Impulsive Stabilization for Systems of Linear Differential and Difference Equations

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1 Introduction

In this project, we are interested in the impulsive stabilization of linear differential and difference equations. In many publications, the impulsive stabilization is subject to the smallness of the impulsive matrix B (the largest absolute eigenvalue is small and in any case does not exceed one) which we are trying to avoid. In this paper, we are trying to find the largest possible eigenvalues of the impulsive matrix B.

The important results we have got in this project is that, in many cases, when some eigenvalues of the system of difference equations satisfy $|\lambda_i| < 1$, then we can stabilize the system using the matrix B with some eigenvalues $|\lambda_j{}'| > 1$. The same result is achieved for differential equations with the length of the solution $X(t)$ not growing for some directions of $X(0)$ for the nonimpulsive system (some eigenvalues of the system of differential equations satisfy $\lambda_i < 0$).

We may illustrate these results using ecosystems. For example, there are two parts of population locations in China: the northern part and the southern part. In the north, due to the cold weather, snakes are not as productive as snakes in the south. Suppose now there is a significant growth of the population of snakes in China, and biologists suggest we must control the growth of the population. If we want to control the growth speed by rearranging the snakes from the south to the north, then the matrix B can indicate us how to rearrange the snakes. In this case, if the largest eigenvalue is less than one, it means we must reduce the population. This can lead to extinction, which is sometimes undesirable in real life. In our project, in some cases, matrix B can have some eigenvalues that are greater than or equal to one, which means we can allow the population of snakes to grow while rearrange it from the south to the north, this is very likely to happen in reality. This is the main idea of this project.

This paper is organized as follows. In Section 2 basic definitions and notations are introduced. In Section 3 basic properties of stochastic matrix are reviewed. In Section 4, we will discuss the impulsive stabilization of differential equations, and largest eigenvalues we can have of impulsive matrix B. In Section 5, we will discuss the impulsive stabilization of systems of difference equations, and the largest eigenvalues we can have of impulsive matrices. Final section gives some historical notes about stochastic matrices, Markov chains, and the biographical data for mathematician A.A. Markov.

2 Definition and Notations

2.1 Definition

Definition 1
Stochastic matrix : A matrix is called **row stochastic** if its entries are nonnegative and the sum of entries in each row equals 1. A matrix is called **column stochastic** if its entries are nonnegative and the sum of entries in each column equals 1.
In this paper, stochastic matrix always refers to a column stochastic matrix.

Definition 2
Doubly stochastic matrix : A doubly stochastic matrix is a matrix $A = (a_{ij})$, such that $a_{ij} \geq 0$ and $\sum_i a_{ij} = \sum_j a_{ij} = 1$ for all i and j. In other words, both the matrix itself and its transpose are stochastic.

Definition 3
Predistance matrix [1] : An $n \times n$ matrix $A = (a_{ij})$ is called a **predistance matrix**, if there exist $p_1, p_2, \ldots \in R^d, n \geq 2$, points in some d- dimensional Euclidean space, such that $a_{ij} = \|p_i - p_j\|^2$. Furthermore, if each off-diagonal entry of the predistance matrix A is strictly positive, then we will say that A is a **nondegenerate predistance matrix**.

2.2 Notations

$e = [1 \quad 1 \quad \ldots \quad 1]^T$

A matrix with strictly positive entries, or $A > 0$, means a matrix with all its entries $a_{ij} > 0$.

$\|\cdot\|$ is the length of a vector: if $V = [\, v_1 \quad v_2 \quad \ldots \quad v_n \,]^T$, then
$\|V\| = \sqrt{|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2}$.

I is the $n \times n$ identity matrix.

3 Stochastic Matrices and Doubly Stochastic Matrices

3.1 Stochastic Matrix

Theorem 3.1 *Let $A \geq 0$, then $A^T e = e$ if and only if A is a stochastic matrix. In other words, a nonnegative matrix A is a stochastic matrix if and only if e is an eigenvector of A^T associated with the eigenvalue $\lambda = 1$.*

Proof : 1) Let us prove that if A is a stochastic matrix, then $A^T e = e$.
Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}.$$

Since A is stochastic, then we have $\sum_{i=1}^{n} a_{ij} = 1$ and $a_{ij} \geq 0$ for any $1 \leq i \leq n, 1 \leq j \leq n$. Then

$$A^T e = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \dots \\ 1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} a_{i1} \\ \sum_{i=1}^{n} a_{i2} \\ \dots \\ \sum_{i=1}^{n} a_{in} \end{bmatrix}.$$

Since $\sum_{i=1}^{n} a_{ij} = 1$ for $1 \leq j \leq n$, then we have $A^T e = e$.
2) Let us prove that if $A^T e = e$, then A is a stochastic matrix.
Assume

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}.$$

Then

$$A^T e = \begin{bmatrix} \sum_{i=1}^{n} a_{i1} & \sum_{i=1}^{n} a_{i2} & \dots & \sum_{i=1}^{n} a_{in} \end{bmatrix}^T = [1 \quad 1 \quad \dots \quad 1]^T = e,$$

so $\sum_{i=1}^{n} a_{ij} = 1$ for $1 \le j \le n$.
By definition, A is a stochastic matrix.

Theorem 3.2 $\lambda = 1$ *is always an eigenvalue of both a row stochastic matrix and a column stochastic matrix.*

Proof : For a row stochastic matrix, the proof is simple.
If A is a row stochastic matrix, then A^T is a column stochastic matrix. By theorem 3.1, we know that if A^T is a column stochastic matrix, then $(A^T)^T e = e$, which means $Ae = e$.
Thus $\lambda = 1$ is always a eigenvalue of a row stochastic matrix.

For a column stochastic matrix, the proof is also simple.
Since $\det(A - \lambda I) = \det((A - \lambda I)^T) = \det(A^T - \lambda I)$,
then A and A^T have the same characteristic polynomial and the same eigenvalues. Thus $\lambda = 1$ is always a eigenvalues of a column stochastic matrix, which completes the proof.

Theorem 3.3 [2] *If A and B are stochastic matrices of the same size, then AB is also a stochastic matrix.*

Proof : 1) Let A, B be two stochastic matrices

$$A^T e = e, B^T e = e,$$

Then we can get $(A \cdot B)^T e = B^T \cdot A^T \cdot e = B^T \cdot \left(A^T \cdot e\right) = B^T \cdot e = e$.
2) We also know that all the entries of matrix A and B are positive, then it is obvious that all the entries of AB are also positive.
So AB is a stochastic matrix.

Theorem 3.4 *If A is a stochastic matrix, then A^n is also a stochastic matrix.*

Proof : We will proof this by induction
1) A is a stochastic matrix.

2) If A^n is a stochastic matrix,
then we have $A^{n+1} = A^n \cdot A$,
where $A^n,\ A$ are stochastic matrices of the same size.
By Theorem 3.3, A^{n+1} is a stochastic matrix as a product of two stochastic matrices.
So if A is a stochastic matrix, A^n is also a stochastic matrix.

Theorem 3.5 *If A, B are stochastic matrices, $a,\ b > 0$, and $a + b = 1$, then $aA + bB$ is also a stochastic matrix.*

Proof : Let A, B be two $n \times n$ stochastic matrices $A = (a_{ij})\,, B = (b_{ij})$.
Since A,B are stochastic matrices, then $\sum_{i=1}^{n} a_{ij} = 1$ and $a_{ij} \geq 0$,
$\sum_{i=1}^{n} b_{ij} = 1$ and $b_{ij} \geq 0$ for any $1 \leq i \leq n, 1 \leq j \leq n$.

a A=$\begin{bmatrix} a \cdot a_{11} & a \cdot a_{12} & \dots & a \cdot a_{1n} \\ a \cdot a_{21} & a \cdot a_{22} & \dots & a \cdot a_{2n} \\ \dots & \dots & \dots & \dots \\ a \cdot a_{n1} & a \cdot a_{n2} & \dots & a \cdot a_{nn} \end{bmatrix}$, b B= $\begin{bmatrix} b \cdot b_{11} & b \cdot b_{12} & \dots & b \cdot b_{1n} \\ b \cdot b_{21} & b \cdot b_{22} & \dots & b \cdot b_{2n} \\ \dots & \dots & \dots & \dots \\ b \cdot b_{n1} & b \cdot b_{n2} & \dots & b \cdot b_{nn} \end{bmatrix}$,

so the sum of the j-th column of $aA + bB$ is

$$\begin{aligned} \sum_{i=1}^{n} a \cdot a_{ij} + b \cdot b_{ij} &= \sum_{i=1}^{n} a \cdot a_{ij} + \sum_{i=1}^{n} b \cdot b_{ij} \\ &= a \cdot \sum_{i=1}^{n} a_{ij} + b \cdot \sum_{i=1}^{n} b_{ij} = a \cdot 1 + b \cdot 1 = a + b = 1. \end{aligned}$$

And it's obvious that the entries of a A + b B are positive, so $aA + bB$ is a stochastic matrix, if A, B are stochastic matrices, $a, b > 0$ and a+b=1.

Theorem 3.6 *All eigenvalues of a stochastic matrix satisfy $|\lambda_i| \leq 1$.*

Proof : Let A be an $n \times n$ stochastic matrix, $A = (a_{ij})$, v be an eigenvector:
v=$[\, v_1 \quad v_2 \quad \dots \quad v_n \,]^T$, such that $\sum_{i=1}^{n} |v_i| = 1$.
We have $\sum_{i=1}^{n} a_{ij} = 1$ and $a_{ij} \geq 0$ for any $1 \leq i \leq n, 1 \leq j \leq n$.
Let

$$W = Av = [\, w_1 \quad w_2 \quad \dots \quad w_n \,]^T,$$

where $w_i = \sum_{j=1}^{n} a_{ij} v_j$.

$Av = \lambda v$ implies

$\sum_{i=1}^{n} |w_i| = \sum_{i=1}^{n} |\lambda \cdot v_i| = \sum_{i=1}^{n} |\lambda| \, |v_i| = |\lambda| \sum_{i=1}^{n} |v_i|$

and $\sum_{i=1}^{n} |w_i| = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} v_j \right|$

$\leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij} v_j| = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \, |v_j| = \sum_{j=1}^{n} |v_j| \sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} |v_j| \cdot 1 = \sum_{j=1}^{n} |v_j|.$

Since $\sum_{i=1}^{n} |v_i| = 1$, then $\sum_{i=1}^{n} |w_i| \leq 1$.

Thus $|\lambda| \sum_{i=1}^{n} |v_i| = \sum_{i=1}^{n} |w_i| \leq 1$, or $|\lambda| \leq \frac{1}{\sum_{i=1}^{n} |v_i|} = 1$.

Finally, $|\lambda| \leq 1$, which completes the proof.

Theorem 3.7 [3] *If $A > 0$, A^T has the greatest eigenvalue $\lambda = 1$ and a positive associated eigenvector, then A is similar to a stochastic matrix.*

Proof : We know that A^T has the greatest eigenvalue $\lambda = 1$, $a_{ij} > 0$

Let $A^T z = z$,

$z = [\, z_1 \quad z_2 \quad \ldots \quad z_n \,]^T$, where $z_i > 0$.

Let P=$\begin{bmatrix} {z_1}^{-1} & 0 & 0 & 0 \\ 0 & {z_2}^{-1} & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & {z_n}^{-1} \end{bmatrix}$, A=$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$.

Then $P^{-1} = \begin{bmatrix} z_1 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & z_n \end{bmatrix}$,

$$P^{-1}AP = \begin{bmatrix} a_{11} z_1 {z_1}^{-1} & a_{12} z_1 {z_2}^{-1} & \ldots & a_{1n} z_1 {z_n}^{-1} \\ a_{21} z_2 {z_1}^{-1} & a_{22} z_2 {z_2}^{-1} & \ldots & a_{2n} z_2 {z_n}^{-1} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} z_n {z_1}^{-1} & a_{n2} z_n {z_2}^{-1} & \ldots & a_{nn} z_n {z_n}^{-1} \end{bmatrix}.$$

So the sum of the j-th column is $\left(\sum_{i=1}^{n} a_{ij} z_i \right) \cdot {z_j}^{-1}$.

We know that $A^T z = z$, so $\left(\sum_{i=1}^{n} a_{ij} z_i\right) = z_j$,
then $\left(\sum_{i=1}^{n} a_{ij} z_i\right) \cdot {z_j}^{-1} = z_j \cdot {z_j}^{-1} = 1$.

Thus the sum of the column of $P^{-1}AP$ is 1 , and obviously $P^{-1}AP$ has no negative entries, which means $P^{-1}AP$ is a column stochastic matrix.
Hence A is similar to a stochastic matrix, which completes the proof.

Remark 3.1 Not all eigenvalues of a stochastic and even of a doubly stochastic matrix are distinct.
Example : Take A as an $n \times n$ identity matrix, then A is a stochastic and even a doubly stochastic matrix, but

$$\lambda_1 = \lambda_2 = \ldots = \lambda_n = 1.$$

Remark 3.2 Stochastic matrices can also have negative eigenvalues and complex eigenvalues.
Example : 1) Let A=$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
Let $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$,
$(-\lambda)^2 - 1 = 0 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda_1 = 1, \lambda_2 = -1$.
So stochastic matrix can have negative eigenvalues.
2): Let A= $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.
Then $|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$.
By simplification, we get $-(\lambda^3 - 1) = 0$,
$\lambda_1 = 1; \lambda_2 = \left(-1 + \sqrt[2]{3}i\right)/2; \lambda_3 = \left(-1 - \sqrt[2]{3}i\right)/2$.
So stochastic matrices can have complex eigenvalues.

Remark 3.3 Not all stochastic matrices are diagonalizable.
Example : Let A=$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix}$.
Since diagonal entries of a triangular matrix are its eigenvalues, so

$$\lambda_1 = 1, \lambda_{2,3} = \frac{1}{2}.$$

$$A - \tfrac{1}{2}I = \begin{bmatrix} \frac{1}{2} - \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} - \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

$\mathrm{Rank}\Big(A - \frac{1}{2}I\Big) = 2$,
so the eigenspace will be one-dimensional.
A is not diagonalizable.

Remark 3.4 For the 2×2 case, if A is a stochastic matrix, and $A \neq I$, then A^{-1} is not a stochastic matrix.

We can support this statement easily.
Let $A = \begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix}$,
where $0 \leq a \leq 1$,$0 \leq b \leq 1$, and $a = 1, b = 0$ will not occur at the same time.
So $A^{-1} = \dfrac{1}{a(1-b)-(1-a)b} \begin{bmatrix} 1-b & -b \\ -(1-a) & a \end{bmatrix}$
As we can see $1 - b \geq 0, a \geq 0, -b \leq 0, -(1-a) \leq 0$.
No matter what sign $\dfrac{1}{(a)(1-b)-(1-a)(b)}$ has, A^{-1} will have some negative entries unless $a = 1, b = 0$. So A^{-1} will not be a stochastic matrix.

3.2 Doubly Stochastic Matrices

Here we present some results on doubly stochastic matrices without proofs.

Theorem 3.8 [1] *If A can be scaled to a doubly stochastic matrix, i.e, $B = DAE$, where D and E are strictly positive diagonal matrices, then B is the only doubly stochastic matrix to which A can be scaled.*

Theorem 3.9 [1] *For every nondegenerate predistance matrix A, there is a doubly stochastic matrix B, such that B=DAE, where D and E are strictly positive diagonal matrices.*

Theorem 3.10 [1] *If A is a nondegenerate matrix, then there exists a unique strictly positive diagonal matrix C, such that CAC is doubly stochastic.*

4 Impulsive Stabilization for Systems of Linear Differential Equations

In this section, we are going to discover some applied results of this project. First we assume A is a symmetric stochastic matrix, and moments of impulses $0 < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots$, satisfy $\lim\limits_{t \to \infty} \tau_k = \infty$.
We consider the linear system of differential equations:

$$X'(t) = AX(t), \tag{1}$$

which is subject to impulsive perturbation

$$X({\tau_k}^+) = BX(\tau_k), \tag{2}$$

with the initial condition

$$X(0) = X_0. \tag{3}$$

We would like to find the greatest possible matrix B, such that the solution of (1), (2), does not significantly grow for any $t > 0$.
In many results the impulsive stabilization is subject to the smallness of B which we are trying to avoid in this project, once the solution of the nonimpulsive system (1) is not growing in all directions (matrix A has $\lambda_k < 0$ for some k).

Theorem 4.1 *Suppose A is a symmetric stochastic matrix, then*

$$\|X(t)\| \le e^t \|X_0\|, \tag{4}$$

where X(t) is the solution of the initial problem (1), (3).

Proof : For an $n \times n$ symmetric stochastic matrix A all eigenvalues are real, so we can always find n normalized orthogonal eigenvectors: $v_1, v_2, \ldots, v_n$ associated with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.
The solution of $X'(t) = AX(t)$ is
$X(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \ldots + c_n e^{\lambda_n t} v_n$,
where $\{v_1, v_2, \ldots, v_n\}$ is an othonormal basis of Euclidean space R^n , $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A.

Since A is a stochastic matrix, then $\lambda_1 = 1$, $|\lambda_i| \le 1$ for $i > 1$.
So $X(t) = c_1 e^t v_1 + c_2 e^{\lambda_2 t} v_2 + \ldots + c_n e^{\lambda_n t} v_n$, where

$$\begin{aligned} X_0 &= c_1 e^0 v_1 + c_2 e^0 v_2 + \ldots + c_n e^0 v_n \\ &= c_1 v_1 + c_2 v_2 + \ldots + c_n v_n. \end{aligned}$$

Thus $\|X_0\| = \sqrt{(c_1 v_1 + c_2 v_2 + \ldots + c_n v_n)^2} = \sqrt{{c_1}^2 + {c_2}^2 + \ldots + {c_n}^2}$, since $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis.

$$\begin{aligned} \|X(t)\| &= \sqrt{(c_1 e^t v_1 + c_2 e^{\lambda_2 t} v_2 + \ldots + c_n e^{\lambda_n t} v_n)^2} \\ &= \sqrt{{c_1}^2 e^{2t} + {c_2}^2 e^{2t\lambda_2} + \ldots + {c_n}^2 e^{2t\lambda_n}} \\ &= e^t \sqrt{{c_1}^2 + {c_2}^2 e^{2t(\lambda_2 - 1)} + \ldots + {c_n}^2 e^{2t(\lambda_n - 1)}} \end{aligned}$$

Since $|\lambda_i| \leq 1$, then $\lambda_i - 1 \leq 0$ and $e^{\lambda_i - 1} \leq e^0 = 1$,
so

$$\sqrt{{c_1}^2 + {c_2}^2 e^{2t(\lambda_2 - 1)} + \ldots + {c_n}^2 e^{2t(\lambda_n - 1)}} \leq \sqrt{{c_1}^2 + {c_2}^2 + \ldots + {c_n}^2} = \|X_0\|,$$

and $\|X(t)\| \leq e^t \cdot \|X_0\|$, which completes the proof.

Theorem 4.2 *Let A be a symmetric stochastic matrix, and $\lim_{t \to \infty} \tau_k = \infty$. We can find a matrix B, such that for any $L > 0$ there exists $T > 0$: for any $t > T$, the solution of* (1), (2) *satisfies*

$$\|X(t)\| \leq L \|X_0\|. \tag{5}$$

Proof : Since A is a stochastic matrix, we can always find n orthonormal eigenvectors: $v_1, v_2, \ldots, v_n$ associated with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.
So the solution of $X'(t) = AX(t)$ is
$X(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 \cdots + c_n e^{\lambda_n t} v_n$,
where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of A, $\{v_1, v_2 \cdots, v_n\}$ is an orthonormal basis. Suppose the largest distance between impulses is τ_{max}. Now we choose matrix B by letting B have the same eigenvectors as A, but associated with different eigenvalues ${\lambda_1}', {\lambda_2}' \cdots, {\lambda_n}'$, which satisfy
$\left|\lambda'_j\right| < e^{-\lambda_j \tau_{max}} \quad 1 \leq j \leq n$.
So $B = PDP^{-1}$, where

$$P = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \text{ and } D = \begin{bmatrix} {\lambda_1}' & 0 & 0 & 0 \\ 0 & {\lambda_2}' & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & {\lambda_n}' \end{bmatrix}.$$

We will prove that B is the matrix which we are looking for.

After adding the impulsive condition $X({\tau_k}^+)$, we have
$X({\tau_1}^+) = B(c_1 e^{\lambda_1 \tau_1} v_1 + c_2 e^{\lambda_2 \tau_1} v_2 \cdots + c_n e^{\lambda_n \tau_1} v_n)$

$= c_1 e^{\lambda_1 \tau_1}(\lambda_1{}' v_1) + c_2 e^{\lambda_2 \tau_1}(\lambda_2{}' v_2) \cdots + c_n e^{\lambda_n \tau_1}(\lambda_n{}' v_n)$
$X(\tau_2{}^+) = c_1 e^{\lambda_1 \tau_2}(\lambda_1{}')^2 v_1 + c_2 e^{\lambda_2 \tau_2}(\lambda_2{}')^2 v_2 + \cdots + c_n e^{\lambda_n \tau_2}(\lambda_n{}')^2 v_n$
$\cdots\cdots\cdots$
$X(\tau_k{}^+) = c_1 e^{\lambda_1 \tau_k}(\lambda_1{}')^k v_1 + c_2 e^{\lambda_2 \tau_k}(\lambda_2{}')^k v_2 + \cdots + c_n e^{\lambda_n \tau_k}(\lambda_n{}')^k v_n$
If $t \in [\tau_k, \tau_{k+1})$, then
$X(t) = c_1 e^{\lambda_1 t}(\lambda_1{}')^k v_1 + c_2 e^{\lambda_2 t}(\lambda_2{}')^k v_2 \cdots + c_n e^{\lambda_n t}(\lambda_n{}')^k v_n,$
$\|X(t)\| = \sqrt{c_1{}^2(\lambda_1{}')^{2k} e^{2\lambda_1 t} + c_2{}^2(\lambda_2{}')^{2k} e^{2\lambda_2 t} \cdots + c_n{}^2(\lambda_n{}')^{2k} e^{2\lambda_n t}},$
and we know that $\|X_0\| = \sqrt{c_1{}^2 + c_2{}^2 \cdots + c_n{}^2}$, since $\{v_1, v_2 \cdots, v_n\}$ is an orthonormal basis.
We can see from the above equation $\|X(t)\|$ gets larger if t gets larger, if $t \in [\tau_k, \tau_{k+1})$, then $\|X(t)\|$ reaches to its maximum at τ_{k+1}.
Besides,
$\|X(\tau_{k+1})\| = \sqrt{c_1{}^2(\lambda_1{}')^{2k} e^{2\lambda_1 \tau_{k+1}} + c_2{}^2(\lambda_2{}')^{2k} e^{2\lambda_2 \tau_{k+1}} \cdots + c_n{}^2(\lambda_n{}')^{2k} e^{2\lambda_n \tau_{k+1}}}$
$\leq \sqrt{c_1{}^2(\lambda_1{}')^{2k} e^{2(k+1)\lambda_1 \tau_{max}} + c_2{}^2(\lambda_2{}')^{2k} e^{2(k+1)\lambda_2 \tau_{max}} \cdots + c_n{}^2(\lambda_n{}')^{2k} e^{2(k+1)\lambda_n \tau_{max}}}$
Since $|\lambda_j| < 1$, then
$\|X(\tau_{k+1})\|$
$\leq \sqrt{c_1{}^2(\lambda_1{}')^{2k} e^{2k\lambda_1 \tau_{max} + 2\tau_{max}} + c_2{}^2(\lambda_2{}')^{2k} e^{2k\lambda_2 \tau_{max} + 2\tau_{max}} \cdots + c_n{}^2(\lambda_n{}')^{2k} e^{2k\lambda_n \tau_{max} + 2\tau_{max}}}$
$= e^{\tau_{max}} \sqrt{c_1{}^2(\lambda_1{}')^{2k} e^{2k\lambda_1 \tau_{max}} + c_2{}^2(\lambda_2{}')^{2k} e^{2k\lambda_2 \tau_{max}} \cdots + c_n{}^2(\lambda_n{}')^{2k} e^{2k\lambda_n \tau_{max}}}.$
As we choose $\left|\lambda_j'\right| < e^{-\lambda_j \tau_{max}}$, so $\left|\lambda_j' e^{\lambda_j \tau_{max}}\right| < 1$.
Then let $a = \max\{\left|\lambda_1' e^{\lambda_1 \tau_{max}}\right|, \left|\lambda_2' e^{\lambda_2 \tau_{max}}\right| \cdots, \left|\lambda_n' e^{\lambda_n \tau_{max}}\right|\}$, where we have $a < 1$.
$\|X(\tau_{k+1})\|$

$$\leq e^{\tau_{max}} a^k \sqrt{c_1{}^2 \left(\frac{\lambda_1{}' e^{\lambda_1 \tau_{max}}}{a}\right)^{2k} + c_2{}^2 \left(\frac{\lambda_2{}' e^{\lambda_2 \tau_{max}}}{a}\right)^{2k} + \cdots + c_n{}^2 \left(\frac{\lambda_n{}' e^{\lambda_n \tau_{max}}}{a}\right)^{2k}}$$

and $\left|\dfrac{\lambda_j{}' e^{\lambda_j \tau_{max}}}{a}\right| \leq 1$, so
$\|X(\tau_{k+1})\| \leq e^{\tau_{max}} a^k \|X_0\|$.
Since $0 < a < 1$, then $L(k) = e^{\tau_{max}} a^k \to 0$ as $k \to \infty$.
Thus for any $L > 0$ we can find $k > 0$, such that $L(k) = e^{\tau_{max}} a^k \leq L$, then choose $t \in [\tau_k, \tau_{k+1})$, so for any $t > T$ we have $\|X(t)\| \leq L \|X_0\|$.

Corollary 4.1 Let us choose $L < 1$, then we can always find

$$\|X(t)\| < L \|X_0\| \quad \text{for} \quad t > T_1, \tag{6}$$

$$\|X(t)\| < L \|X(T_1)\| < L^2 \|X_0\| \quad \text{for} \quad t > T_2, \tag{7}$$

and so on, where $X(t)$ is the solution of (1) and (2). This means

$$\lim_{t\to\infty} \|X(t)\| = 0 \tag{8}$$

for the choice of matrix B as in the proof of Theorem 4.2.

Let us demonstrate that eigenvalues of B are not necessarily less than 1. From Theorem 4.2, we know that we can always find such a matrix B, which has the same eigenvectors as A, but associated with different eigenvalues, $|\lambda_j{}'| < e^{-\lambda_j \tau_{max}}$. Here λ_j is the eigenvalue of A, $\lambda_j{}'$ is the eigenvalue of B associated with the same eigenvector as λ_j.

Corollary 4.2 Suppose A has at least k negative eigenvalues, then there exists a matrix B, such that B has at least k eigenvalues, $\lambda_j{}' > 1$.
Proof : $e^{-\lambda_j \tau_{max}} > 1$, where
$\lambda_j < 0$, and we know $\tau_{max} > 0$.
Then we can always choose $\lambda_j{}' > 1$,
since matrix B has its eigenvalues, such that $|\lambda_j{}'| < e^{-\lambda_j \tau_{max}}$.

Remark 4.1 In the 2×2 case, we can always guarantee such a matrix B with strictly positive entries.
Proof : Since A is a symmetric stochastic matrix,
let $A = \begin{bmatrix} a & 1-a \\ 1-a & a \end{bmatrix}$, where $0 \le a \le 1$.
First, let us find its eigenvalues.
Then, let $|A - \lambda I| = 0$.
$\begin{vmatrix} a-\lambda & 1-a \\ 1-a & a-\lambda \end{vmatrix} = 0 \Rightarrow (a-\lambda)(a-\lambda) - (1-a)(1-a) = 0$
$\Rightarrow a^2 - 2\lambda a + \lambda^2 - 1 + 2a - a^2 = 0 \quad \Rightarrow \lambda^2 - 2\lambda a + (2a-1) = 0$
$(\lambda - 1)(\lambda - 2a + 1) = 0 \qquad \Rightarrow \lambda_1 = 1; \lambda_2 = 2a - 1.$
Then, we can find its eigenvectors :
1) With $\lambda_1 = 1$, we have $A - \lambda I = \begin{bmatrix} a-1 & 1-a \\ 1-a & a-1 \end{bmatrix}$.
So we can choose $v_1 = [1 \quad 1]^T$.
2) With $\lambda_2 = 2a - 1$, we have $A - \lambda I = \begin{bmatrix} -a+1 & 1-a \\ 1-a & -a+1 \end{bmatrix}$.
So we can choose $v_2 = [1 \quad -1]^T$.

Thus we get $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and $P^{-1} = \dfrac{1}{-1-1} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \dfrac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$,
$D = \begin{bmatrix} {\lambda_1}' & 0 \\ 0 & {\lambda_2}' \end{bmatrix}$.
So $B = PDP^{-1}$

$$= \tfrac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} {\lambda_1}' & 0 \\ 0 & {\lambda_2}' \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \tfrac{1}{2} \begin{bmatrix} {\lambda_1}' + {\lambda_2}' & {\lambda_1}' - {\lambda_2}' \\ {\lambda_1}' - {\lambda_2}' & {\lambda_1}' + {\lambda_2}' \end{bmatrix}$$

$\Rightarrow {\lambda_1}' > {\lambda_2}' > 0$
or $\quad {\lambda_1}' > -{\lambda_2}' > 0$.
Therefore we can guarantee a matrix B with strictly positive entries, by choosing ${\lambda_1}' > {\lambda_2}' > 0$ or ${\lambda_1}' > -{\lambda_2}' > 0$.
However, since for stabilization we should have ${\lambda_1}' < 1$, then the requirement of the positiveness for entries of matrix B reduces possible eigenvalues ${\lambda_i}'$.

Remark 4.2 In 2×2 case, if A has positive eigenvalues $\lambda_1 = 1, 0 < \lambda_2 < 1$, then we *can not* find such a matrix B with one of its eigenvalues ${\lambda_i}' \geq 1$.

Suppose ${\lambda_1}' \geq 1$.
Then we have $Bw = {\lambda_1}' w$.
Let $w = c_1 e^{\lambda_1 \tau_1} v_1 + c_2 e^{\lambda_2 \tau_1} v_2$,
where $\{v_1, v_2\}$ is an orthonormal basis composed by A's eigenvectors.
Since the solution of $X'(t) = AX(t)$ is
$X(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 \Rightarrow \|X_0\| = \sqrt{{c_1}^2 + {c_2}^2}$.
After adding the impulsive condition we have

$$\begin{aligned} X({\tau_1}^+) &= B(c_1 e^{\lambda_1 \tau_1} v_1 + c_2 e^{\lambda_2 \tau_1} v_2) \\ &= Bw \\ &= {\lambda_1}' w \\ &= {\lambda_1}'(c_1 e^{\lambda_1 \tau_1} v_1 + c_2 e^{\lambda_2 \tau_1} v_2). \end{aligned}$$

$$\begin{aligned} \text{And } \|X({\tau_1}^+)\| &= {\lambda_1}' \sqrt{{c_1}^2 e^{2\lambda_1 \tau_1} + {c_2}^2 e^{2\lambda_2 \tau_1}} \\ &\geq \lambda_1 e^{\lambda_2 \tau_1} \sqrt{{c_1}^2 + {c_2}^2} > \|X_0\|. \end{aligned}$$

So, it is impossible to find such a B matrix.

5 Impulsive Stabilization for Systems of Linear Difference Equations

In this section, again we assume A is a symmetric stochastic matrix, and consider the linear system of difference equations:

$$X_{n+1} = AX_n, \tag{9}$$

which is subject to impulsive perturbations at each k-th step

$$X_{mk^+} = BX_{mk}, m = 1, 2, 3, \ldots \tag{10}$$

Here X_0 is given.
This time, we are still interested in the largest possible eigenvalues of the impulsive matrix B which can reduce the growth of the solution of (9), (10).

Theorem 5.1 *Suppose A is a symmetric stochastic matrix, X_0 is an eigenvector of A. Then for any solution of system* (9) *we have either $\|X_n\| = \|X_0\|$ or $\|X_n\| \to 0$ as $n \to \infty$.*

Proof : As we know from theorem 3.6, the stochastic matrix A always has eigenvalues $|\lambda_i| \leq 1$.
1) If X_0 is an eigenvector associated with eigenvalue λ, $|\lambda| = 1$,
then $\|X_1\| = \|AX_0\| = |\lambda| \, \|X_0\| = \|X_0\|$
$\|X_2\| = \|AX_1\| = |\lambda| \, \|X_1\| = \|X_1\| = \|X_0\|$
$\cdots\cdots$
$\|X_n\| = \|X_{n-1}\| = \cdots = \|X_1\| = \|X_0\|$

2) If X_0 is an eigenvector associated with eigenvalue λ, $|\lambda| < 1$,
then $\|X_1\| = \|AX_0\| = |\lambda| \, \|X_0\|$
$\|X_2\| = \|AX_1\| = |\lambda| \, \|X_1\| = |\lambda| \, |\lambda| \, \|X_0\| = |\lambda|^2 \, \|X_0\|$
$\cdots\cdots$
$\|X_n\| = |\lambda| \, \|X_{n-1}\| = \cdots = |\lambda|^{n-1} \, \|X_1\| = |\lambda|^n \, \|X_0\|$
Since $|\lambda|^n \to 0$ as $n \to \infty$ for any $|\lambda| < 1$, then $\|X_n\| \to 0$ as $n \to \infty$, which completes the proof.

Comments on Theorem 5.1
The following theorem can give us a more general idea of the length of $X(n)$,

when $n \to \infty$. This theorem is due to [4], [5].

Theorem If X is the solution of (9), then either $X(n) = 0$ for all large n or

$$\lambda_i = \lim_{n\to\infty} \sqrt[n]{\|X(n)\|}, \tag{11}$$

where λ_i is one of the eigenvalues of matrix A.

Theorem 5.2 *Suppose A is a symmetric stochastic matrix. We can find matrix B, such that for any $L > 0$ there exists n_0: for $n > n_0$, the solution of* (9), (10) *satisfies $\|X_n\| \le L\,\|X_0\|$.*

Proof : Since A is an $n \times n$ symmetric stochastic matrix, then we can always find n normalized orthogonal eigenvectors $v_1, v_2, \ldots, v_n$ associated with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Thus X_0 can be written as $X_0 = \sum_{i=1}^{n} c_i v_i$, and $\|X_0\| = \sqrt{\sum_{i=1}^{n} {c_i}^2}$, since $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis.

Let B have the same eigenvectors as A, but associated with different eigenvalues ${\lambda_1}', {\lambda_2}', \cdots, {\lambda_n}'$, where $|{\lambda_j}'| < |\lambda_j|^{-k}, 1 \le j \le n$.

So $B = PDP^{-1}$, where

$P = [\, v_1 \quad v_2 \quad \cdots \quad v_n \,]$ and $D = \begin{bmatrix} {\lambda_1}' & 0 & 0 & 0 \\ 0 & {\lambda_2}' & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & {\lambda_n}' \end{bmatrix}$

Let us prove B is the matrix which satisfies the requirements of the theorem.

We have $X_k = A^k X_0 = A^k \sum_{i=1}^{n} c_i v_i = \sum_{i=1}^{n} c_i {\lambda_i}^k v_i$,

$X_{k^+} = BX_{kn} = B \sum_{i=1}^{n} c_i {\lambda_i}^k v_i = \sum_{i=1}^{n} c_i {\lambda_i}' {\lambda_i}^k v_i$,

……

$X_{mk^+} = \sum_{i=1}^{n} c_i ({\lambda_i}' {\lambda_i}^k)^m v_i$, $\|X_{mk^+}\| = \sqrt{\sum_{i=1}^{n} {c_i}^2 ({\lambda_i}')^{2m} {\lambda_i}^{2mk}}$,

$X_{(m+1)k} = \sum_{i=1}^{n} c_i ({\lambda_i}')^m {\lambda_i}^{(m+1)k} v_i$, and $\left\|X_{(m+1)k}\right\| = \sqrt{\sum_{i=1}^{n} {c_i}^2 ({\lambda_i}')^{2m} {\lambda_i}^{2(m+1)k}}$.

For any n_0, such that $mk \le n_0 < (m+1)k$,

we have $X_{n_0} = \sum_{i=1}^{n} c_i (\lambda_i')^m {\lambda_i}^{n_0} v_i$, so $\|X_{n_0}\| = \sqrt{\sum_{i=1}^{n} {c_i}^2 ({\lambda_i}')^{2m} {\lambda_i}^{2n_0}}$.
As A is a stochastic matrix, then $|\lambda_i| \leq 1$. Besides, $mk \leq n_0 < (m+1)k$,
so we can see that $\sqrt{\sum_{i=1}^{n} {c_i}^2 ({\lambda_i}')^{2m} {\lambda_i}^{2(m+1)k}} \leq \sqrt{\sum_{i=1}^{n} {c_i}^2 ({\lambda_i}')^{2m} {\lambda_i}^{2n_0}} \leq \sqrt{\sum_{i=1}^{n} {c_i}^2 ({\lambda_i}')^{2m} {\lambda_i}^{2mk}}$,
which means $\left\|X_{(m+1)k}\right\| \leq \|X_{n_0}\| \leq \|X_{mk^+}\|$.
Let $a = \max\{\left|{\lambda_1}'{\lambda_1}^k\right|, \left|{\lambda_2}'{\lambda_2}^k\right|, \ldots, \left|{\lambda_n}'{\lambda_n}^k\right|\}$,
then $\|X_{mk^+}\| = a^m \sqrt{\sum_{i=1}^{n} {c_i}^2 \left(\frac{{\lambda_i}'{\lambda_i}^k}{a}\right)^{2m}} \leq a^m \sqrt{\sum_{i=1}^{n} {c_i}^2} \Rightarrow \|X_{mk^+}\| \leq a^m \|X_0\|$
Since we choose $|{\lambda_i}'| < |\lambda_i|^{-k}$
$\Rightarrow \left|{\lambda_i}'{\lambda_i}^k\right| < 1 \Rightarrow a < 1$
$\Rightarrow L(m) = a^m \rightarrow 0$ as $m \rightarrow \infty$
Thus for any constant L, there exsits $m_0 > 0$, such that $L(m_0) = a^{m_0} \leq L$. Then we can choose $n_0 \in [m_0 k, (m_0+1)k)$, so for any $n > n_0$, $\|X_{n_0}\| < L\|X_0\|$, which completes the proof.

Remark 5.1 The proof of Theorem 5.2 is given for doubly stochastic matrices, however with insignificant changes it is appropriate for any unitarily diagonalizable matrix.

Now we want to find the largest possible eigenvalue that matrix B can have. From Theorem 5.2, we know that we can always find such a matrix B with the same eigenvectors as A, but associated with different eigenvalues, $|{\lambda_j}'| < |\lambda_j|^{-k}$, $0 < j \leq n$. Here λ_j is the eigenvalue of matrix A, and ${\lambda_j}'$ is the eigenvalue of matrix B, which is associated with the same eigenvector as λ_j.

Remark 5.2 For any given symmetric stochastic matrix A, we *can not* guarantee such a matrix B with all its eigenvalues greater than or equal to 1.

Really, If A is an identity matrix, all A's eigenvalues equal to one,
which means $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1$.
$\Rightarrow |{\lambda_j}'| < |\lambda_j|^{-k} = 1$.

Corollary 5.2 If A has at least k eigenvalues, such that $|\lambda_i| < 1$, then we can find a matrix B associated with at least k eigenvalues that are greater than 1 .
Proof : Since A is a symmetric stochastic matrix, so $|\lambda_i| < 1$ for $i = 1, 2 \cdots k$, which means $|\lambda_j|^{-k} > 1$.
Thus we can always choose $|\lambda_j{}'| \in (1, |\lambda_j|^{-k})$ for $i = 1, 2 \cdots, k$.

6 Some Historical Notes

6.1 Historical Reference About Stochastic Matrices and Markov Chains

The theory of stochastic matrices was launched by the work on Markov chains, which were named in honor of the Russian mathematician A.A. Markov.

The following historical data is due to [6].

The basic concept of a Markov chain was introduced by A.A. Markov in 1907, and since that time the literature on the subject has grown remarkably. Fundamental investigations by Kolmogorov in 1930's extended the mathematical theory to chains with an infinite number of states; Doeblin and Doob made important contributions during the period of 1935 to 1945. The present state of the theory of Markov chains is summarized by Chung [7].

By 1950 it was well recognized that Markov chain is a useful model for a multitude of physical processes, and an increasing number of applications of the mathematical theory have been made to problems in such fields as physics, chemistry, biology, and operations research. In these applications it is generally assumed that the matrix of transition probabilities is known, although, since 1954, questions of hypothesis testing and maximum-likelihood estimation have been investigated. These latter results are summarized by Billingsley [8], who gives extensive references.

Decision problems involving Markov chains have received increasing attention during the last decade. In 1953, L. S. Shapley [9], using a game-theoretic formulation, studied one of the earliest sequential decision models in a Markov chain with alternative transition probabilities, which were assumed to be known. Similar game formulations have been examined more recently by Zachrisson [10] and Shor [11]. A more general class of Markovian decision models with known transition probabilities have been investigated by Blackwell [12], Derman [13], Howard [14], and others, using the techniques of linear and dynamic programming. These models have been extended to semi-Markov processes by Howard [15] and Jewell [16,17]. Further references are given by Jewell [17].

6.2 Some Biographical Data for A.A. Markov

Most of the biographical data below is due to [18].
Andrei A. Markov was a gifted Russian mathematician, a disciple of the renowned Pafnuty Lvovich Chebyshev. At the age of 30 Markov became a professor at St. Petersburg University and a member of St. Petersburg Academy of Sciences. He published more than 120 scientific papers on number theory, continuous fraction theory, differential equations, probability theory, and statistics. His classical textbook, "Calculus of Probabilities," was published four times in Russian and was translated into German. Many of his papers were devoted to creating a new field of research, Markov chains. The solution of many fundamental problems of modern science and technology would not be possible without his contributions. It is fitting that these chains bear his name, acknowledging his trailblazing role in the development of Markov chains. In fact, as early as 1926, just twenty years after his initial discoveries, a paper by Russian mathematician S. N. Bernstein used the phrase "Markov chain".

Andrei A. Markov was born on June 14, 1856 in the town of Ryazan. In the early 1860s, the Markovs moved to St. Petersburg, where Andrei entered a classical gymnasium in 1866. He was rather poor in many subjects, but not in mathematics, toward which he showed enthusiasm and which he even studied on his own. For some time he believed he had invented a new method for the solution of linear differential equations. He informed A. N. Korkin and E. I. Zolotarev, prominent Russian mathematicians of the time, about his "discovery", but they explained to the young man that the method was in fact not new. Nevertheless, his discovery was important as it initiated his lasting relationship with Korkin and Zolotarev, two professors at the St. Petersburg University.

Markov graduated from the gymnasium in 1874 and entered the Faculty of Mechanics and Mathematics of St. Petersburg University, where Chebyshev, Korkin and Zolotarev lectured. In addition to lecturing at the university, Korkin and Zolotarev conducted special courses for the best students, which were held chiefly at their own homes. Markov was the most active attendee of these courses. In 1877, he was awarded a gold medal for his research on the topic, proposed by the faculty, "On Solution of Differential Equations with the Help of Continued Fractions".

In 1880, Markov defended his master's thesis, "On the Binary Square Forms with Positive Determinants", and, in 1884, his Doctorate, "On Certain Applications of the Algebraic Continuous Fractions". In 1886, Markov, by Chebyshev's proposal, was elected for his significant contributions to science as an adjunct member of St. Petersburg Academy of Sciences. In 1890, he was promoted to extraordinary academician, and in 1896, ordinary academician.

In 1880, as an associate professor at the university, Markov lectured on "Introduction to Analysis" and "Differential and Integral Calculus". After Chebyshev resigned from the university in 1883, Markov started teaching the probability theory course. In 1886, he was elected as an extraordinary professor, and in 1893, as an ordinary professor. In 1905, celebrating the 25th anniversary of his work at St. Petersburg University, Markov was awarded an honorary professorship. Shortly after that, he retired from the university but continued to lecture on probability theory and the theory of continuous fractions.

At an Academy session in 1889, Markov read his paper, "On a question by D. I. Mendeleev". This paper appeared the following year. It contained the proof of the now famous Markov inequality for algebraic polynomials. In 1900, the first edition of his textbook, "Calculus of Probabilities" was published. In 1923 Norbert Wiener became the first to treat rigorously a continuous Markov process. The foundation of a general theory was provided during the 1930s by Andrei Kolmogorov.

7 Comments on References

In [2], Chapter 2.10 "Stochastic Matrices and Their Applications", was applied.
In [3], chapter 2.5 "Stochastic Matrices", was applied.
in [6], chapter 1.1 "Historical Perspective", was applied.

8 Future Work

1. In this project, we only considered a symmetric stochastic matrix A, which is diagonalizable. In the future work, a general type of stochastic matrix A should be considered, which may be non-diagonalizable or may have complex eigenvalues.

2. In our project, we only considered the eigenvalues of the impulsive matrix B, but we did not require B to be a stochastic matrix. In our future work, we will need to think whether it is possible or not to find a stochastic impulsive matrix B for any given stochastic matrix A.

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