On Impulsive Beverton-Holt Difference Equations and their Applications

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Abstract

The asymptotic properties of the impulsive Beverton-Holt difference equation

$$x_{n+1} = \frac{\alpha_n x_n}{1 + B_n x_n}, \; x_{pk}^{+} = b_k x_{pk} - d_k, \; n, k = 1, 2, \dots,$$

where p is a fixed positive integer, are considered. The results are applied to an impulsive logistic equation with nonconstant coefficients

$$\dot{x}(t) = x(t)(r(t) - a(t)x(t)), \quad x(\tau_k) = b_k x(\tau_k^-) - d_k, \; \lim_{k \to \infty} \tau_k = \infty.$$

In particular, sufficient extinction and non-extinction conditions are obtained for both equations.

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1 Introduction

The simplest discrete model describing population growth is the Beverton and Holt equation [1]

$$x_{n+1} = \frac{\alpha x_n}{1 + B x_n} \tag{1}$$

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(which is also sometimes called the Pielou logistic difference equation). In some bottom-feeding fish populations, like North Atlantic plaice and haddock studied in [1], recruitment appears to be essentially unaffected by fishing, and this is true over a wide range of fishing effort. These species have very high fertility rates and very low survivorship to adulthood. The Beverton-Holt model is equivalent to the discrete Verhulst equation

$$x_{n+1} = \frac{rx_n}{x_n + A},$$

which is a discrete version of the logistic differential equation [14, 15]

$$\dot{x} = rx\left(1 - \frac{x}{K}\right),\ x(t_0) = x_0, \tag{2}$$

where $x(t)$ is the size of the resource population, r is an intrinsic growth rate, K is a carrying capacity, or saturation level; for more sophisticated models of the population growth see [2-5].

We can extend the Beverton-Holt model to the case when the recruitment is affected by some harvesting (e.g., fishing or hunting). Let us assume that at every p-th step the system is subject to a perturbation which incorporates the proportional decrease (only bx_{pk} is left at pk-th step, where $0 < b < 1$ and x_{pk} is the value before the perturbation) and a deduction d which does not depend on the size of the population x_{pk}. After the perturbation at step pk the size of the population x_{pk}^+ becomes

$$x_{pk}^+ = bx_{pk} - d, \quad k = 1, 2, \dots\ , \tag{3}$$

where x_{pk} is the size of the population at step pk before the impulsive perturbation.

The impulsive conditions include both cases of the proportional and constant harvesting. A proportional part can be interpreted as fishery or hunting with a constant fishing (hunting) effort, while a constant part can correspond to hunting with a constant number of licences or predation with a constant number of predators that control a certain territory. Alternatively, if the model describes a farm with finite resources, the proportional reduction can correspond to some kind of taxation, while the constant part can describe a producer's consumption. In both examples the problem of non-extinction of the population becomes a description of a reasonable hunting rate or of a judicious taxation.

As will be shown in the sequel, (3),(1) can be treated as a discrete analogue of the logistic equation (2) subject to short-time harvesting which is described by impulsive perturbations

$$x(\tau_n) = bx(\tau_n^-) - d, \quad \lim_{n\to\infty} \tau_n = \infty.$$

We will study the Beverton-Holt model with nonconstant coefficients and variable impulsive conditions of a linear type

$$x_{n+1} = \frac{\alpha_n x_n}{1 + B_n x_n}, \quad x_{pk}^+ = b_k x_{pk} - d_k,\ 0 < b_k \le 1, d_k \ge 0. \tag{4}$$

In the model with nonconstant coefficients (4) we assume that the hunting rate and effort are not constant (however do not depend on the size of the population). Let us note that if $d_k \equiv 0$ then all solutions of (4), with $x_0 > 0$, are positive, while $d_k > 0$ can lead to the extinction of the population.

The results on (4) are applied to the logistic equation with nonconstant growth rates and carrying capacities

$$\dot{x} = x(t)(r(t) - a(t)x(t)) \tag{5}$$

and variable impulsive conditions

$$x(\tau_n) = b_n x(\tau_n^-) - d_n, \lim_{n\to\infty} \tau_n = \infty. \tag{6}$$

Logistic equations with impulsive harvesting and general impulsive perturbations were recently considered in [6].

The model (5),(6) is more general than most of previously considered impulsively perturbed logistic systems. First, the impulsive conditions (6) do not describe the proportional harvesting only. Thus the solution can become negative in some finite time which corresponds to the extinction of the population. Secondly, functions $a(t), r(t)$ are not considered to be piecewise continuous but locally essentially bounded only. No additional condition (say, the periodicity of a, r) is imposed.

Discrete and continuous models of mathematical biology are closely connected (see, for example, the monograph [7] and the recent paper [8]): similar methods are sometimes applied to study linear difference and differential equations, continuous models are constructed on the base of discrete ones and vice versa. Impulsive differential equations incorporate elements of continuous and discrete systems. The main results of the present paper are concerned with the difference equation (4), the detailed analysis is due to the recent publications [9, 10]. Based on these results, the asymptotics of the impulsive differential equation (5),(6) is studied.

The paper is organized as follows. Section 2 includes relevant notations and auxiliary theorems on Riccati difference equations from [9, 10]. Two modifications of these results (Lemmas 2,5) are applied to the analysis of impulsive difference equations. In Section 3 an impulsive Beverton-Holt equation with constant and non-constant parameters is considered. Conditions for positiveness of solutions and asymptotics are described. In Section 4 the equivalence of the positiveness of a solution of (5),(6) and the positiveness of the solution of a specially constructed impulsive Beverton-Holt equation is established. The results of Section 3 are applied to the study of the logistic equation with impulsive harvesting, including positiveness of solutions and limit sets. Comparison results and a numerical example are presented. Section 5 discusses possible developments.

We note that delay logistic equations with a steady proportional harvesting were studied in [11], however, methods were completely different.

2 Preliminaries

We will need the following auxiliary results concerning the first order Riccati difference equation

$$w_{n+1} = 1 - \frac{R}{w_n}, \quad n \geq 0, R \neq 0. \tag{7}$$

These results were obtained in [9] and can be also found in the monograph [10].

Definition. The set of initial conditions $w_0 \in \mathbf{R}$ through which the denominator w_n will become zero for some $n \geq 0$ is called **the forbidden set F** of Eq.(7).

Definition. A point $\bar{w}$ is called an equilibrium point if $\bar{w} = 1 - \frac{R}{\bar{w}}$. An equilibrium point is called **a global attractor** if for every $w_0 \notin \mathbf{F}$ we have $\lim_{n\to\infty} w_n = \bar{w}$. An equilibrium point is called **a repeller** if there exists such $r > 0$ that for any $w_0 \notin \mathbf{F}$ satisfying $|w_0 - \bar{w}| < r$ there exists such N that $|w_N - \bar{w}| \geq r$.

Lemma 1 *[10] Let $R < \frac{1}{4}$, $R \neq 0$.*
Then

$$w_- = \frac{1 - \sqrt{1-4R}}{2} \quad \textit{and} \quad w_+ = \frac{1 + \sqrt{1-4R}}{2} \tag{8}$$

are the only equilibrium points of Eq. (7). The forbidden set F of Eq. (7) is the sequence of points

$$f_n = \left(\frac{w_+^{n-1} - w_-^{n-1}}{w_+^n - w_-^n}\right) w_+ w_-, \quad n = 1, 2, \dots .$$

If $w_0 \notin F \cup \{w_-\}$, then w_+ is a global attractor, w_- is a repeller, and the solution is given by

$$w_n = \frac{(w_0 - w_-)w_+^{n+1} - (w_+ - w_0)w_-^{n+1}}{(w_0 - w_-)w_+^n - (w_+ - w_0)w_-^n}, \ n = 1, 2, \dots . \tag{9}$$

In addition to asymptotic behavior of solutions we will need the following result.

Lemma 2 *Let $R < \frac{1}{4}$ and $w_0 > w_- > 0$ (we keep the notation w_-, w_+ of Lemma 1). Then every next element of the sequence is closer to w_+ than the previous one $|w_{n+1} - w_+| < |w_n - w_+|$ and $w_n > w_-$. Moreover, the sequence $\{w_n\}$ is monotone: if $w_n < w_+$, then $w_n \leq w_{n+1} < w_+$; if $w_n > w_+$, then $w_n \geq w_{n+1} > w_+$.*

Proof. First let us prove that $w_n > w_-$ and $\operatorname{sgn}(w_n - w_+) = \operatorname{sgn}(w_0 - w_+)$. To this end let us note that $w_+ w_- = R$. Assuming $w_n > w_-$ we have

$$w_{n+1} = 1 - \frac{R}{w_n} = 1 - \frac{w_+ w_-}{w_n} > 1 - \frac{w_+ w_-}{w_-} = 1 - w_+ = w_-,$$

so by induction $w_n > w_-$.

If $w_n < w_+$, then

$$w_{n+1} = 1 - \frac{w_+ w_-}{w_n} < 1 - \frac{w_+ w_-}{w_+} = 1 - w_- = w_+.$$

Similarly, if $w_n > w_+$, then $w_{n+1} > w_+$.

Further, since $w_{n+1} = 1 - \frac{R}{w_n}$ and $R = w_- w_+$, then

$$|w_{n+1} - w_+| = \left|1 - \frac{R}{w_n} - \frac{1}{2} - \frac{\sqrt{1-4R}}{2}\right| = \left|\frac{1 - \sqrt{1-4R}}{2} - \frac{w_- w_+}{w_n}\right|$$

$$= \left|w_- - \frac{w_- w_+}{w_n}\right| = \left|\frac{w_-(w_n - w_+)}{w_n}\right| = \left|\frac{w_-}{w_n}\right| |w_n - w_+| < |w_n - w_+|,$$

which completes the proof.

Lemma 3 *[10] Let* $R = \frac{1}{4}$*. Then the only equilibrium of Eq. (7) is* $\bar{w} = \frac{1}{2}$*. The forbidden set* F *is the sequence of points* $f_n = \frac{n-1}{2n}$ *for* $n = 1, 2, \ldots$*, which converges to the equilibrium from the left. The equilibrium point is a sink (attractor) for all* $w_0 > \bar{w}$ *and is a repeller for all* $w_0 < \bar{w}$*, where* $w_0 \notin F$*.*

Lemma 4 *[10] Let* $R > \frac{1}{4}$ *and let* $\phi \in (0, \frac{\pi}{2})$ *be such that*

$$\cos\phi = \frac{1}{2\sqrt{R}} \quad \textit{and} \quad \sin\phi = \frac{\sqrt{4R-1}}{2\sqrt{R}}.$$

Then the forbidden set F *is a sequence of points*

$$f_n = \frac{1}{2} - \frac{\sqrt{4R-1}}{2}\cot(n\phi), \ \ n = 1, 2, \ldots \ .$$

In addition, denote

$$r = \sqrt{4R - 1 + (2w_0 - 1)^2}.$$

Let $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ *be such that*

$$\cos\theta = \frac{\sqrt{4R-1}}{r}, \quad \sin\theta = \frac{2w_0 - 1}{r}.$$

Then for any $w_0 \notin F$

$$w_n = \sqrt{R}\frac{\cos(n\phi + \phi - \theta)}{cos(n\phi - \theta)}.$$

There is no equilibrium points in this case.

Consider also the first order Riccati difference equation with a nonconstant coefficient

$$w_{n+1} = 1 - \frac{R_n}{w_n}, \quad n \geq 0. \tag{10}$$

Suppose $\{c_n\}$ is a sequence of numbers. Consider the following problem: when does there exist a solution of (10) satisfying $w_n > c_n$? The following lemma presents sufficient conditions.

Lemma 5 *Suppose* $R = \sup_n R_n < \frac{1}{4}$ *and* $\{c_n\}$ *is a sequence. Denote*

$$m_n = \frac{1 - \sqrt{1 - 4R_n}}{2}, \quad M_n = \frac{1 + \sqrt{1 - 4R_n}}{2},$$

$$c = \sup_n c_n, \quad m = \sup_n m_n, \quad M = \inf_n M_n = \frac{1}{2}(1 + \sqrt{1 - 4R}), \quad \mathcal{M} = \sup_n M_n. \tag{11}$$

Suppose $c < M$*. Then any solution with the initial condition*

$$w_0 > \max\{c, m\}$$

satisfies $w_n > c_n$*. Besides,*

$$M \leq \liminf_{t\to\infty} w_n \leq \limsup_{t\to\infty} w_n \leq \mathcal{M}. \tag{12}$$

Proof. Let $w_0 > \max\{c, m\}$. As the first step, we will prove $w_n > c_n$.

Suppose first $w_0 < M$. Then $w_1 = 1 - \frac{R_0}{w_0}$ satisfies

$$M - w_1 = M - 1 + \frac{R_0}{w_0} < M_0 - 1 + \frac{R_0}{w_0} = \frac{1}{2} + \frac{\sqrt{1-4R_0}}{2} - 1 + \frac{m_0 M_0}{w_0}$$

$$= -\frac{1}{2} + \frac{\sqrt{1-4R_0}}{2} + \frac{m_0 M_0}{w_0} = m_0 \left(\frac{M_0}{w_0} - 1 \right) > 0,$$

so $w_1 < M$. Similarly, if $w_n < M$, then $w_{n+1} < M$. In addition (see the proof of Lemma 2), $|w_{n+1} - M_n| < |w_n - M_n|$, so if $w_n < M$ then

$$M - w_{n+1} = (M - M_n) + (M_n - w_{n+1}) < (M - M_n) + (M_n - w_n) = M - w_n.$$

Consequently the sequence $\{w_n\}$ is increasing as far as $w_n < M$. Hence if $w_0 < M$, then $w_n \geq w_0 > c \geq c_n$.

Now suppose $w_0 \geq M$. If all $w_n \geq M$ then obviously $w_n > c_n$; if there exists such n that $w_n < M$, then the argument is similar to the previous case $w_0 < M$.

We note that since $R_n < \frac{1}{4}$ then all m_n are positive, thus w_n are also positive.

At the second step, we will prove $M \leq \liminf_{n\to\infty} w_n \leq \limsup_{n\to\infty} w_n \leq \mathcal{M}$ for any initial condition $w_0 > m$. Note that $m_n M_n = R_n$, $m_n + M_n = 1$. If $w_n < M_n$, then

$$w_{n+1} = 1 - \frac{m_n M_n}{w_n} < 1 - \frac{m_n M_n}{M_n} = 1 - m_n = M_n.$$

Similarly, if $w_n > M_n$, then $w_{n+1} > M_n > M$.

In addition, since $w_{n+1} = 1 - \frac{R_n}{w_n}$ and $R = m_n M_n$ then

$$|w_{n+1} - M_n| = \left| 1 - \frac{R_n}{w_n} - \frac{1}{2} - \frac{\sqrt{1-4R_n}}{2} \right| = \left| \frac{1 - \sqrt{1-4R_n}}{2} - \frac{m_n M_n}{w_n} \right|$$

$$= \left| m_n - \frac{m_n M_n}{w_n} \right| = \left| \frac{m_n (w_n - M_n)}{w_n} \right| = \left| \frac{m_n}{w_n} \right| |w_n - M_n| < |w_n - M_n|,$$

so $w_n < M_n$ implies $w_{n+1} > w_n$. If $w_n > M_n$, then $w_n > w_{n+1} > M_n$ and still $w_n < \mathcal{M}$, $w_{n+1} < \mathcal{M}$.

Thus if w_n belongs to the interval $[M, \mathcal{M}]$, then all $w_k \in [M, \mathcal{M}]$, $k \geq n$, belong to this interval: if $M \leq w_k < M_{k+1}$, then $M \leq w_{k+1} < M_{k+1} \leq \mathcal{M}$, if $\mathcal{M} \geq w_k \geq M_{k+1}$, then $\mathcal{M} \geq w_{k+1} \geq M_{k+1} \geq M$. Consequently, as far as w_n enters the interval $[M, \mathcal{M}]$, so are all w_k, $k \geq n$, which implies (12).

Let us consider two additional cases: all $w_n < M$ and all $w_n > \mathcal{M}$.

If $w_n < M$ for all n, then necessarily $w_n < M_n$. Consider the sequence ν_n, with $\nu_0 = w_0$ and

$$\nu_{n+1} = 1 - \frac{R}{\nu_n}, \quad \text{where} \quad R = \sup_n R_n.$$

$$\text{Then } \nu_1 = 1 - \frac{R}{\nu_0} < 1 - \frac{R_n}{\nu_0} = 1 - \frac{R_n}{w_0} = w_1.$$

Further, if $\nu_n < w_n$ then

$$\nu_{n+1} = 1 - \frac{R}{\nu_n} < 1 - \frac{R_n}{\nu_n} < 1 - \frac{R_n}{w_n} = w_{n+1}.$$

Consequently, $\nu_n < w_n$. Since $\lim_{n\to\infty} \nu_n = M$ (M is an attractor of the Riccati equation with a constant coefficient) then $\liminf_{n\to\infty} w_n \geq M$. If all $w_n < M$ then obviously $\limsup_{n\to\infty} w_n \leq M \leq \mathcal{M}$.

Similarly the case when $w_n > \mathcal{M}$ can be considered, which completes the proof.

3 An Impulsive Beverton-Holt Difference Equation

First let us consider the impulsive Beverton-Holt difference equation with constant parameters (1),(3). Since (1) leads to a similar mapping for $n = pk$

$$x_{p(k+1)} = \frac{\bar{\alpha} x_{pk}}{1 + \bar{B} x_{pk}},$$

then without loss of generality we can assume an impulsive perturbation at each step ($p = 1$)

$$x_{n+1} = b\left(\frac{\alpha x_n}{1 + B x_n}\right) - d = \frac{b\alpha x_n - d - dBx_n}{1 + Bx_n} = \frac{(b\alpha - dB)x_n - d}{1 + Bx_n}.$$

Let us denote $\beta = b\alpha - dB$ and study the asymptotic properties and non-extinction conditions for the equation

$$x_{n+1} = \frac{\beta x_n - d}{1 + B x_n}, \tag{13}$$

which is equivalent to the impulsive Beverton-Holt difference equation (1),(3) with $p = 1$.

Denote

$$R = \frac{\beta + dB}{(\beta + 1)^2}, \qquad w_+ = \frac{1 + \sqrt{1 - 4R}}{2}, \quad w_- = \frac{1 - \sqrt{1 - 4R}}{2}. \tag{14}$$

Theorem 1 *Suppose $R < \frac{1}{4}$, where R is defined in (14).*
1) If $(\beta + 1)(1 + \sqrt{1 - 4R}) < 2$ then every solution of (13) becomes negative in a finite time.
2) If $\beta > 1$, then any solution of (13), with

$$x_0 > \max\left\{0, \frac{\beta + 1}{B} w_- - \frac{1}{B}\right\}, \tag{15}$$

is positive for $n \geq 0$ and tends to $p = \dfrac{\beta + 1}{B} w_+ - \dfrac{1}{B}$ as $n \to \infty$, where w_-,w_+ are defined in (14).

Proof. After the transformation

$$x_n = \frac{\beta + 1}{B} w_n - \frac{1}{B} \tag{16}$$

the difference equation (13) takes the form

$$w_{n+1} = 1 - \frac{R}{w_n}, \tag{17}$$

with $R = \dfrac{\beta + dB}{(\beta + 1)^2}$. Let us note that the positive solutions x_n of (13) after transformation (16) correspond to such w_n that $(\beta + 1)w_n - 1 > 0$, or $w_n > \frac{1}{\beta + 1}$.

Thus by Lemma 1 w_+ is a global attractor of the difference equation (17) while w_- is a repeller, where w_+, w_- are defined in (14). If $w_+ < \frac{1}{\beta+1}$, i.e. $(\beta+1)(1+\sqrt{1-4R}) < 2$, then the global attractor corresponds to negative values of x and thus every solution becomes negative in some finite time, which corresponds to the extinction of the population. This completes the proof of 1).

Now assume $(\beta+1)(1+\sqrt{1-4R}) > 2$, or $\sqrt{1-4R} > \frac{2}{\beta+1} - 1$. This is obviously satisfied if $\beta > 1$ (the right hand side is negative while the left hand side is positive). Thus the conditions of 2) imply that the global attractor w_+ satisfies $w_+ > \frac{1}{\beta+1}$.

Besides, if $w_0 > w_-$ and $w_0 > \frac{1}{\beta+1}$ then all the values of w_n are above $\max\{w_-, \frac{1}{\beta+1}\}$ (since $f(w) = 1 - \frac{R}{w}$ is concave, monotone and its fixed points are w_-, w_+). After substituting bounds for w_n in (16) we obtain that any solution with the initial value satisfying (15) is positive. Since $p = \frac{\beta+1}{B}w_+ - \frac{1}{B}$ is a global attractor of the difference equation (13) then $\lim_{n\to\infty} x_n = p$ which completes the proof of 2).

Remark. It is to be noted that if $\beta < 1$ then $\sqrt{1-4R} > \frac{2}{\beta+1} - 1$ is equivalent to

$$1 - 4R > \frac{4}{(\beta+1)^2} - \frac{4}{\beta+1} + 1.$$

After substituting $R = \frac{\beta+dB}{(\beta+1)^2}$ and multiplying by $(\beta+1)^2$ we have $4 + 4(\beta+dB) < 4(\beta+1)$ which implies the impossible inequality $dB < 0$. Thus the condition $\beta \geq 1$ is a necessary one for the existence of a positive solution.

Theorem 2 *Let $R = \frac{1}{4}$, where R is defined in (14).*
1) If $\beta < 1$ then every solution of (13) becomes negative in a finite time.
2) If $\beta > 1$ and $x_0 \geq \frac{\beta-1}{2B}$, then any solution of (13) is positive for $n \geq 0$ and tends to $p = \dfrac{\beta-1}{2B}$ as $n \to \infty$.

Proof is similar to the proof of Theorem 1, where Lemma 3 is applied. Here there is the only equilibrium state $\bar{x} = \frac{\beta+1}{2B} - \frac{1}{B} = \frac{\beta-1}{2B}$. If $\beta > 1$ then any solution with $x_0 \geq \bar{x}$ is positive for $t \geq t_0$ and tends to the equilibrium.

Theorem 3 *Let $R > \frac{1}{4}$, where R is defined in (14). Then for any initial value a solution of (13) becomes negative in a finite time.*

Proof. Similar to Theorems 1,2, the equation (13) can be reduced to (17); if a solution of (17) becomes negative then so is the corresponding solution of (13). According to Lemma 4

$$w_n = \sqrt{R}\frac{\cos(n\phi + \phi - \theta)}{\cos(n\phi - \theta)},$$

where $0 < \phi < \frac{\pi}{2}$. Thus there exists such n that $\frac{\cos(n\phi+\phi-\theta)}{\cos(n\phi-\theta)} < 0$ (beginning with some $k > 1$ it is enough to find the smallest n such that $\cos(n\phi + \phi - \theta) < 0$, then $\cos(n\phi - \theta) > 0$). Thus there exists a negative w_n and so a negative x_n, which completes the proof.

Now let us proceed to the impulsive Beverton-Holt equation with variable parameters (4) which can be reduced to

$$x_{n+1} = \frac{\beta_n x_n - d_{n+1}}{1 + B_n x_n}, \quad \beta_n = b_{n+1}\alpha_n - d_{n+1}B_n, \tag{18}$$

similar to (13):

$$x_{n+1} = b_{n+1}\left(\frac{\alpha_n x_n}{1+B_n x_n}\right) - d_{n+1} = \frac{b_{n+1}\alpha_n x_n - d_{n+1} - d_{n+1}B_n x_n}{1+B_n x_n}$$

$$= \frac{(b_{n+1}\alpha_n - d_{n+1}B_n)x_n - d_{n+1}}{1+B_n x_n} = \frac{\beta_n x_n - d_{n+1}}{1+B_n x_n}.$$

Theorem 4 *Denote*

$$R_n = \frac{b_{n+1}\alpha_n B_{n+1}B_{n-1}(\beta_n+1)}{(\beta_{n+1}+1)(B_{n+1}\beta_n + B_n)(B_n\beta_{n-1}+B_{n-1})}, \quad n \geq 1, \tag{19}$$

$$\rho = \inf_n R_n, \quad R = \sup_n R_n, \quad m = \frac{1-\sqrt{1-4R}}{2}, \quad M = \frac{1+\sqrt{1-4R}}{2}, \quad \mathcal{M} = \frac{1+\sqrt{1-4\rho}}{2}.$$

If $R < \frac{1}{4}$, $\quad m < M$ *and*

$$\mathcal{B} = \sup_{n\geq 1}\left\{\frac{B_n\beta_{n-1}+B_{n-1}}{(\beta_n+1)^2 B_{n-1}}\right\} < M,$$

then any solution of (18) with the initial condition x_0*, such that*

$$\frac{(B_1x_1+1)B_0}{B_1\beta_0+B_1} > \max\{\mathcal{B}, m\}, \quad \text{where} \quad x_1 = \frac{\beta_0 x_0 - d_1}{1+B_0x_0},$$

is positive.

Proof. In the equation (18) let us make a substitution

$$x_n = \frac{\beta_n+1}{B_n}z_n - \frac{1}{B_n}. \tag{20}$$

Then

$$z_n = \frac{B_n x_n + 1}{\beta_n+1}, \; z_{n+1} = \frac{B_{n+1}x_{n+1}+1}{\beta_{n+1}+1}, \quad \text{so}$$

$$x_{n+1} = \frac{\beta_n x_n + \frac{\beta_n}{B_n} - \frac{\beta_n}{B_n} - d_{n+1}}{1+B_n x_n} = \frac{\beta_n}{B_n} - \frac{\frac{\beta_n}{B_n}+d_{n+1}}{1+B_n x_n}$$

$$= \frac{\beta_n}{B_n} - \frac{\beta_n + B_n d_{n+1}}{(\beta_n+1)B_n z_n} = \frac{\beta_n}{B_n} - \frac{b_{n+1}\alpha_n}{(\beta_n+1)B_n z_n}.$$

Consequently,

$$z_{n+1} = \frac{B_{n+1}\beta_n}{B_n(\beta_{n+1}+1)} - \frac{b_{n+1}\alpha_n B_{n+1}}{B_n(\beta_{n+1}+1)^2 z_n} + \frac{1}{\beta_{n+1}+1} = \frac{B_{n+1}\beta_n + B_n}{B_n(\beta_{n+1}+1)} - \frac{b_{n+1}\alpha_n B_{n+1}}{B_n(\beta_{n+1}+1)^2 z_n}.$$

The equation (18) has a positive solution if and only if the latter equation has a solution satisfying

$$z_n > \frac{1}{\beta_n+1}. \tag{21}$$

Let us make an additional substitution to obtain a Riccati equation

$$w_n = \frac{z_n B_{n-1}(\beta_n + 1)}{B_n \beta_{n-1} + B_{n-1}}, \quad w_{n+1} = \frac{z_{n+1} B_n(\beta_{n+1} + 1)}{B_{n+1}\beta_n + B_n}, \ n \geq 1. \tag{22}$$

Thus

$$w_{n+1} = 1 - \frac{b_{n+1}\alpha_n B_{n+1}}{(\beta_{n+1}+1)(B_{n+1}\beta_n + B_n) z_n} = 1 - \frac{b_{n+1}\alpha_n B_{n+1} B_{n-1}(\beta_n+1)}{(\beta_{n+1}+1)(B_{n+1}\beta_n + B_n)(B_n\beta_{n-1} + B_{n-1}) w_n}$$

Denoting R_n as in (19) we obtain the Riccati equation with a nonconstant coefficient which was considered in Lemma 5.

The original solution x_n is positive if and only if

$$\beta_0 x_0 > d_1 \qquad \text{and} \qquad w_n > \frac{B_n\beta_{n-1} + B_{n-1}}{(\beta_n+1)^2 B_{n-1}}, \quad n \geq 1.$$

The rest of the proof is an application of Lemma 5.

Finally, let us proceed to comparison results. To this end consider a comparison equation

$$y_{n+1} = \frac{\gamma_n y_n - \delta_{n+1}}{1 + \kappa_n y_n}. \tag{23}$$

Thus we immediately get the following results.

Theorem 5 *1) Let $\beta_n \geq \gamma_n, B_n \leq \kappa_n, d_k \leq \delta_k, x_0 \geq y_0$. Then*
a) if a solution of (23) is positive, then a solution of (18), with $x_0 = y_0$, is also positive;
b) generally, if $\{x_n\}$ is a solution of (18) and $\{y_n\}$ is a solution of (23), with $x_0 = y_0$, then

$$x_n \geq y_n.$$

2) Let $\beta_n \leq \gamma_n, B_n \geq \kappa_n, d_k \geq \delta_k, x_0 \leq y_0$. Then
a) if (23) has no nonnegative solutions satisfying $y_0 < N$, so does (18);
b) if the solution of (23) is negative for $n > n_0$, then the solution of (23), with $x_0 = y_0$, is also negative for $n > n_0$;
c) generally, if $\{x_n\}$ is a solution of (18) and $\{y_n\}$ is a solution of (23), with $x_0 = y_0$, then

$$x_n \leq y_n.$$

Theorems 1-3,5 imply the following result.

Theorem 6 *Let*

$$d = \inf_i d_i \geq 0, \ D = \sup_i d_i, a = \inf_i B_i, \ A = \sup_i B_i, \underline{\beta} = \inf_i \beta_i, \bar{\beta} = \sup_i \beta_i.$$

1) If $\underline{\beta} > 1$ and

$$R = \frac{\underline{\beta} + AD}{(\underline{\beta}+1)^2} \leq \frac{1}{4},$$

then (18) has a positive solution; moreover, all solutions satisfying the initial condition

$$x_0 > \max\left\{0, \frac{(\underline{\beta}+1)(1-\sqrt{1-4R})}{2a} - \frac{1}{a}\right\}$$

are positive for $n \geq 1$.
2) If

$$\frac{\bar{\beta} + ad}{(\bar{\beta}+1)^2} > \frac{1}{4},$$

then (18) has no positive solutions.

4 Applications to Logistic Impulsive Differential Equations

Now consider a logistic equation with nonconstant coefficients

$$\dot{x}(t) = x(t)(r(t) - a(t)x(t)) \tag{24}$$

and linear impulsive perturbations

$$x(\tau_k) = b_k x(\tau_k^-) - d_k \tag{25}$$

under the following conditions:
(a1) $r(t), a(t)$ are locally essentially bounded functions, $r(t) \geq 0, a(t) \geq 0$;
(a2) $\lim_{n\to\infty} \tau_n = \infty$, $0 < b_k \leq 1$, $d_k \geq 0$, the inequalities mean that impulsive perturbations are of harvesting type.

This includes the cases of proportional and periodic constant harvesting.

Equation (24) has an explicit solution in every interval without an impulse $[\tau_k, \tau_{k+1}^-)$:

$$x(t) = \frac{x(\tau_k)\exp\{\int_{\tau_k}^t r(s)\ ds\}}{1 + x(\tau_k)\int_{\tau_k}^t a(s)\exp\{\int_{\tau_k}^s r(\zeta)d\zeta\}ds}. \tag{26}$$

In order to study the impulsive equation (24),(25) first consider a logistic equation with constant coefficients and a constant distance between impulses:

$$\dot{x}(t) = rx\left(1 - \frac{x}{K}\right),\ t \geq t_0, \tag{27}$$

$$x(\tau_j) = bx(\tau_j^-) - d, \tau_k = t_0 + k\tau, \tag{28}$$

under the following conditions:
(c1) $r, K > 0,\ \tau > 0$;
(c2) $0 < b \leq 1, d \geq 0, bK > d$.
We will also consider the initial value problem, with

$$x(t_0) = x_0. \tag{29}$$

In addition, let us introduce the following notation

$$\beta = e^{r\tau}\left(b - \frac{d}{K}\right), B = \frac{e^{r\tau}}{K}. \tag{30}$$

Definition. We will say that $S \subset \mathbf{R}$ is a limit set of the problem equation (27)-(29) if for any $y \in S$ and $t_1 > t_0$ there exists such $t_2 > t_1$ that $x(t_2) = y$.

Theorem 7 *Suppose (c1)-(c2) are satisfied and $R < \frac{1}{4}$, where β, B, R are defined in (30), (14).*
1) If $(\beta+1)(1+\sqrt{1-4R}) < 2$ then every solution of (27)-(29) becomes negative in a finite time.
2) If $\beta > 1$, then any solution of (27)-(29), with

$$x_0 > \max\left\{0, \frac{\beta+1}{B}w_- - \frac{1}{B}\right\},$$

is positive for $t > t_0$ and the limit set of (27)-(29) is an interval (p,q), where

$$p = \frac{\beta+1}{B}w_+ - \frac{1}{B},\ q = \frac{pe^{r\tau}}{1+\frac{p}{K}e^{r\tau}},$$

where w_-,w_+ are defined in (14).

Proof. The solution of (27)-(29) has the following form

$$x(t) = \frac{x_0 e^{r(t-t_0)}}{1+\frac{x_0}{K}e^{r(t-t_0)}},\quad t_0 \le t < t_0+\tau = \tau_1,\ \text{thus}\quad x(\tau_1^-) = \frac{x_0 e^{r\tau}}{1+\frac{x_0}{K}e^{r\tau}}\ \text{and}$$

$$x(\tau_1) = \frac{bx_0 e^{r\tau}}{1+\frac{x_0}{K}e^{r\tau}} - d = \frac{bx_0e^{r\tau} - d - d\frac{x_0}{K}e^{r\tau}}{1+\frac{x_0}{K}e^{r\tau}} = \frac{(b-\frac{d}{K})e^{r\tau}x_0 - d}{1+\frac{x_0}{K}e^{r\tau}}.$$

Similarly, if we denote $x_n = x(\tau_n)$, then

$$x_{n+1} = \frac{(b-\frac{d}{K})e^{r\tau}x_n - d}{1+\frac{1}{K}e^{r\tau}x_n}. \tag{31}$$

Let us consider solutions (27)-(29) with $x_0 < K$ which is not really a restriction: if $b < 1$,$d > 0$ then any solution eventually becomes less than K. Suppose $x_0 > K, x(t) > K$ and $y(t)$ is a solution of the problem (27), (29) without impulses. Then $x(t) \le y(t)$ and $\lim_{t\to\infty} y(t) = K$. Hence $\lim_{t\to\infty} x(t) = K$. The limit case in (28) implies $K = bK - d$ which contradicts $b < 1$,$d > 0$.

Thus any solution with $x_0 < K$ is positive for any $t > t_0$ if and only if the difference equation (31), with any x_0 satisfying $0 < x_0 < K$, has a positive solution. Thus Theorem 1 can be applied. Since $p = \frac{\beta+1}{B}w_+ - \frac{1}{B}$ is a global attractor of the difference equation (13) then $\lim_{k\to\infty} x(\tau_k) = p$. The limit set is a set of all values of $x(t)$ such that

$$x(t) = \varphi(t) = \frac{pe^{rt}}{1+\frac{p}{K}e^{rt}},\ 0 < t < \tau.$$

Since $\varphi(t)$ is continuous, then the limit set is $(p, \varphi(\tau))$, which completes the proof of 2).

Thus we can apply Theorem 1.

By applying Theorems 2,3 we deduce the following result.

Theorem 8 *Suppose (c1)-(c2) are satisfied.*
1) Let $R = \frac{1}{4}$, where R is defined in (14).
a) If $\beta < 1$ then every solution of (27)-(29) becomes negative in a finite time.
b) If $\beta > 1$ and $x_0 \ge \frac{\beta-1}{2B}$, then any solution of (27)-(29) is positive for $t > t_0$ and the limit set of (27)-(29) is (p,q), where

$$p = \frac{\beta-1}{2B},\ q = \frac{pe^{r\tau}}{1+\frac{p}{K}e^{r\tau}}$$

2) Let $R > \frac{1}{4}$. Then for any initial value a solution of (27)-(29) becomes negative in a finite time.

Now let us proceed to the impulsive logistic equation with variable coefficients (24),(25). Denote

$$r_n = \int_{\tau_n}^{\tau_{n+1}} r(s)\ ds, \quad B_n = \int_{\tau_n}^{\tau_{n+1}} a(s) \exp\left\{ \int_{\tau_n}^{s} r(\zeta) d\zeta \right\} ds, \quad \beta_n = b_{n+1} r_n - d_{n+1} B_n \qquad (32)$$

The following result establishes the relation between a solution of the logistic impulsive equation and an impulsive Beverton-Holt difference equation.

Theorem 9 *Suppose (a1)-(a2) are satisfied and $a(t_0) > 0$.*
Then the solution of (24),(25),(29), with an initial condition $0 < x_0 < \frac{r(t_0)}{a(t_0)}$, is positive if and only if the solution of the difference equation

$$x_{n+1} = \frac{(b_{n+1} e^{r_n} - d_{n+1} B_n) x_n - d_{n+1}}{1 + B_n x_n} = \frac{\beta_n x_n - d_{n+1}}{1 + B_n x_n} \qquad (33)$$

with the same initial condition is positive.

Proof. By the solution representation (26)

$$x(\tau_{n+1}^-) = \frac{x(\tau_n) \exp\{\int_{\tau_n}^{\tau_{n+1}} r(s) ds\}}{1 + x(\tau_n) \int_{\tau_n}^{\tau_{n+1}} a(s) \exp\{\exp\{\int_{\tau_n}^{s} r(\zeta) d\zeta\} ds} = \frac{x(\tau_n) e^{r_n}}{1 + B_n x(\tau_n)}.$$

After the impulse we have

$$x(\tau_{n+1}) = b_{n+1} x(\tau_{n+1}^-) - d_{n+1} = b_{n+1} \frac{x(\tau_n) e^{r_n}}{1 + B_n x(\tau_n)} - d_{n+1} = \frac{(b_{n+1} e^{r_n} - d_{n+1} B_n) x(\tau_n) - d_{n+1}}{1 + B_n x(\tau_n)},$$

thus using notation (32) we obtain (33). Besides, equality (24) and inequalities $0 < x_0 < \frac{r(t_0)}{a(t_0)}$ imply that the solution grows between impulse points. Thus the solution of (24),(25),(29), with an initial condition $0 < x_0 < \frac{r(t_0)}{a(t_0)}$, is positive if and only if the solution of the difference equation (33) with the same initial condition is positive, which completes the proof of the theorem.

Corollary 9.1. Suppose (a1)-(a2) are satisfied and in (24)-(25) $b_n \equiv b, d_n \equiv d$,

$$\int_{\tau_n}^{\tau_{n+1}} a(s) \exp\left\{ \int_{\tau_n}^{s} r(\zeta) d\zeta \right\} ds \equiv B, \quad \int_{\tau_n}^{\tau_{n+1}} r(s)\ ds \equiv r,$$

where b, d, a, r are positive constants. Then the solution of (24),(25),(29), with an initial condition $0 < x_0 < \frac{r(t_0)}{a(t_0)}$, is positive if and only if the solution of the difference equation

$$x_{n+1} = \frac{(be^r - dB) x_n - d}{1 + a x_n}, \qquad (34)$$

with the same x_0, is positive.

Corollary 9.2. Suppose (a1)-(a2) are satisfied. If there exists such n that in (32) $\beta_n < 0$ then all solutions of (24)-(25) become negative after $t > \tau_{n+1}$.

By applying Theorem 4 we obtain

Theorem 10 *Suppose (a1)-(a2) hold and*

$$R_n = \frac{b_{n+1} r_n B_{n+1} B_{n-1}(\beta_n + 1)}{(\beta_{n+1}+1)(B_{n+1}\beta_n + B_n)(B_n\beta_{n-1} + B_{n-1})}, \quad n \geq 1,$$

$$\rho = \inf_n R_n, \quad R = \sup_n R_n, \quad m = \frac{1-\sqrt{1-4R}}{2}, \quad M = \frac{1+\sqrt{1-4R}}{2}, \quad \mathcal{M} = \frac{1+\sqrt{1-4\rho}}{2}.$$

If $R < \frac{1}{4}, \quad m < M$ *and*

$$\mathcal{B} = \sup_{n \geq 1} \left\{ \frac{B_n \beta_{n-1} + B_{n-1}}{(\beta_n + 1)^2 B_{n-1}} \right\} < M,$$

then any solution of (24)-(25) with the initial condition x_0*, such that*

$$\frac{(B_1 x_1 + 1)B_0}{B_1 \beta_0 + B_1} > \max\{\mathcal{B}, m\}, \quad \textit{where} \quad x_1 = \frac{\beta_0 x_0 - d_1}{1 + B_0 x_0}$$

is positive.

Now let us proceed to comparison results. To this end consider an additional logistic equation

$$\dot{y}(t) = y(t)(r_1(t) - a_1(t) y(t)) \tag{35}$$

with linear impulsive perturbations

$$y(\tau_k) = \bar{b}_k y(\tau_k^-) - \bar{d}_k \tag{36}$$

and initial condition

$$y(t_0) = y_0. \tag{37}$$

Theorem 11 *Suppose (a1)-(a2) are satisfied for the equations (24)-(25) and (35)-(36).*
1) Let $r(t) \geq r_1(t), a(t) \leq a_1(t), b_k \geq \bar{b}_k, d_k \leq \bar{d}_k, x_0 \geq y_0$*. Then*
a) if all solutions of (35),(36),(37) are positive, then all solutions of (24), (25), (29) are positive;
b) if for some N *all solutions of (35),(36) satisfying* $N < y(t_0) < \frac{r_1(t_0)}{a_1(t_0)}$ *are positive then all solutions of (24),(25) satisfying* $N < x(t_0) < \frac{r(t_0)}{a(t_0)}$ *are positive;*
c) generally, if x *is a solution of (24),(25),(29) and* y *is a solution of (35),(36),(37), then*

$$x(t) \geq y(t).$$

2) Let $r(t) \leq r_1(t), a(t) \geq a_1(t), b_k \leq \bar{b}_k, d_k \geq \bar{d}_k, x_0 \leq y_0$*. Then*
a) if (35),(36) has no nonnegative solutions satisfying $y(t_0) < N$*, so does (24),(25);*
b) if the solution of (35),(36), with the initial condition (37) is negative for $t > t_1$ *then the solution of (24), (25), (29) is also negative for* $t > t_1$*;*
c) generally, if x *is a solution of (24), (25), (29) and* y *is a solution of (35),(36), (37), then*

$$x(t) \leq y(t).$$

Proof. The statements 1 c), 2c) of the theorem are an immediate result of the solution representation between impulses (26), which can be rewritten in the form

$$x(t) = \frac{1}{\frac{1}{x(\tau_k)} + \int_{\tau_k}^{t} a(s) \exp\{-\int_s^t r(\zeta) d\zeta\} ds}$$

and the inequality $bx - d \geq b_1 x - d_1$, if $b \geq b_1$, $d \leq d_1$. The other statements are a consequence of 1c), 2c).

Theorem 12 *Suppose (a1)-(a2) are satisfied, a_i and r_i are defined in (32) and*

$$b = \inf_i b_i > 0,\ B = \sup_i b_i,\ d = \inf_i d_i \geq 0,\ D = \sup_i d_i, a = \inf_i B_i,\ A = \sup_i B_i,$$

$$r = \inf_i r_i,\ \rho = \sup_i r_i,\ \underline{\beta} = be^{\rho} - da, \bar{\beta} = Be^{r} - DA.$$

1) If $\underline{\beta} > 1$ and

$$R = \frac{be^r}{(\underline{\beta}+1)^2} \leq \frac{1}{4},$$

then (24)-(25) has a positive solution; moreover, all solutions satisfying the initial condition

$$x(t_0) > \max\left\{0, \frac{(\underline{\beta}+1)(1-\sqrt{1-4R})}{2a} - \frac{1}{a}\right\}$$

are positive for $t > t_0$.
2) If

$$\frac{Be^{\rho}}{(\bar{\beta}+1)^2} > \frac{1}{4},$$

then (24)-(25) has no positive solutions.

Proof. This result is a consequence of the comparison Theorem 11 and Corollary 9.1. In the case 1) the comparison equation is

$$\dot{y}(t) = y(t)\left(\frac{r}{\tau_{i+1}-\tau_i}\chi_{[\tau_i,\tau_{i+1})}(t) - \frac{A}{\int_{\tau_i}^{\tau_{i+1}} \exp\{\frac{r(s-\tau_i)}{\tau_{i+1}-\tau_i}\} ds}\chi_{[\tau_i,\tau_{i+1})}(t) y(t)\right), \tag{38}$$

where $\chi_{[z,t)}$ is a characteristic function of the interval $[z,t)$, with linear impulsive perturbations

$$y(\tau_k) = By(\tau_k^-) - d. \tag{39}$$

If we denote $b_i = b, d_i = D$,

$$r_1(t) = \frac{r}{\tau_{i+1}-\tau_i}\chi_{[\tau_i,\tau_{i+1})}(t),\ a_1(t) = \frac{A}{\int_{\tau_i}^{\tau_{i+1}} \exp\{\frac{r(s-\tau_i)}{\tau_{i+1}-\tau_i}\} ds}\chi_{[\tau_i,\tau_{i+1})}(t),$$

then $r(t) \geq r_1(t), a(t) \leq a_1(t), b_k \geq b, d_k < D$. Consequently, if (38),(39) has a positive solution then (24)-(25) has a positive solution. Finally, we apply Corollary 4.1 to the equation (38),(39).

In case 2) the comparison equation is

$$\dot{y}(t) = y(t) \left(\frac{\rho}{\tau_{i+1} - \tau_i} \chi_{[\tau_i, \tau_{i+1})}(t) - \frac{a}{\int_{\tau_i}^{\tau_{i+1}} \exp\{\frac{\rho(s-\tau_i)}{\tau_{i+1}-\tau_i}\} ds} \chi_{[\tau_i, \tau_{i+1})}(t) y(t) \right),$$

where $\chi_{[z,t)}$ is a characteristic function of the interval $[z, t)$, with linear impulsive perturbations

$$y(\tau_k) = By(\tau_k^-) - d.$$

The rest of the proof is similar to case 1).

The following example illustrates Theorems 7,8.

Example. Consider the logistic equation (27) with $K = 1$ and impulsive conditions (28) with b=0.9,d=0.2, $\tau = 1$. If $r = 1$ then $R \approx 0.67, R > \frac{1}{4}$ which means that all solutions become negative in a finite time. If $r = 2$ then $R \approx 0.17 < \frac{1}{4}$. Fig. 1 illustrates the vanishing solution in the first case and a positive solution in the second case. Both solutions correspond to the same initial value $x(0) = 0.2$.

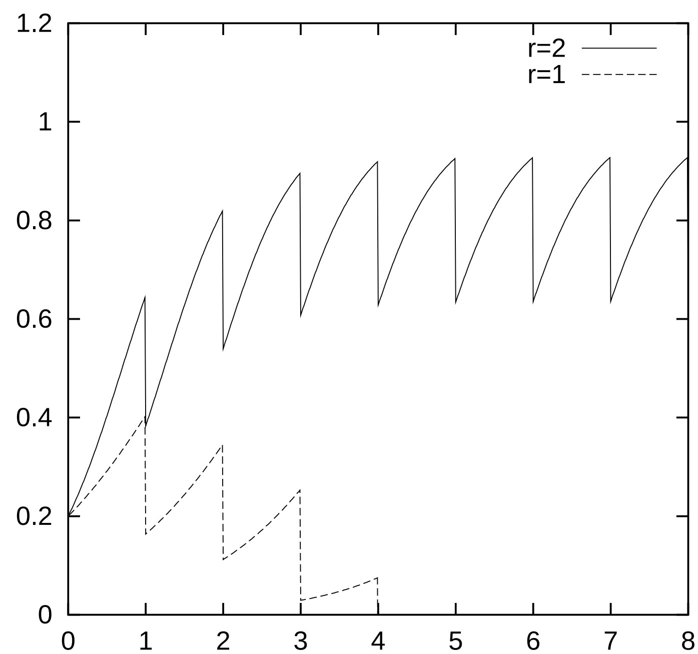

Figure 1: The behavior of solutions of (27), (28) with $K = 1$, $b = 0.9$, $d = 0.2$, $\tau = 1$, $x(0) = 0.2$ and $r = 1$ or $r = 2$. In the first case the solution becomes negative for $t > 4$, in the second case the solution is positive.

5 Conclusions and Open Problems

We have demonstrated that the impulsive Beverton-Holt equation can be applied to the study of the asymptotics for a logistic equation with impulsive harvesting. The analysis of the difference equation led to explicit extinction and non-extinction results, comparison theorems and provided a complete description of the asymptotic behavior in the case of constant coefficients.

In the framework of discrete population models with impulsive harvesting the following models can be studied.

1. If we assume that there is a time delay of time period 1 in the response of the growth rate per individual to a density change, we obtain the delay difference model [2], p. 79:
$$x_{n+1} = \frac{a_n x_n}{1 + B_n x_{n-1}}.$$
If in addition impulsive conditions are imposed, we get a second order difference equation
$$x_{n+1} = \frac{\alpha_n x_n - \beta_n x_{n-1} - d_n}{1 + B_n x_{n-1}}$$
which is a particular case of the rational second order difference equation; such an equation was studied in [10]. We can also consider the recruitment type coefficient $b_n > 1$; this models the situation when both natural population growth and recruitment compensate an independent consumption d_n. The situation with $d_n < 0$ and $0 < b_n < 1$ models the situation when a constant recruitment and natural population growth are balanced by proportional harvesting (in the latter case all solutions are positive).

2. In the harvesting model we assumed that the harvesting rate depends on the present state of the population. Let us consider a model with a harvesting (hunting) strategy where we do not know the exact size of the population x_n. However it is required to determine the hunting quota. There is always a delay in processing and distributing field information, so the quota must be set long before the hunting season begins. This means that the harvesting rate depends on the previous value x_n at $n+1$-th step (we use the population size information from the previous step x_n). Let the population x_n be harvested at $(n+1)$-th step at the rate $(1-b_n)+d_n$. This leads to the first order difference equation of the form
$$x_{n+1} = \frac{\alpha_n x_n}{1 + B_n x_n} - (1-b_n)x_n - d_n, \;\; c_n > 0, 0 < b_n \leq 1, d_n > 0.$$

3. Other discrete models can be considered subject to impulsive perturbations, for example, the second Verhulst difference equation
$$x_{n+1} = \frac{r x_n^2}{x_n^2 + A}$$
under linear impulsive perturbations takes the form
$$x_{n+1} = \frac{\beta x_n^2 - Ad}{x_n^2 + A}, \;\; \beta = br - d.$$

4. The analysis presented in this paper and proposed in items 1-3 can be applied to various differential equations
$$x'(t) = f(t,x)$$
with impulsive conditions (6). By applying the comparison technique one can deduce sufficient non-extinction conditions even in cases when the explicit solution is not available.

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