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Model-updating for self-adjoint quadratic eigenvalue problems

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Abstract

This paper concerns quadratic matrix functions of the form $L(\lambda) = M\lambda^2 + D\lambda + K$ where M, D, K are Hermitian $n \times n$ matrices with $M > 0$. It is shown how new systems of the same type can be generated with some eigenvalues and/or eigenvectors updated and this is accomplished without "spill-over" (i.e. other spectral data remain undisturbed). Furthermore, symmetry is preserved. The methods also apply for Hermitian matrix polynomials of higher degree.

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1. Introduction

This paper concerns quadratic matrix functions of the form

$$L(\lambda) = M\lambda^2 + D\lambda + K, \tag{1}$$

where M, D, K are Hermitian $n \times n$ matrices. It will be assumed throughout that M is positive definite (written $M > 0$). Positivity properties for D and K occur in many problem areas, but are not necessary for the methods developed here. Matrix functions of this kind appear frequently in problems of classical mechanics where M, D, K are known as the mass, damping, and stiffness matrices, respectively, and are generally real and symmetric.

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The set of all eigenvalues of $L(\lambda)$ (zeros of the determinant, $\det L(\lambda)$) form the *spectrum*, $\sigma(L)$, of $L(\lambda)$, and are of great physical interest. There are $2n$ eigenvalues (counting algebraic multiplicities) and their location in the complex plane (necessarily symmetric about the real line) determines vital physical properties of any underlying system.

The model-updating problem considered here is briefly as follows: Given spectral information on $L(\lambda)$, suppose that the locations of some (or all) eigenvalues are seen to be unfavourable. What changes in M, D, K will produce a favourable re-location of the spectrum? The same question may be posed for re-assigning unfavourable eigenvectors. Frequently, "unfavourable" eigenvalue distributions concern either clustered eigenvalues, or eigenvalues close to the imaginary axis, and adjustments are to be made to moderate such properties. However, although some constraints are necessary, there is no hypothesis in this work requiring that updates be "small" in any sense. When posed in the context of the "transfer function" $L(\lambda)^{-1}$, the problem concerning eigenvalues could also be described as that of "pole placement". Once the desired updates in coefficients have been determined, feedback methods may serve for implementation.

Problems of this kind have been considered by several authors (see [1,2,3,5,6,10], etc.). In particular, interesting solutions are proposed in the first two of these papers and, as in this work, solutions are sought which *maintain the symmetry of the coefficient matrices after disturbance*. The techniques proposed here are different, and more general in the sense that the condition $K > 0$ is not imposed and the coefficients may be complex Hermitian (not just real and symmetric). We take advantage of the detailed study of inverse problems begun in [8] and recently studied more closely in [13,11], for example.

In the theory developed in these references, the notion of a *self-adjoint* triple of spectral data plays a vital role. For the fundamental existence theorem (which is constructive) see Theorem 10.6 of [8] and papers quoted there. These triples play a vital part in the updating strategy proposed. We also remark that the theory and techniques developed here apply immediately to self-adjoint polynomial functions $L(\lambda)$ of higher degree – under much the same conditions.

Section 2 is devoted to a summary of necessary results from the theory of Hermitian matrix poynomials, and Section 3 gives the general updating strategy proposed. Here, we incorporate some ideas developed recently by Chu and Xu in [2]. Some remarks on confining perturbations to D and K only are included in Section 3.1. We then show in Section 4 how the strategy can be applied very simply to the basic problems of updating a single real eigenvalue, and of updating a pair of non-real conjugate complex eigenvalues. It will be seen that these particular problems can be solved quite easily with modest computational expense.

If the number of eigenvalues and/or eigenvectors to be updated is large compared to n, it may be advisable to compute complete spectral data for the unperturbed problem. But in many cases, this information is likely to be already known. Indeed, much of this information may be necessary before sensible large-scale "updates" can be formulated. However, the methods of Sections 4.1 and 4.2 do not require a complete spectral analysis, and will generally be much more efficient from the computational point of view. A start is made on the treatment of multiple semisimple eigenvalues in Sections 4.3 and 4.4 concerns the special case of "elliptic" problems, i.e. having no real eigenvalues. Section 5 concerns estimation of the magnitudes of the perturbed coefficient matrices in the case when only eigenvalues are updated.

The methods developed are illustrated on small artificial problems, but it is clear that accessible software can be utilized more generally. For simplicity, and because this arises most frequently in practice, all examples concern systems with real and symmetric coefficients. There is no explicit restriction on the size of $L(\lambda)$, but stability problems are to be expected as the size increases, as this will generally imply clustering of eigenvalues (see also Example 1 below).

2. Spectral data

The complexity of our problem is increased if $L(\lambda)$ has *both* real eigenvalues and eigenvalues in complex-conjugate pairs and the theory is formulated assuming that both are present. Our main focus is on the generic case of simple eigenvalues and the easily formulated (and understood) methods for updating them. The theory can be extended to admit multiple semisimple eigenvalues (i.e. having the same geometric and algebraic multiplicities) but, so that attention is not diverted from the most important case of simple eigenvalues, our discussion is limited. The inclusion of *defective* eigenvalues would be more technical and obscure the main ideas.

So suppose that there are $2r$ real semisimple eigenvalues ($0 \leqslant r \leqslant n$). When $r < n$ the non-real (semisimple) eigenvalues in the upper half of the complex plane are determined by a complex diagonal matrix $J_c = U_1 + \mathrm{i}W$ of size $(n-r) \times (n-r)$ with $W > 0$. The complex conjugate eigenvalues make up the diagonal entries of $\overline{J_c}$. Then the $2r$ real eigenvalues which are distributed between the diagonal entries of two $r \times r$ real diagonal matrices U_2 and U_3. The way in which these two matrices are formed will be discussed in what follows.

A complex (canonical) diagonal $2n \times 2n$ matrix including all the eigenvalues is now

$$J = \begin{bmatrix} J_c & 0 & 0 & 0 \\ 0 & U_2 & 0 & 0 \\ 0 & 0 & U_3 & 0 \\ 0 & 0 & 0 & \overline{J_c} \end{bmatrix} = \begin{bmatrix} U_1 + \mathrm{i}W & 0 & 0 & 0 \\ 0 & U_2 & 0 & 0 \\ 0 & 0 & U_3 & 0 \\ 0 & 0 & 0 & U_1 - \mathrm{i}W \end{bmatrix}. \tag{2}$$

A right eigenvector (say $x_j \neq 0$) can be associated with each diagonal entry of J (each eigenvalue), and these form the columns of an associated $n \times 2n$ matrix of eigenvectors, say X. Here, with our hypotheses on the spectrum, we may define an $n \times 2n$ matrix of eigenvectors of $L(\lambda)$ in the form

$$X = \begin{bmatrix} X_{c1} & X_{R1} & X_{R2} & X_{c2} \end{bmatrix}, \tag{3}$$

where X_{c1} and X_{c2} are $n \times (n-r)$ matrices of (generally) non-real eigenvectors corresponding to the eigenvalues of J_c and $\overline{J}_c$, respectively.[1] Matrices X_{R1} and X_{R2} are $n \times r$ (generally complex) matrices of eigenvectors corresponding to the real eigenvalues in U_2 and U_3, respectively. Note that the structure of X is consistent with that of J in (2).[2]

The two matrices (X, J) form a *Jordan pair* of matrices for $L(\lambda)$ and necessarily satisfy the condition

$$\det \begin{bmatrix} X \\ XJ \end{bmatrix} \neq 0. \tag{4}$$

Canonical structures of the function $L(\lambda)$ require the definition of a third matrix. With X, J formulated as above, this matrix takes the form

$$P = \begin{bmatrix} 0 & 0 & 0 & I_{n-r} \\ 0 & I_r & 0 & 0 \\ 0 & 0 & -I_r & 0 \\ I_{n-r} & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

and we observe that $P^* = P$ and $(JP)^* = JP$. This imposes a constraint on the distribution of the *r*eal eigenvalues between U_2 and U_3. The eigenvalues in U_2, U_3 have *positive type* and

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[1] When M, C, K are real and symmetric, we may take $X_{c2} = \overline{X}_{c1}$.

[2] It is known that $L(\lambda)$ of (1) can always be written as the product of two factors of first degree (see [7]). It follows from this factorization theory that, in addition, $\begin{bmatrix} X_{c1} & X_{R1} \end{bmatrix}$ and $\begin{bmatrix} X_{R2} & X_{c2} \end{bmatrix}$ are necessarily non-singular.

negative type, respectively. A simple procedure for determining these types is described in Section 4.1. If it known *a priori* that there are no real eigenvalues, then corresponding blocks of J, X, and P simply do not appear. (See [7], [8], or Chapter 12 of [9] for the theory, and [11] for an expository discussion.)

It is shown in [11] that if M, D, K are Hermitian and $M > 0$, then X can be defined so that the two conditions

$$XPX^* = 0, \quad X(JP)X^* = M^{-1} > 0 \tag{6}$$

hold. Then (X, J, PX^*) is known as a *self-adjoint* triple.

Although we have made the simplifying assumption that all eigenvalues are distinct, an algorithm for computing a self-adjoint triple making *no* hypotheses on the spectrum could be based on the basic constructive existence theorem – Theorem 10.6 of [8]. In the real-symmetric case, another approach can be found in [12].

Given a self-adjoint triple, the *moments* of the system are then the Hermitian matrices

$$\Gamma_j = X(J^j P)X^* \tag{7}$$

for all integers j for which J^j is defined. Furthermore, it can be shown that, when conditions (6) hold, Hermitian coeffcients of $L(\lambda)$ are determined recursively in terms of the moments (and hence X, J, P) as follows:

$$M = \Gamma_1^{-1}, \quad D = -M\Gamma_2 M, \quad K = -M\Gamma_3 M + D\Gamma_1 D. \tag{8}$$

This immediately solves the full inverse spectral problem: Given viable spectral data (in the form of a self-adjoint triple), the system coefficients can be computed using these formulae.

Example 1. When $n = 1$ and $L(\lambda) = m\lambda^2 + b\lambda + c$ (with $m > 0$) is semisimple (so that the two eigenvalues are distinct) there are self-adjoint triples of the following forms:

(a) When there are real zeros λ_1, λ_2 with $\lambda_1 > \lambda_2$,

$$X = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(b) When there are non-real zeros $\mu \pm \mathrm{i}(\lambda_1 - \lambda_2)/2$ with $\lambda_1 > \lambda_2$

$$X = \begin{bmatrix} e^{\mathrm{i}\pi/4} & e^{-\mathrm{i}\pi/4} \end{bmatrix}, \quad J = \begin{bmatrix} \mu + \mathrm{i}\frac{\lambda_1 - \lambda_2}{2} & 0 \\ 0 & \mu - \mathrm{i}\frac{\lambda_1 - \lambda_2}{2} \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In both cases (8) implies $m^{-1} = XJPX^* = \lambda_1 - \lambda_2$. This suggests that a smooth transition from real to non-real eigenvalues, or vice versa, will generally induce singularities in the coefficients.

3. A general strategy

It will be convenient to denote the coefficients of the target updated system by $\widehat{M}$, $\widehat{D}$, $\widehat{K}$. Consider the mass matrix first. Suppose that the eigenvalues and eigenvectors to be modified have been identified. Furthermore, we assume that (as described in Section 4.1) these eigenvectors are normalised as for inclusion in a complete self-adjoint triple (X, J, PX^*). Let J_1 be the sub-matrix of J associated with the data to be modified and J_0 be its (unknown) complement in J. Similarly, let the columns of a matrix X of a complete spectral triple be divided into submatrices X_1 (associated with the updating) and its complement X_0. Let the corresponding square submatrices of P be P_0, P_1.

Then the spectral decomposition of M from (8) gives

$$M^{-1} = XJPX^* = S_0 + S_1, \tag{9}$$

where we write $S_0 := X_0 J_0 P_0 X_0^*$, $S_1 = X_1 J_1 P_1 X_1^*$. Now changes are to be made to J_1 and/or X_1, but in such a way that the matrix P_1 is unchanged. Thus we admit only changes from real to real eigenvalues (maintaining sign characteristics), and non-real conjugate pairs of eigenvalues to non-real conjugate pairs. (In addition, the changes must be such that the new pair (X, J) is a Jordan pair, i.e. (4) still holds. But this is not a sensitive issue.)

Suppose that, after adjustment, the components X_1 and J_1 making up S_1 take the values $\widehat{X}_1$ and $\widehat{J}_1$, respectively, and define $\widehat{S}_1 = \widehat{X}_1 \widehat{J}_1 P_1 \widehat{X}_1^*$. Then the new mass matrix satisfies $\widehat{M}^{-1} = S_0 + \widehat{S}_1$, and it follows from (9) that

$$\widehat{M}^{-1} = M^{-1} + (\widehat{S}_1 - S_1). \tag{10}$$

It is important to note that, in general, the rank of S_1 and $\widehat{S}_1$ is just the number of updates being implemented. Thus, when updating one real eigenvalue, or a conjugate pair of eigenvalues, these matrices will have rank one or two, respectively. After computation of these low-rank matrices, $\widehat{M}$ can be computed at the further expense of two inversions of $n \times n$ matrices.

For updating the damping matrix, D, we introduce the (generally) low-rank matrices

$$T_0 := X_0 J_0^2 P_0 X_0^*, \quad T_1 := X_1 J_1^2 P_1 X_1^*$$

and we have $\Gamma_2 = T_0 + T_1$. Now let the updated T_1 be $\widehat{T}_1$ and, from (8),

$$D = -M(T_0 + T_1)M, \quad \widehat{D} = -\widehat{M}(T_0 + \widehat{T}_1)\widehat{M}.$$

Some algebraic manipulation to eliminate T_0 now leads to

$$\widehat{D} = \widehat{M}[M^{-1}DM^{-1} - (\widehat{T}_1 - T_1)]\widehat{M}. \tag{11}$$

To update the stiffness matrix, K, introduce $U_0 = X_0 J_0^3 P_0 X_0^*$ and $U_1 = X_1 J_1^3 P_1 X_1^*$ so that $\Gamma_3 = U_0 + U_1$, and then define the updated matrix $\widehat{U}_1 = \widehat{X}_1 \widehat{J}_1^3 P_1 X_1^*$. Then it follows from (8) that

$$K = -M(U_0 + U_1)M + DM^{-1}D, \quad \widehat{K} = -\widehat{M}(U_0 + \widehat{U}_1)\widehat{M} + \widehat{D}\widehat{M}^{-1}\widehat{D}.$$

Elimination of U_0 now leads to

$$\widehat{K} = -\widehat{M}[M^{-1}(DM^{-1}D - K)M^{-1} + (\widehat{U}_1 - U_1)]\widehat{M} + \widehat{D}\widehat{M}^{-1}\widehat{D}. \tag{12}$$

Coefficients of the updated system can now be generated by working successively through Eqs. (10)–(12).

At first glance, it may appear to be necessary to compute a complete self-adjoint triple for $L(\lambda)$. However, the final formulae (10)–(12) involve only the given coefficients M, D, K, the normalized data to be perturbed, and the updates.

These formulae may appear quite formidable, but note that after computing the spectral data to be modified, equation (9) (for the new mass matrix) requires inversion of two $n \times n$ matrices, but no more inversions are required for the damping and stiffness matrices of (11) and (12). Notice also that the differences $\widehat{S}_1 - S_1$, $\widehat{T}_1 - T_1$, $\widehat{U}_1 - U_1$ are easily formed.

Three constraints on this strategy should be emphasised:

1. The canonical matrix P is common to both the initial and the updated systems. (In particular, there is no change in the total number of real eigenvalues.)

2. If eigenvectors are updated, then the $n \times 2n$ updated matrix of eigenvectors, say $\widehat{X}$, must satisfy $\widehat{X} P \widehat{X}^* = 0$. (This implies, in particular, that it is not possible to update one and only one eigenvector: such an update will destroy this orthogonality condition.)
3. If the updated leading coefficient, $\widehat{M}$, is to be positive definite (like M) then (cf. Eq. (9)) there is a constraint on the updated data of the form $S_0 + \widehat{S}_1 > 0$.

3.1. *Confining perturbations to D and K*

Our analysis has assumed that all three coefficient matrices M, D, and K are accessible for the implementation of updating. Given our hypothesis that $M > 0$, it is possible to confine the perturbations to the damping and stiffness matrices, but at the expense of perturbing eigenvectors. This may be particularly useful if the precise nature of the eigenvectors is unimportant. The main advantage of such a strategy is likely to be that updates in D and K can be achieved by state and velocity feedback mechanisms only (and not acceleration). It can also be argued that, since eigenvalue/eigenvector problems are defined by a homogeneous equation, it is natural to insist on a common normalization for the perturbed and unperturbed systems.

Suppose that the updating process of $L(\lambda)$ has been completed and results in a system $\widehat{L}(\lambda) = \widehat{M}\lambda^2 + \widehat{D}\lambda + \widehat{K}$. Since $M > 0$ and $\widehat{M} > 0$ both matrices have unique positive definite square-roots, $M^{\frac{1}{2}}$ and $\widehat{M}^{\frac{1}{2}}$. It is clear that the system

$$M^{\frac{1}{2}} \widehat{M}^{-\frac{1}{2}} \widehat{L}(\lambda) \widehat{M}^{-\frac{1}{2}} M^{\frac{1}{2}}$$

has the spectrum of $\widehat{L}$ and the leading coefficient, M, of the original system, L. The cost of this re-normalization is to replace each eigenvector, x_j of $\widehat{L}(\lambda)$ by the transformed vector $M^{-\frac{1}{2}} \widehat{M}^{\frac{1}{2}} x_j$. This simplifies, of course, if the original system happened to be *monic*, i.e. had $M = I$.

4. Special cases of eigenvalue updating

This section includes basic eigenvalue-by-eigenvalue steps in computing the components of a self-adjoint triple . We consider the two basic cases: updating a real eigenvalue, and updating a conjugate pair of non-real eigenvalues. Obviously, if several updates of these types are to be made, they can either be made successively, or they can be made collectively in one application of the general strategy.

4.1. *Updating a real eigenvalue*

Suppose that we know a *real* eigenvalue $\lambda \neq 0$ and an associated eigenvector x_0 (possibly complex). For some reason, we are unhappy with this eigenvalue – it is to be "updated".

Using the data λ, x_0 and the coefficients M, D, K, first compute the real number

$$x_0^* D x_0 + 2\lambda(x_0^* M x_0), \tag{13}$$

which we write in the form $\epsilon\kappa^2$ where $\epsilon = \pm 1$ and $\kappa > 0$. Thus, ϵ is just the sign of this real number, and κ^2 is its absolute value.

Normalized data for this eigenvalue/eigenvector pair is now

$$\lambda, \quad x = x_0/\kappa, \quad \epsilon \tag{14}$$

and ϵ is called the *sign characteristic* of the eigenvalue λ_1. Using this data form the $n \times n$ matrices of rank one

$$S_1 = \epsilon\lambda xx^*, \quad T_1 = \epsilon\lambda^2 xx^*, \quad U_1 = \epsilon\lambda^3 xx^*. \tag{15}$$

Eigenvalue λ is now to be updated. Say, $\lambda \to \hat{\lambda}$. We form the corresponding matrices

$$\widehat{S}_1 = \epsilon\hat{\lambda} xx^*, \quad \widehat{T}_1 = \epsilon\hat{\lambda}^2 xx^*, \quad \widehat{U}_1 = \epsilon\hat{\lambda}^3 xx^* \tag{16}$$

(and note that the sign characteristic has not changed).

Then the updated coefficient matrices are given by Eqs. 10,11,12. The system $\widehat{L}(\lambda) = \widehat{M}\lambda^2 + \widehat{D}\lambda + \widehat{K}$ has the desired spectrum.

Example 2. We consider a problem from Section 7.4 of [6]. There, updating is accomplished by first generating an updated non-symmetric system using a method from control theory. Then "optimisation is required to produce symmetric matrices". In fact, an isospectral symmetric system is generated by an iterative process based on a technique devised by Minas and Inman, [15].

The inital system has real-symmetric coefficients

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 3 & 0 & 0 \\ & & 2 & 0 \\ & & & 4 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & -1 & -2 & 0 \\ & 2 & 0 & 0 \\ & & 4 & -2 \\ & & & 3 \end{bmatrix}, \quad K = \begin{bmatrix} 5 & -1 & -2 & 0 \\ & 2 & -1 & 0 \\ & & 4 & -1 \\ & & & 2 \end{bmatrix}.$$

Standard algorithms show (as in [6]) that the eigenvalues are[3]

$$\begin{aligned} &-1.2669, \quad -4.3499, \quad -0.1876 \pm 0.4487\mathrm{i}, \\ &-0.4179 \pm 0.7143\mathrm{i}, \quad -0.7944 \pm 0.8500\mathrm{i}. \end{aligned} \tag{17}$$

Corresponding eigenvectors are

−0.5592	−0.2147	−0.2975 − 0.0323i	0.1064 − 0.0969i	−0.1112 − 0.2309i
0.0905	0.0156	−0.5753	0.5355	0.2040 − 0.0766i
0.2382	0.0618	−0.4208 − 0.0660i	−0.0468 − 0.2136i	−0.3254 − 0.3482i
−0.0791	−0.0074	−0.4185 − 0.1779i	−0.3668 + 0.3220i	0.0467 + 0.2870i

Here, we find the system with the eigenvalues of (17) except that -1.2669 is to be replaced by -1.5. The eigenvectors are, of course, to be preserved. The procedure described above leads to the system with coefficients:

$$\widehat{M} = \begin{bmatrix} 1.0598 & -0.0290 & -0.0510 & 0.338 \\ & 3.0141 & 0.0247 & -0.0164 \\ & & 2.0434 & -0.0288 \\ & & & 4.0191 \end{bmatrix},$$

$$\widehat{D} = \begin{bmatrix} 5.5679 & -1.1802 & -2.4158 & 0.1941 \\ & 2.0412 & 0.1205 & -0.0402 \\ & & 4.2963 & -2.1269 \\ & & & 3.0379 \end{bmatrix},$$

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[3] For convenience, numerical results are displayed to four (truncated) decimal places. This does not apply to the computations themselves.

$$\widehat{K} = \begin{bmatrix} 6.3478 & -1.2012 & -2.8255 & 0.1589 \\ & 2.0300 & -0.8768 & -0.0237 \\ & & 4.5056 & -1.0973 \\ & & & 2.0187 \end{bmatrix}.$$

It can be verified numerically that this system has the prescribed properties.

4.2. *Updating a conjugate pair of eigenvalues*

Now suppose that we know a conjugate pair of eigenvalues, μ and $\bar{\mu}$ together with eigenvectors v_0 and w_0, respectively. If the system has real and symmetric coefficients, M, D, K, then one may take $w_0 = \bar{v}_0$, but this is no longer the case if at least one of the coeficients is Hermitian but not real and symmetric. We will maintain the more general hypothesis.

Calculate the (generally complex) number

$$k = w_0^* D v_0 + 2\mu w_0^* M v_0. \tag{18}$$

Let κ denote one of the square roots of k and then form the normalized data:

$$\begin{aligned} &\text{eigenvalue } \mu \text{ with eigenvector } v = v_0/\kappa, \\ &\text{eigenvalue } \bar{\mu} \text{ with eigenvector } w = w_0/\bar{\kappa}. \end{aligned} \tag{19}$$

Define matrices S_1, T_1, U_1 (which are the analogues of those in (15) for a real eigenvalue):

$$S_1 = \begin{bmatrix} v & w \end{bmatrix} \begin{bmatrix} 0 & \mu \\ \bar{\mu} & 0 \end{bmatrix} \begin{bmatrix} v & w \end{bmatrix}^*, \tag{20}$$

$$T_1 = \begin{bmatrix} v & w \end{bmatrix} \begin{bmatrix} 0 & \mu \\ \bar{\mu} & 0 \end{bmatrix}^2 \begin{bmatrix} v & w \end{bmatrix}^*, \tag{21}$$

$$U_1 = \begin{bmatrix} v & w \end{bmatrix} \begin{bmatrix} 0 & \mu \\ \bar{\mu} & 0 \end{bmatrix}^3 \begin{bmatrix} v & w \end{bmatrix}^*. \tag{22}$$

Consider updated data $\mu \to \hat{\mu}$ and its conjugate $\bar{\mu} \to \overline{\hat{\mu}}$. Formulate the updated matrices $\widehat{S}_1, \widehat{T}_1, \widehat{U}_1$ accordingly:

$$\widehat{S}_1 = \begin{bmatrix} v & w \end{bmatrix} \begin{bmatrix} 0 & \hat{\mu} \\ \overline{\hat{\mu}} & 0 \end{bmatrix} \begin{bmatrix} v & w \end{bmatrix}^* \tag{23}$$

and similarly for $\widehat{T}_1$ and $\widehat{U}_1$.

The updated coefficient matrices $\widehat{M}, \widehat{D}, \widehat{K}$ are now obtained by applying the formulae 10,11,12 as in the case of an updated real eigenvalue and, once again, symmetry of the system is maintained.

Example 3. We use the data of Example 2 again and, as suggested in [6], we make the updates $(-0.1876 \pm 0.4487\mathrm{i}) \to (-0.2 \pm 0.5\mathrm{i})$.

The updated system is found to be

$$\widehat{M} = \begin{bmatrix} 0.9954 & -0.0264 & -0.0129 & -0.0260 \\ & 2.8496 & -0.0752 & -0.1559 \\ & & 1.9634 & -0.0725 \\ & & & 3.8679 \end{bmatrix},$$

$$\widehat{D}=\begin{bmatrix}4.9998 & -1.0108 & -2.0005 & 0.0070\\ & 1.8867 & -0.0284 & -0.0231\\ & & 3.9973 & -1.9876\\ & & & 3.0947\end{bmatrix},$$

$$\widehat{K}=\begin{bmatrix}5.0014 & -0.9940 & -1.9983 & 0.0094\\ & 2.0109 & -0.9936 & 0.0461\\ & & 4.0020 & -0.9881\\ & & & 2.0607\end{bmatrix}.$$

4.3. *Multiple semisimple eigenvalues*

For simplicity, attention is confined to multiple *real* eigenvalues. Indeed, since the strategy is very like that for distinct eigenvalues, an example is used for demonstration. The essential feature of the process is that the orthogonality condition $XPX^* = 0$ of (6) must be maintained and, to this end, the matrix P remains unchanged while J is updated.

Example 4. The system $L(\lambda)$ with $M = I_3$,

$$D=\begin{bmatrix}5/4 & -\sqrt{2}/4 & -3/4\\ & 1/2 & -\sqrt{2}/4\\ & & 5/4\end{bmatrix},\quad K=\begin{bmatrix}1/4 & \sqrt{2}/4 & 1/4\\ & 1/2 & \sqrt{2}/4\\ & & 1/4\end{bmatrix}$$

has two eigenvalues equal to zero and both have positive type. There are also two real eigenvalues, -1 and -2, both of negative type, and a complex pair, ±i.

We update *one* of the pair of zero eigenvalues to the value -0.5 (retaining the positive type) without disturbing the other eigenvalues or the eigenvectors. Following the “general strategy” it is found that

$$\widehat{M}=\begin{bmatrix}1.3214 & -0.1515 & -0.1071\\ & 1.0714 & 0.0505\\ & & 1.0357\end{bmatrix},\quad \widehat{D}=\begin{bmatrix}2.3214 & -0.7576 & -1.2500\\ & 0.6429 & -0.1515\\ & & 1.4643\end{bmatrix},$$

$$\widehat{K}=\begin{bmatrix}1.1429 & 0.1010 & -0.2857\\ & 0.5714 & 0.5051\\ & & 0.5714\end{bmatrix}.$$

It can be verified numerically that this system has the desired spectral properties.

A converse problem might be: Perturb one of two distinct real eigenvalues (with independent eigenvectors) to create a system with a double semisimple real eigenvalue. This is obviously feasible.

4.4. *Elliptic systems*

Consider the case in which all eigenvalues appear in non-real conjugate pairs (both before and after updating). Such systems are said to be *elliptic*, and have special properties worthy of some separate discussion. The first observation is that, with real-symmetric elliptic systems, Eqs. (2) and (3) take the more simple form

$$J = J_c \oplus \overline{J_c},\quad X = \begin{bmatrix}X_c & \overline{X_c}\end{bmatrix}.$$

Example 5 (*This example is taken from* [4]). The oscillations of a mass–spring system lead to analysis of the 3×3 function $L(\lambda) = M\lambda^2 + D\lambda + K$ where

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 6 \end{bmatrix}, \quad K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 10 \end{bmatrix}.$$

It is found that the eigenvalues are the diagonal elements of J with

$$J_c = \begin{bmatrix} -0.0826 + \mathrm{i}(1.4502) & 0 & 0 \\ 0 & -0.7530 + \mathrm{i}(0.8576) & 0 \\ 0 & 0 & -0.5144 + \mathrm{i}(1.2469) \end{bmatrix}.$$

We observe that the problem is, indeed, elliptic.

Clearly, we have $P = \left[\begin{smallmatrix} 0 & I_3 \\ I_3 & 0 \end{smallmatrix}\right]$. Then repeated use of the strategy of Section 4.2 (or the “collective” procedure described in [12]) gives the normalized eigenvectors as the columns of

$$X_c = \begin{bmatrix} 0.3775 - 0.4667\mathrm{i} & 0.2440 - 0.0283\mathrm{i} & 0.1447 - 0.0656\mathrm{i} \\ -0.1482 - 0.0455\mathrm{i} & 0.4103 - 0.3670\mathrm{i} & 0.0186 - 0.2322\mathrm{i} \\ -0.0290 - 0.0066\mathrm{i} & -0.0343 + 0.1257\mathrm{i} & 0.2083 - 0.1720\mathrm{i} \end{bmatrix}.$$

This completes the preparatory steps. (It can now be checked that $\Gamma_0 = 0$ and $\Gamma_1 = M^{-1}$, for example.)

Suppose now that the first eigenvalue, $-0.0826 + \mathrm{i}(1.4502)$, is thought to be too close to the imaginary axis. We generate an updated system with this eigenvalue replaced by $-\mu + \mathrm{i}(1.4502)$ where $\mu > 0.0826$ and with the eigenvectors unchanged. So the new matrix $\widehat{J}$ is formed from

$$\widehat{J_c} = \begin{bmatrix} -\mu + \mathrm{i}(1.4502) & 0 & 0 \\ 0 & -0.7530 + \mathrm{i}(0.8576) & 0 \\ 0 & 0 & -0.5144 + \mathrm{i}(1.2469) \end{bmatrix}$$

and X_c, X are unchanged. It can then be verified numerically that $X(\widehat{J}P)X^* > 0$. We give the results below rounded to two decimal places for convenience.

When the critical eigenvalue is $-0.1826 + \mathrm{i}(1.4502)$ equations (8) give the coefficient matrices:

$$M = \begin{bmatrix} 0.99 & -0.03 & -0.01 \\ & 2.02 & 0.01 \\ & & 5.00 \end{bmatrix}, \quad D = \begin{bmatrix} 0.20 & -0.10 & -0.03 \\ & 3.02 & -1.00 \\ & & 6.00 \end{bmatrix},$$

$$K = \begin{bmatrix} 2.03 & -0.98 & 0.03 \\ & 2.96 & -0.02 \\ & & 9.99 \end{bmatrix}.$$

When the critical eigenvalue is $-0.2826 + \mathrm{i}(1.4502)$ the coefficient matrices become

$$M = \begin{bmatrix} 0.97 & -0.06 & -0.03 \\ & 2.04 & 0.02 \\ & & 5.01 \end{bmatrix}, \quad D = \begin{bmatrix} 0.41 & -0.21 & -0.07 \\ & 3.04 & -1.00 \\ & & 5.99 \end{bmatrix},$$

$$K = \begin{bmatrix} 2.07 & -0.97 & 0.05 \\ & 2.92 & -0.04 \\ & & 9.99 \end{bmatrix}.$$

These results suggest that the update can be achieved in large measure by adjustments to the damping matrix, D, only.

In this example there has been no perturbation of the eigenvectors; i.e. the columns of X, or X_c. As noted above, if X is also to be perturbed, then we are faced with the complication of maintaining the condition $X^*PX = 0$. However, Prells has shown in [16] (see also [13]) how this can be controlled. Thus, the normalizations required in this theory imply that (for an elliptic system) the real and imaginary parts of X_c are strongly connected. Thus, if we write the real and imaginary parts, $X_c = X_{\mathrm{R}} + \mathrm{i}X_{\mathrm{I}}$ then, *necessarily*, there is a real orthogonal matrix Θ such that

$$X_c = X_{\mathrm{R}}(I - \mathrm{i}\Theta). \tag{24}$$

Indeed, when X is properly normalized, X_{R} is non-singular and $X_{\mathrm{R}}^{-1}X_{\mathrm{I}}$ is an orthogonal matrix. Furthermore, the positive definite properties of the coefficients M, D, K depend on Θ, and not on X_R (see Theorem 9 of [13]).

However, for present purposes, the important observation is that, if X_c has the form (24), then the condition $X^*PX = 0$ is automatically satisfied. Furthermore, if updates are made in X_{R} only then, by not changing Θ, positivity properties of M, D, K will be preserved.

5. Estimating the perturbations

It is not surprising that, in the present context, the coefficient matrices M, D, K generally depend continuously on variations in the spectral data. For simplicity, we examine this more closely in the case when only a few (significantly less than n) eigenvalues are to be updated.

The first observation is that, if the change in the matrix of eigenvalues, J, has low rank, then the same is true of the moments Γ_j of (7). However, the resulting perturbations of M, D, K from (8) are less clear. To examine this, recall that neither P nor the eigenvector matrix X are to be updated. Then we have

$$\widehat{M}^{-1} - M^{-1} = X\widehat{J}PX^* - XJPX^*$$

from which it is easily deduced that

$$M - \widehat{M} = (\widehat{M} - M)X(\widehat{J} - J)PX^*M + MX(\widehat{J} - J)PX^*M.$$

Using the spectral matrix norm define $\kappa = \|X(\widehat{J} - J)PX^*M\|$, and it follows that

$$\|\widehat{M} - M\| \leqslant \kappa\|\widehat{M} - M\| + \kappa\|M\|.$$

Thus, if $\kappa < 1$ the relative change in the mass matrix under the perturbation satisfies:

$$\frac{\|\widehat{M} - M\|}{\|M\|} \leqslant \frac{\kappa}{1-\kappa} \tag{25}$$

and $\kappa \to 0$ as $\widehat{J} \to J$. Using standard arguments it is easily seen that, also, $\|\widehat{D} - D\| = \mathrm{O}(\kappa)$ and $\|\widehat{K} - K\| = \mathrm{O}(\kappa)$ as $\kappa \to 0$. Numerical experiments suggest that there can be wide variation in the "constants" associated with these "O" symbols – a subject that may justify further examination.

We compute an estimate for κ when just one real eigenvalue is perturbed, or when one conjugate pair is perturbed. Of course, we take advantage of the resulting low rank of $\widehat{J} - J$. Thus, if λ_j is a real eigenvalue and is replaced by a real $\hat{\lambda}_j$ (the associated real eigenvector x_j remaining unchanged),

$$X(\widehat{J}-J)PX^*M = \pm(\hat{\lambda}_j-\lambda_j)x_jx_j^{\mathrm{T}}M$$

implies

$$\begin{aligned}\kappa = |\hat{\lambda}_j-\lambda_j|\|x_jx_j^{\mathrm{T}}M\| &= |\hat{\lambda}_j-\lambda_j|\mathrm{sup}_{x\neq 0}\frac{\|x_jx_j^{\mathrm{T}}Mx\|}{\|x\|},\\ &= |\hat{\lambda}_j-\lambda_j|\mathrm{sup}_{x=1}|x_j^{\mathrm{T}}Mx|\|x_j\|,\\ &\leqslant |\hat{\lambda}_j-\lambda_j|\|Mx_j\|\|x_j\|,\end{aligned} \tag{26}$$

using Schwartz' inequality at the last step.

Now suppose that a non-real eigenvalue pair $\lambda_j, \bar{\lambda}_j$ is updated; $\lambda_j \to \hat{\lambda}_j$. A little calculation yields

$$X(\widehat{J}-J)PX^*M = 2\mathrm{Re}(x_j(\hat{\lambda}_j-\lambda_j)x_j^{\mathrm{T}})M,$$

from which it can be deduced that

$$\kappa \leqslant 2|\hat{\lambda}_j-\lambda_j|\|Mx_j\|\|x_j\|. \tag{27}$$

A priori estimates of the change in M, for example, under the shift of eigenvalues can be obtained by combining (25) with (26) or (27).

6. Conclusions

A general method for spectral updating of $L(\lambda)$ (or "pole placement" of the transfer function $L(\lambda)^{-1}$) has been presented and illustrated with numerical examples. In general only the data to be updated is required, but the eigenvector data must be carefully normalised. The theoretical spectral analysis of self-adjoint matrix polynomials is used to advantage here in a computational setting. The reader is reminded that, although second degree polynomials are the objects of study of this paper, the methods apply immediately to self-adjoint matrix polynomials of any degree with positive definite leading coefficient. A general-purpose algorithm for the computation of a self-adjoint triple (see equations (6)) would be of great advantage in problems of this kind and could be a topic of further investigation.

Attention has been confined to self-adjoint systems, but it is clear that, if symmetry is not an issue, then the techniques used here can be applied more widely to non-self-adjoint problems. (In particular, the notion of positive and negative real eigenvalue types does not arise.) Updates of selected parts of the spectrum and/or eigenvectors can be made once a *Jordan triple* of eigenvalues and eigenvectors for the undisturbed system has been determined (see [7,8,14], for example). This may be another topic for further investigation.

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