcal{G}$*, or* $\mathcal{H}$*, which the authors cite as the only other possibilities in* [1, Theorem 1, p. 58]*. Hence, the aforementioned two counterexamples close the door on the investigation of the Diophantine equation of the title.*

The following result will provide a segue to the conclusion of this note.

Proposition 1 *Let* $k, x \in \mathbb{N}$ *with* $k > 2$ *and let* p *be an odd prime. Then*

$$D = (k-2)x^2 + 1 = k^p - x^2$$

if and only if $p = 5$*,* $x = 11$*, and* $k = 3$*, namely* $D = 122$*.*

Proof. This is a consequence of Theorem 1, since $(k^p - 1)/(k-1) = x^2$. □

The interesting thing about Proposition 1 is that it has, as a consequence, the only solution of $3^p = 2x^2 + 1$ is that given therein. This takes us full circle to the counterexample to [1] given in [6], which, as we have seen, is actually a consequence of results from Ljunggren in the early 1940s. Hence, this is a fitting place to conclude our discussion.

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Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta
Canada, T2N 1N4
URL: http://www.math.ucalgary.ca/˜ ramollin/
E-mail: ramollin@math.ucalgary.ca