FUNCTIONAL DIFFERENTIAL EQUATIONS
VOLUME 10
2003, NO
PP. –

DELAY DIFFERENTIAL LOGISTIC EQUATIONS WITH A NONLINEAR HARVESTING FUNCTION

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Abstract. We study a delay logistic equation with a nonlinear harvesting term

$$\dot{N}(t) = r(t)N(t)\left[a - bN(h(t))\right] - \frac{c(t)N^{\alpha}(g(t))}{1 + d(t)N^{\beta}(p(t))}.$$

Sufficient conditions for positiveness, boundedness and extinction of solutions are obtained for this equation.

Key Words. Delay logistic equations, harvesting, positive solutions, extinction of a population

AMS(MOS) subject classification. 34K12, 92D25

1. Introduction. A classic model of population dynamics which has important applications in real world problems can be written as

$$\frac{dN}{dt} = r(N(t),t)N(t) - E(N(t),t), \tag{1}$$

where $N(t)$ is the population density, $r(N(t),t) > 0$ is the intrinsic growth rate of the population, and $E(N,t)$ is a harvesting strategy for the population. Harvesting includes fisheries, forest-stand and wildlife management problems, as well as mass rearing of insects for biological control.

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Here are two examples of populations subject to nonlinear harvesting.

EXAMPLE 1. **Grazing ecosystems.** *Let us consider a plant population that grows according to the logistic growth model and is grazed by cattle. Traditionally, due to May [15, 16], the harvesting rate is modeled by a function of the form* $E(N) = qsN/(N + 0.05d)$. *The resulting model is*

$$\frac{dN}{dt} = N(a - bN) - \frac{qsN}{N + 0.05d},$$

where s, q, *and* d *are positive constants.*

EXAMPLE 2. **Population outbreaks.** *Population outbreaks are characterized by a rapid change in population density over several orders of magnitude. Population outbreaks often cause serious ecological and economic problems. The most recent example of the outbreak is West Nile Virus Outbreak. A simple model by Ludwig et al [14] of the spruce budworm population dynamics includes a logistic population growth, a functional response of polyphagous predators (birds), and is described by the following differential equation*

$$\frac{dN}{dt} = rN(a - bN) - \frac{\alpha N^2}{\gamma^2 + \beta N^2}.$$

The existence of break points holds important implications for control of the system. It is well known that continuous changes in harvesting rates can cause discontinuous collapse of insect pests.

The introduction of delays into existing population ecology equations is supported by general arguments that the interacting species somehow rely on resources and harvesting that have been accumulated in the past.

Since time lags were introduced in population dynamics by Hutchison [11], there has been an enormous increase in the study of these models (see [4]-[13]).

Eq. (1) with delay can be written in the following form

$$\frac{dN}{dt} = r(N(h(t)), t)N(t) - E(N(g(t)), t). \tag{2}$$

In papers [2]-[3] we set up delay differential equations similar to (2) with a linear harvesting function $E = c(t)N(g(t))$. A proportional harvesting function without lags was introduced in [9]-[12].

Here we will obtained sufficient conditions for positiveness, boundedness and extinction of solutions to the equation (2).

The present paper extends the results recently obtained in [2] for a differential equation with a linear harvesting function to the equations with nonlinear harvesting. The results of this type for the *delay* logistic equation with harvesting have never been stated before.

We will apply oscillation properties of a linear delay differential equation with positive and negative coefficients recently obtained in [1].

2. Preliminaries. Consider a scalar delay differential equation

$$\dot{N}(t) = r(t)N(t)\left[a - bN(h(t))\right] - \frac{c(t)N^{\alpha}(g(t))}{1 + d(t)N^{\beta}(p(t))}, \quad t \geq 0, \tag{3}$$

with the initial function and the initial value

$$N(t) = \varphi(t),\ t < 0,\ N(0) = N_0, \tag{4}$$

under the following conditions:
(a1) $a > 0, b > 0, \alpha \geq 1, \beta \geq 0$;
(a2) $r(t) \geq 0, c(t) \geq 0, d(t) \geq 0$ are Lebesgue measurable and locally essentially bounded functions, $\int_0^\infty c(s)ds = \infty$;
(a3) $h(t), g(t), p(t)$ are Lebesgue measurable functions, $h(t) \leq t$, $g(t) \leq t$, $p(t) \leq t$, $\lim_{t\to\infty} h(t) = \infty$, $\lim_{t\to\infty} g(t) = \infty$, $\lim_{t\to\infty} p(t) = \infty$;
(a4) $\varphi : (-\infty, 0) \to R$ is a Borel measurable bounded function, $\varphi(t) \geq 0$, $N_0 > 0$.

Definition 1. *An absolutely continuous in each interval* $[0, b]$ *function* $N : R \to R$ *is called* **a solution of the problem** *(3), (4), if it satisfies equation (3) for almost all* $t \in [0, \infty)$ *and equalities (4) for* $t \leq 0$.

Definition 2. *We say that a solution of a differential equation is* **nonoscillatory** *if it is eventually positive or eventually negative.*

We will present here lemmas which will be used in the proof of the main results.

Consider the linear delay differential equation

$$\dot{x}(t) + c(t)x(g(t)) = 0, \quad t \geq 0. \tag{5}$$

Lemma 1. [8] *Suppose that for the functions* c, g *hypotheses (a2)-(a3) hold.*
1) If

$$\sup_{t>0} \int_{g(t)}^{t} c(s)ds \leq \frac{1}{e}, \tag{6}$$

then equation (5) has a nonoscillatory solution.

If in addition, $0 \leq \varphi(t) < N_0$, *then the solution of initial value problem (5)-(4), where* $N(t)$ *in (4) is replaced by* $x(t)$, *is positive.*
2) For every nonoscillatory solution $x(t)$ *of (5) we have* $\lim_{t\to\infty} x(t) = 0$.

Consider also the following linear delay equation with positive and negative coefficients

$$\dot{x}(t) + c(t)x(g(t)) - a(t)x(t) = 0, \quad t \geq 0. \tag{7}$$

DEFINITION 3. *A solution $X(t,s)$ of the problem*

$$\dot{x}(t) + c(t)x(g(t)) - a(t)x(t) = 0, \quad t \geq s,$$

$$x(t) = 0, \quad t < s, \; x(s) = 1,$$

is called **a fundamental function** *of (7).*

LEMMA 2. [1] *Suppose that for the functions $c(t), g(t)$ conditions (a2)-(a3) hold, a is a locally bounded function, $a(t) \geq 0$,*

$$c(t) \geq a(t), \quad \int_0^\infty [c(t) - a(t)]\, dt = \infty, \tag{8}$$

and

$$\limsup_{t\to\infty} a(t)(t - g(t)) < 1. \tag{9}$$

1) If there exists a nonoscillatory solution of (7), then for some t_0 and $t \geq t_0$ we have $X(t,s) > 0, \; t \geq s \geq t_0$, where $X(t,s)$ is a fundamental function of (7).
2)For every nonoscillatory solution $x(t)$ of (7) we have $\lim_{t\to\infty} x(t) = 0$.

3. Main results. In addition to (a1)-(a4) consider the following hypothesis:
(a5) $h(t)$ is a nondecreasing continuous function.

If in Eq. (3) we neglect the harvesting term, i.e. assume $c(t) \equiv 0$, then the positive equilibrium becomes $\frac{a}{b}$. Suppose the initial function and the initial value are less than this equilibrium. In practical models this means that hunting is the only reason for the population decrease and under certain conditions for the extinction of a population. The following result presents lower and upper bounds for the solution.

THEOREM 1. *Suppose (a1)-(a5) hold,*

$$\varphi(t) \leq N_0 < \frac{a}{b}, \quad t < 0, \tag{10}$$

and

$$\left(\frac{a}{b}\right)^{\alpha-1} \exp\left\{a(\alpha-1)\sup_{t>0}\int_{h(t)}^t r(s)ds\right\} \times \tag{11}$$

$$\sup_{t>0} \int_{g(t)}^{t} c(s) \exp\left\{ a \left[\exp\left\{ a \sup_{t>0} \int_{h(t)}^{t} r(\xi) d\xi \right\} - 1 \right] \int_{g(s)}^{s} r(\tau) d\tau \right\} ds \leq \frac{1}{e}.$$

Then for any solution of (3),(4) we have

$$0 < N(t) \leq \frac{a}{b} \exp\left\{ a \sup_{t>0} \int_{h(t)}^{t} r(s) ds \right\}. \tag{12}$$

Proof. Suppose (12) is not valid. Then either there exists $\bar{t} > 0$ such that

$$0 < N(t) \leq \frac{a}{b} \exp\left\{ a \sup_{t>0} \int_{h(t)}^{t} r(s) ds \right\},\ 0 \leq t < \bar{t},$$

$$N(\bar{t}) = \frac{a}{b} \exp\left\{ a \sup_{t>0} \int_{h(t)}^{t} r(s) ds \right\}, \dot{N}(\bar{t}) > 0, \tag{13}$$

or there exists $\bar{t} > 0$ such that

$$0 < N(t) \leq \frac{a}{b} \exp\left\{ a \sup_{t>0} \int_{h(t)}^{t} r(s) ds \right\},\ 0 \leq t < \bar{t},\ N(\bar{t}) = 0. \tag{14}$$

Suppose we have the first possibility for a solution $N(t)$ of (3),(4). Denote by

$$t_1 < t_2 < \ldots < t_k < \ldots$$

a sequence of all points t_k, such that

$$N(h(t_k)) = \frac{a}{b}, \dot{N}(h(t_k)) > 0.$$

Equality (13) and inequality (10) imply that the set $\{t_k\}$ is not empty. Suppose t^* is a point of a local maximum for $N(t)$.

We will prove that if $N(t^*) > \frac{a}{b}$, then $t^* \in \bigcup_k [h(t_k), t_k]$.

Let t_k be the greatest among all points of the sequence $\{t_k\}$ satisfying $h(t_k) < t^*$.

Suppose first $N(t) \leq \frac{a}{b}$ for some t and $h(t_k) < t \leq t_k$. The definitions of t_k and t^* imply $t^* < t$ and hence $t^* \in [h(t_k), t_k]$.

Now suppose $N(t) > \frac{a}{b}$ for $h(t_k) < t \leq t_k$.

Let there exist the smallest point $t' > t^*$ such that $N(t') = \frac{a}{b}$. Then (3) implies $\dot{N}(t) < 0,\ t_k \leq t < t'$. Hence in this interval $N(t)$ has no maximal points. Thus $h(t_k) < t^* < t_k$.

If such t' does not exist then $\dot{N}(t) \le 0, t > t_k$ and again $h(t_k) < t^* < t_k$. Eq.(3) implies now that

$$\dot{N}(t) \le ar(t)N(t),\ h(t_k) \le t \le t^*,\ N(h(t_k)) = \frac{a}{b}.$$

Then

$$N(t^*) \le \frac{a}{b} \exp\left\{ a \int_{h(t_k)}^{t^*} r(s)ds \right\}$$

$$\le \frac{a}{b} \exp\left\{ a \int_{h(t_k)}^{t_k} r(s)ds \right\} \le \frac{a}{b} \exp\left\{ a \sup_{t>0} \int_{h(t)}^{t} r(s)ds \right\},\ \dot{N}(t^*) = 0,$$

which contradicts our assumption (13).

Suppose now there exists $\bar{t} > 0$ such that (14) holds. After substituting

$$N(t) = \exp\left\{ \int_0^t r(s)\left[a - bN(h(s))\right] ds \right\} x(t) \tag{15}$$

in (3),(4) we have the following system:

$$\dot{x}(t) = -\frac{c(t)N^{\alpha-1}(g(t))}{1 + d(t)N^{\beta}(p(t))} \exp\left\{ -\int_{g(t)}^{t} r(s)\left[a - bN(h(s))\right] ds \right\} x(g(t)), t > 0, \tag{16}$$

$$x(t) = \varphi(t), t < 0, x(0) = N_0 \tag{17}$$

(we assume $r(t) = 0$ if $t < 0$). Consider now an initial value problem for a linear delay differential equation:

$$\dot{y}(t) = -v(t)y(g(t)), t > 0, \tag{18}$$

$$y(t) = \psi(t), t < 0, y(0) = y_0, \tag{19}$$

where

$$v(t) = \frac{c(t)N^{\alpha-1}(g(t))}{1 + d(t)N^{\beta}(p(t))} \exp\left\{ -\int_{g(t)}^{t} r(s)\left[a - bN(h(s))\right] ds \right\}.$$

It is evident that if $\psi(t) = \varphi(t), y_0 = N_0$, then the solutions of (16),(17) and (18),(19) coincide. Inequalities (14) and (11) imply that

$$\int_{g(t)}^{t} v(s)ds = \int_{g(t)}^{t} \frac{c(s)N^{\alpha-1}(g(s))}{1 + d(s)N^{\beta}(p(s))} \exp\left\{ \int_{g(s)}^{s} r(\tau)\left[bN(h(\tau)) - a\right] d\tau \right\} ds$$

$$\leq \left(\frac{a}{b}\right)^{\alpha-1} \exp\left\{a(\alpha-1)\sup_{t>0}\int_{h(t)}^{t} r(s)ds\right\}$$

$$\times \sup_{t>0}\int_{g(t)}^{t} c(s)\exp\left\{a\left[\exp\left\{a\sup_{t>0}\int_{h(t)}^{t} r(\xi)d\xi\right\}-1\right]\int_{g(s)}^{s} r(\tau)d\tau\right\}ds \leq \frac{1}{e}.$$

Inequality (10) and definition (17) imply

$$\varphi(t) \leq N_0.$$

Thus Lemma 1 yields that if $\psi(t)=\varphi(t), y_0=N_0$, then $y(t)>0, t>0$. Hence $x(t)>0,\ t>0$. Consequently by (15) we have $N(t)>0,\ t>0$, which contradicts assumption (14) which completes the proof. □

Corollary 1. *Consider an autonomous equation*

$$\dot{N}(t)=N(t)\left[a-bN(t-h)\right]-\frac{cN^{\alpha}(t-g)}{1+dN^{\beta}(t-p)},\quad t\geq 0, \tag{20}$$

with the initial conditions (4), where

$$a>0, b>0, c>0, d>0, h\geq 0, g\geq 0, p\geq 0, \alpha\geq 1, \beta\geq 0.$$

Suppose (10) holds and

$$gc\left(\frac{a}{b}\right)^{\alpha-1}\exp\left\{ag\left(\exp\left\{ah\right\}-1\right)+a(\alpha-1)h\right\}\leq\frac{1}{e}.$$

Then for a solution of (20),(4) we have

$$0<N(t)\leq\frac{ae^{ah}}{b}.$$

Theorem 2. *Suppose (a1)-(a5) hold, then for every eventually positive solution of (3)-(4) there exists $t_0\geq 0$ such that (12) holds for $t\geq t_0$.*

Proof. Suppose $N(t)$ is an eventually positive solution of (3),(4). If $N(t)\leq\frac{a}{b}$, $t\geq t_0$, for some $t_0\geq 0$, then the statement of Theorem 2 is true.

Suppose now that $N(t)>\frac{a}{b}$, $t\geq t_1$, for some $t_1\geq 0$. Equality (3) implies that $\dot{N}(t)\leq -c(t)N^{\alpha}(g(t))\leq 0,\ t\geq t_2$, for some $t_2\geq t_1$. Then there exists $N=\lim_{t\to\infty}N(t)$. Suppose $N>0$. Then

$$N(t)=N(t_2)+\int_{t_2}^{t}\dot{N}(s)ds\leq N(t_2)-\int_{t_2}^{\infty}c(s)N^{\alpha}(g(s))ds=-\infty.$$

Hence $N=0$ and then $\lim_{t\to\infty} N(t)=0$. We have a contradiction with our assumption.

Consequently there exists a sequence $\{t_n\}$, $\lim_{n\to\infty} t_n=\infty$, such that $N(h(t_n))=\frac{a}{b}$. The end of the proof is similar to the corresponding part of the proof of Theorem 1. □

REMARK 1. *The same result for a more general logistic equation without a harvesting term was obtained in [13].*

Consider now a modified equation (3):

$$\dot N(t)=r(t)N(t)\left[a-b_0N(t)-bN(h(t))\right]-\frac{c(t)N^{\alpha}(g(t))}{1+d(t)N^{\beta}(p(t))},\quad t\ge 0. \tag{21}$$

THEOREM 3. *Suppose $b_0>0$, hypotheses (a1)-(a4) hold,*

$$\varphi(t)\le N_0<\frac{a}{b_0}, \tag{22}$$

and

$$\left(\frac{a}{b_0}\right)^{\alpha-1}\sup_{t>0}\int_{g(t)}^{t} c(s)\exp\left\{\left[\frac{ab}{b_0}\right]\int_{g(s)}^{s} r(\tau)d\tau\right\}ds\le\frac{1}{e}. \tag{23}$$

Then for any solution of (21),(4) we have

$$0<N(t)\le\frac{a}{b_0}. \tag{24}$$

Proof. We apply the scheme of the proof of Theorem 1. Suppose (24) is not correct. Then either there exists $\bar t>0$ such that

$$0<N(t)\le\frac{a}{b_0},\ 0\le t<\bar t,\ N(\bar t)=\frac{a}{b_0},\ \dot N(\bar t)>0, \tag{25}$$

or there exists $\bar t>0$ such that

$$0<N(t)\le\frac{a}{b_0},\ 0\le t<\bar t, N(\bar t)=0. \tag{26}$$

Suppose the first possibility (25) holds. Then for $0<t<\bar t$ we have

$$\dot N(t)\le r(t)N(t)[a-b_0N(t)],\ N(0)=N_0.$$

Denote by x a solution of the following problem

$$\dot x(t)=r(t)x(t)[a-b_0x(t)],\ x(0)=N_0. \tag{27}$$

Then

$$N(t) \leq x(t) < \frac{a}{b_0},\ 0 \leq t \leq \bar{t},$$

since the solution of Eq.(27) tends to $\frac{a}{b_0}$ and is always less than $\frac{a}{b_0}$.

We have a contradiction with assumption (25).

Suppose now that for $\bar{t} > t_0$ (26) holds. Substituting in (21),(4)

$$N(t) = \exp\left\{\int_0^t r(s)\left[a - b_0 N(s) - bN(h(s))\right] ds\right\} x(t) \tag{28}$$

we have the following system:

$$\dot{x}(t) = -v(t)x(g(t)),\ t > 0, \tag{29}$$

$$x(t) = \varphi(t),\ t < 0, x(0) = N_0,$$

where

$$v(t) = \frac{c(t)N^{\alpha-1}(g(t))}{1 + d(t)N^{\beta}(p(t))} \exp\left\{-\int_{g(t)}^t r(s)\left[a - b_0 N(s) - bN(h(s))\right] ds\right\}.$$

Inequalities (23) and (26) imply that $\int_{g(t)}^t v(s)ds \leq$

$$\leq \int_{g(t)}^t \frac{c(s)N^{\alpha-1}(g(s))}{1 + d(s)N^{\beta}(p(s))} \exp\left\{\int_{g(s)}^s r(\tau)\left[bN(h(\tau)) + b_0 N(\tau) - a\right] d\tau\right\} ds$$

$$\leq \left(\frac{a}{b_0}\right)^{\alpha-1} \sup_{t>0} \int_{g(t)}^t c(s) \exp\left\{\left[\frac{ab}{b_0}\right] \int_{g(s)}^s r(\tau)d\tau\right\} ds \leq \frac{1}{e}.$$

Similarly to the proof of Theorem 1, Lemma 1 implies $N(t) > 0,\ 0 \leq t \leq \bar{t}$. This contradiction proves the theorem. □

Corollary 2. *Consider an autonomous equation*

$$\dot{N}(t) = N(t)\left[a - b_0 N(t) - bN(t-h)\right] - \frac{cN^{\alpha}(t-g)}{1 + dN^{\beta}(t-p)},\quad t \geq 0, \tag{30}$$

with initial conditions (4), where $a > 0, b > 0, c > 0, d > 0, h \geq 0, g \geq 0, p \geq 0, \alpha \geq 1, \beta \geq 0$. *Suppose the following conditions hold:*

$$\varphi(t) \leq N_0 < \frac{a}{b_0}$$

and

$$\left(\frac{a}{b_0}\right)^{\alpha-1} gc \exp\left\{\frac{agb}{b_0}\right\} \le \frac{1}{e}.$$

Then for every solution of (30),(4) we have

$$0 < N(t) \le \frac{a}{b_0}.$$

Corollary 3. *Consider an autonomous equation*

$$\dot{N}(t) = N(t)[a - b_0 N(t)] - \frac{cN^{\alpha}(t-g)}{1 + dN^{\beta}(t-p)}, \quad t \ge 0, \tag{31}$$

with initial conditions (4), where $a > 0, b_0 > 0, c > 0, d > 0, g \ge 0, p \ge 0, \alpha \ge 1, \beta \ge 0$. *Suppose the following conditions hold:*

$$\varphi(t) \le N_0 < \frac{a}{b_0}$$

and

$$\left(\frac{a}{b_0}\right)^{\alpha-1} gc \le \frac{1}{e}.$$

Then for every solution of (31),(4) we have

$$0 < N(t) \le \frac{a}{b_0}.$$

Theorem 4. *Suppose* $b_0 > 0$, *(a1)-(a4) hold. Then for every eventually positive solution of (21)-(4) there exists* $t_0 \ge 0$ *such that (24) holds for* $t \ge t_0$.
Proof is similar to the proof of Theorem 2.

Now let us obtain sufficient extinction conditions for solutions of logistic equations with harvesting. To this end consider the following equation:

$$\dot{N}(t) = r(t)N(t)\left[a - bN(h(t))\right] - \frac{c(t)N(g(t))}{1 + d(t)N^{\beta}(p(t))}, \quad t \ge 0. \tag{32}$$

Theorem 5. *Suppose conditions (a1)-(a5) hold (with* $\alpha = 1$*).*
Suppose in addition

$$\limsup_{t\to\infty} a(t - g(t)) < 1, \;\; C(t) \ge a, \;\; \int_0^{\infty} (C(t) - a)dt = \infty,$$

where $C(t) = \dfrac{c(t)}{1 + d(t)\left(\frac{a}{b}\right)^{\beta} \exp\left\{a\beta \sup_{t>0} \int_{h(t)}^{t} r(s)ds\right\}}.$

Then for any solution of (32)-(4) either $\lim_{t\to\infty} N(t) = 0$ *or there exists* $\bar{t} > 0$ *such that* $N(\bar{t}) < 0$.

Proof. It is sufficient to prove that for every positive solution $N(t)$ of (32)-(4) we have $\lim_{t\to\infty} N(t) = 0$.

Suppose $N(t) > 0$ is a solution of (32)-(4). Equation (32) and Theorem 2 imply

$$\dot{N}(t) + C(t)N(g(t)) - aN(t) \le 0.$$

Lemma 2 yields that there exists $t_0 \ge 0$, such that the fundamental function $X(t,s)$ of the equation

$$\dot{x}(t) + C(t)x(g(t)) - ax(t) = 0 \tag{33}$$

is positive for $t \ge s \ge t_0$. Then the variation of constant formula [8] implies

$$N(t) = x(t) + \int_{t_0}^{t} X(t,s)f(s)ds,$$

where $x(t)$ is a solution of (33) with the initial condition $x(t) = N(t),\ t \le t_0$ and $f(t)$ is a nonpositive function. Hence $0 < N(t) \le x(t)$.

Lemma 2 implies $\lim_{t\to\infty} x(t) = 0$. Then also $\lim_{t\to\infty} N(t) = 0$. ∎

COROLLARY 4. *Suppose conditions of Theorem 1 and Theorem 5 hold. Then any solution* $N(t)$ *of (3)-(4) is positive and satisfies* $\lim_{t\to\infty} N(t) = 0$.

COROLLARY 5. *Suppose for Eq.(20)* $a > 0, b > 0, c > 0, d > 0, h \ge 0, g \ge 0, p \ge 0, \alpha = 1, \beta \ge 0$. *If*

$$ag < 1, c > a\left(1 + d\left(\frac{a}{b}\right)^{\beta} \exp\{a\beta h\}\right)$$

then for any solution of (20)-(4) either $\lim_{t\to\infty} N(t) = 0$ *or there exists* $\bar{t} > 0$ *such that* $N(\bar{t}) < 0$.

COROLLARY 6. *Suppose conditions of Corollaries 1 and 5 hold. Then any solution* $N(t)$ *of (20)-(4) is positive and* $\lim_{t\to\infty} N(t) = 0$.

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