A Stabilization Criterion for Matrices

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Abstract

Given a simple linear system $\dot{x}(t) = Ax(t)$ which is unstable in the sense that A has eigenvalues in the open left half of the complex plane, it is shown how the system can be dilated to a *stable* system of larger size. The cases of real matrices and complex matrices are considered separately.

1 Introduction

Let $\mathbb{R}^{m\times n}$ denote the linear space of all real $m\times n$ matrices. We are concerned with *stable* and *unstable* matrices from $\mathbb{R}^{n\times n}$, i.e. those whose eigenvalues are all in the open left half of the complex plane, and those for which this is not the case, respectively. Note that, depending on the precise definition of "stability", pure-imaginary eigenvalues of a matrix may be either stable or unstable. Here, it is convenient to classify *all* pure-imaginary eigenvalues as unstable.

Our problem is easily stated: given an unstable matrix $A \in \mathbb{R}^{n\times n}$ and a matrix $D \in \mathbb{R}^{m\times m}$, when do there exist matrices $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$ such that the matrix

$$M := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1}$$

is stable? When this is the case we say that A and D are "merged stably", in other words A can be stabilized with a dilation.

An important application of this result is to the design of a "dynamic controller" to stabilize the unstable system $\dot{x}(t) = Ax(t)$. Thus, a positive solution to our problem ensures the existence of a stable dilated system:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

It will be seen that, in our construction, a necessary condition for a stable dilation is that m (the size of D) is not less than the dimension of a maximal unstable invariant subspace of A.

In Section 4 we consider the corresponding problem for complex matrices.

2 Preliminaries

In this section we review some necessary spectral theory of real square matrices. Reference [1] is a good source for this material, and we follow the style of that monograph. In general a matrix $M \in \mathbb{R}^{n\times n}$ has real eigenvalues and also non-real eigenvalues arising in complex conjugate pairs. The set of all eigenvalues of M is known as the *spectrum* of M and is denoted by $\sigma(M)$. To each *real* eigenvalue λ_0 of M corresponds a *root subspace* in $\mathbb{R}^n$ defined by

$$\mathcal{R}_{\lambda_0} = \mathrm{Ker}(\lambda_0 I - M)^n,$$

and to each pair of eigenvalues $\sigma \pm i\tau, \tau \neq 0$, corresponds a root subspace

$$\mathcal{R}_{\sigma\pm i\tau} = \mathrm{Ker}\left((\sigma^2+\tau^2)I - 2\sigma M + M^2\right)^p$$

for a sufficiently large integer p. Then $\mathbb{R}^n$ is the direct sum of all such subspaces, $\mathcal{R}_{\lambda_0}$ and $\mathcal{R}_{\sigma\pm i\tau}$. An *eigenspace* $\mathcal{E}$ is a direct sum of root subspaces of M.

The *stable subspace* of M is the eigenspace associated with all eigenvalues of M in the **open** left-half of the complex plane. Similarly, the *unstable subspace* of M is the eigenspace associated with all eigenvalues of M in the **closed** right-half of the complex plane. We say that matrix M is *stable* or *unstable* according as the stable subspace is all of $\mathbb{R}^n$ or a proper subspace of $\mathbb{R}^n$.

Now we re-formulate our problem more precisely:

Problem 1: Given an unstable $A \in \mathbb{R}^{n\times n}$ with a maximal unstable eigenspace of dimension $r > 0$ and a $D \in \mathbb{R}^{m\times m}$ with $m \geq r$, when do there exist matrices $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$ such that matrix M of (1) is stable?

Sufficient conditions for A and D to be stably merged will be provided in the main theorem below. However, our proof depends on an interesting result concerning vibrating systems, sometimes referred to as the Kelvin-Tait-Chetaev (or KTC) theorem. A general form of this result appears as Theorem 7 in the paper [3]. Here, we quote a special case in the form of a lemma. (The notations $P > 0$, $P \geq 0$ and $P < 0$ mean that P is positive definite, positive semidefinite, or negative definite, respectively.)

Another useful notation will be needed: if $A \in \mathbb{R}^{n\times n}$ then $\mathcal{S}(A) := \frac{1}{2}(A + A^T)$ and $\mathcal{I}(A) := \frac{1}{2}(A - A^T)$ are called the symmetric and skew-symmetric parts of A, respectively.

Lemma: Consider the matrix polynomial $L(\lambda) := \lambda^2 I + \lambda F + K$ where $K > 0$ and F, K are real. If $\mathcal{S}(F) > 0$ then L is stable, i.e. all eigenvalues of $L(\lambda)$ are in the open left half-plane.

The interesting feature of this result is that stability is ensured whatever $\mathcal{I}(F)$, the skew-symmetric part of F, may be. A natural tool for application to problems of this type is the Lyapunov theory. This is not used explicitly here, but note that it *is* used in [3] for the proof of Lemma 1.

3 The main result

Some preliminary constructions will be developed before stating the main result. Let A and D have the properties stated in **Problem 1**.

Since A has a maximal unstable eigenspace of dimension $r > 0$, A^T also has an unstable eigenspace, $\mathcal{N}$, of dimension r. Then A^T has a *real* Jordan canonical-form $A_s^T \dot{+} A_u^T$ where A_s is stable and all eigenvalues of A_u have nonnegative real parts. Thus, A itself has a real Jordan form

$$\begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix} \quad \begin{matrix} n-r \\ r \end{matrix} \tag{2}$$

Let D have a stable eigenspace $\mathcal{F}$ of dimension $m - r$ (possibly trivial). Then D has a real Jordan form

$$D = \begin{bmatrix} D_\nu & 0 \\ 0 & D_s \end{bmatrix} \quad \begin{matrix} r \\ m-r \end{matrix} \tag{3}$$

in which D_s is stable when $m > r$. The real Jordan matrix D_ν is "neutral" in the sense that, at this stage, there is no commitment with regard to the distribution of its eigenvalues.

Now let us state the main theorem:

Theorem 1 *Let $A \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{m\times m}$ be given as in* **Problem 1**. *If also*

D has a stable eigenspace $\mathcal{F}$ of dimension $m - r$ and, in (2) and (3), (4)

$$\mathcal{S}(D_\nu) < -\mathcal{S}(A_u), \tag{5}$$

then A and D can be merged stably, i.e. there exist real matrices B and C which solve Problem 1.

Notice first that condition (4) is empty if $m = r$. Also, since A_u is likely to have positive spectrum, the condition (5) generally forces the real Jordan matrix D_ν of (3) (and hence D itself) to also be stable.

Proof: It follows from (2) and (3) that there exist nonsingular real matrices P_1 and P_2 of size n and m, respectively, such that

$$P_1^{-1}AP_1 = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix}, \quad P_2^{-1}DP_2 = \begin{bmatrix} D_\nu & 0 \\ 0 & D_s \end{bmatrix}. \tag{6}$$

(If $m = r$ the block D_s does not appear, and similar reductions appear in the following argument.)

Let $\tilde{B}, \tilde{C} \in \mathbb{R}^{r \times r}$ be arbitrary, and define

$$B := P_1 \begin{bmatrix} 0 & 0 \\ \tilde{B} & 0 \end{bmatrix} P_2^{-1} \in \mathbb{R}^{n \times m}, \quad C := P_2 \begin{bmatrix} 0 & \tilde{C} \\ 0 & 0 \end{bmatrix} P_1^{-1} \in \mathbb{R}^{m \times n}. \tag{7}$$

Then

$$M := \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} A_s & 0 & \vdots & 0 & 0 \\ 0 & A_u & \vdots & \tilde{B} & 0 \\ \cdots & & \cdots & & \cdots \\ 0 & \tilde{C} & \vdots & D_\nu & 0 \\ 0 & 0 & \vdots & 0 & D_s \end{bmatrix} \begin{bmatrix} P_1^{-1} & 0 \\ 0 & P_2^{-1} \end{bmatrix}.$$

Clearly, the spectrum of M is the union of the spectra of A_s, D_s and the $2r \times 2r$ matrix

$$\tilde{M} := \begin{bmatrix} A_u & \tilde{B} \\ \tilde{C} & D_\nu \end{bmatrix}.$$

By definition A_s and D_s are stable, so the problem reduces to assigning the matrices $\tilde{B}$ and $\tilde{C}$ so that $\tilde{M}$ is stable.

To see that this can be done consider the characteristic polynomial $c(\lambda) = \det(\lambda I - \tilde{M})$. Using a Schur complement (see [2], for example):

$$\begin{aligned} c(\lambda) &= \det \begin{bmatrix} \lambda I - A_u & -\tilde{B} \\ -\tilde{C} & \lambda I - D_\nu \end{bmatrix} \\ &= \det(\lambda I - A_u) \cdot \det\left((\lambda I - D_\nu) - \tilde{C}(\lambda I - A_u)^{-1} \tilde{B} \right). \end{aligned}$$

Since $\tilde{C}$ is free for choice, assume that it commutes with A_u ($\tilde{C} = I$, for example). Then

$$\begin{aligned} c(\lambda) &= \det(\lambda I - A_u) \cdot \det\left((\lambda I - D_\nu) - (\lambda I - A_u)^{-1} \tilde{C}\tilde{B} \right) \\ &= \det\left((\lambda I - A_u)(\lambda I - D_\nu) - \tilde{C}\tilde{B} \right) \\ &= \det\left(\lambda^2 I - \lambda(A_u + D_\nu) + (A_u D_\nu - \tilde{C}\tilde{B}) \right). \end{aligned}$$

Now take any positive definite K of size r and choose $\tilde{C}$ (commuting with A_u) and $\tilde{B}$ so that

$$A_u D_\nu - \tilde{C}\tilde{B} = K. \tag{8}$$

This is clearly possible; choose $\tilde{C} = I$ and then $\tilde{B} = A_u D_\nu - K$, for example. In this case we obtain

$$c(\lambda) = \det\left(\lambda^2 I - \lambda(A_u + D_\nu) + K\right).$$

Recalling the hypothesis (5), we see that the lemma can be applied to conclude that all the zeros of $c(\lambda)$ (which are also the eigenvalues of $\tilde{M}$) are in the open left half plane. This concludes the proof. □

Example 1: Let

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}, \qquad D = \begin{bmatrix} -3 & 2 & -2 \\ 2 & -3 & -2 \\ 0 & 0 & -5 \end{bmatrix},$$

with

$$\sigma(A) = \{1,\, 4\} \quad \text{and} \quad \sigma(D) = \{-5,\, -5,\, -1\}.$$

A corresponding matrix of eigenvectors for D is (see (6))

$$P_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Thus, A_s and D_ν do not appear and $A_u = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$. The unstable subspace of A is $\mathcal{E}_u(A) = \mathbb{R}^2$.

Since D is stable of size three and is diagonalizable there are three natural choices for a two-dimensional stable invariant subspace of D - and hence for D_s. We choose the span of the first two eigenvectors of D (the first two columns of P_2) and obtain

$$D_\nu = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}, \quad D_s = \begin{bmatrix} -1 \end{bmatrix}.$$

Checking condition (5) we obtain

$$\mathcal{S}(A_u + D_\nu) = \mathcal{S}(A + D_\nu) = \begin{bmatrix} -3 & 1 \\ 1 & -2 \end{bmatrix} < 0,$$

and confirm that A and D can be merged stably. (If, on the other hand, we choose $D_\nu = \begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix}$, then condition (5) is not satisfied.)

To construct the matrices B and C we take $\tilde{C} = K = I_2$ in (8), so that

$$\tilde{B} = A_u D_\nu - K = \begin{bmatrix} -11 & -5 \\ -10 & -16 \end{bmatrix}.$$

Since $P_1 = I_2$, equations (7) yield

$$B = \begin{bmatrix} \tilde{B} & 0 \end{bmatrix} P_2^{-1} = \begin{bmatrix} -3 & 3 & -8 \\ 3 & -3 & -13 \end{bmatrix}, \quad C = P_2 \begin{bmatrix} \tilde{C} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

The (truncated) eigenvalues of the resulting 5×5 matrix M are given by: $\sigma(M) = \{-3.732,\ -0.5 \pm i(0.866),\ -0.268,\ -1\}$. We see that M is, indeed, stable. □

4 A related problem

A more direct (and closely connected) problem can be posed as follows, and can be resolved by the techniques developed above:

Problem 2: Given an $A \in \mathbb{R}^{n\times n}$ with an unstable eigenspace of dimension $r > 0$, find *stable* matrices $D \in \mathbb{R}^{r\times r}$, and matrices $B \in \mathbb{R}^{n\times r}$ and $C \in \mathbb{R}^{r\times n}$ such that matrix M of (1) is stable.

It is clear that D can be chosen to be a negative definite (real) diagonal matrix. As in (6), write the real Jordan form:

$$P_1^{-1} A P_1 = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix}, \tag{9}$$

and then, with $\tilde{B}, \tilde{C} \in \mathbb{R}^{r\times r}$,

$$B := P_1 \begin{bmatrix} 0 \\ \tilde{B} \end{bmatrix} \in \mathbb{R}^{n\times r}, \quad C := \begin{bmatrix} 0 & \tilde{C} \end{bmatrix} \in \mathbb{R}^{r\times n}. \tag{10}$$

As in the proof above it is found that M is stable if and only if the $2r \times 2r$ matrix

$$\tilde{M} := \begin{bmatrix} A_u & \tilde{B} \\ \tilde{C} & D \end{bmatrix}$$

is stable where $\tilde{C} \in \mathbb{R}^{r\times r}$ is a nonsingular matrix commuting with A_u. This is the case if, for any $K > 0$ in $\mathbb{R}^{r\times r}$, $\tilde{B} = \tilde{C}^{-1}(A_u D_s - K)$ and $D < -\mathcal{S}(A_u)$. We have:

Theorem 2 *Let $A \in \mathbb{R}^{n\times n}$ have an unstable eigenspace of dimension $r > 0$ and a real Jordan form as in* (9)*. Let $D \in \mathbb{R}^{r\times r}$ be a diagonal matrix, let $\tilde{C} \in \mathbb{R}^{(n-r)\times(n-r)}$ be any nonsingular matrix commuting with A_u and, with any positive definite $K \in \mathbb{R}^{r\times r}$, define $\tilde{B} = \tilde{C}^{-1}(A_u D - K)$. Then, with B and C defined as in* (10)*,*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{R}^{(n+r)\times(n+r)}$$

is stable provided that $D < -\mathcal{S}(A_u)$.

Example 2: Let us apply Theorem 2 to find stable dilations of size $2n$ for the identity, $I \in \mathbb{R}^{n\times n}$. They are found to have the form

$$\begin{bmatrix} I & C^{-1}(D-K) \\ C & D \end{bmatrix},$$

where C, D, $K \in \mathbb{R}^{n\times n}$, C is nonsingular, $K > 0$, and the real diagonal D satisfies $D < -I$.

Notice also that such a dilation cannot be symmetric, for this would imply $D = CC^T + K > 0$, contradicting the condition $D < -I$. □

5 Complex matrices

The stabilization result has been established for real matrices. But our problem can be posed equally well in the context of complex matrices. Furthermore, the line of argument used above extends immediately to complex matrices A, B, C, D. The main adjustments are, first, to simply replace the notion of the "symmetric" and "skew-symmetric" parts of a real matrix by the "hermitian" and "skew-hermitian" parts of a complex matrix and, second, to use the complex Jordan canonical form instead of the real form. Thus, if $A \in \mathbb{C}^{n\times n}$ the hermitian and skew-hermitian parts of A are the hermitian matrices

$$\mathcal{H}(A) := \frac{1}{2}(A + A^*), \quad \mathcal{K}(A) := \frac{1}{2i}(A - A^*),$$

and $A = \mathcal{H}(A) + i\mathcal{K}(A)$. In particular, the theorem of [3] used to justify the Lemma is formulated in these terms (i.e. for complex matrices). In the modified Lemma we have K hermitian and positive definite, and $\mathcal{H}(F) > 0$.

For the statement of the next theorem it is to be understood that A has a complex Jordan form $\begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix}$ with A_u unstable of size r and A_s stable of size $n-r$. Also, D has a complex Jordan form $\begin{bmatrix} D_\nu & 0 \\ 0 & D_s \end{bmatrix}$ in which D_s is stable.

Theorem 3 *Let $A \in \mathbb{C}^{n\times n}$ have a maximal unstable eigenspace of dimension $r > 0$ and $D \in \mathbb{C}^{m\times m}$ with $m \geq r$. If also*

$$D \text{ has a stable eigenspace of dimension } m - r \text{ and} \tag{11}$$

$$\mathcal{H}(D_\nu) < -\mathcal{H}(A_u), \tag{12}$$

then A and D can be merged stably, i.e. there exist complex matrices B and C which solve our Problem 1.

A version of Theorem 2 for complex matrices is left for the interested reader.

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