LINEARIZATION OF MATRIX POLYNOMIALS EXPRESSED IN POLYNOMIAL BASES

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Abstract

This paper concerns regular matrix polynomials $P(\lambda)$ when represented in various polynomial bases (other than the monomials 1, λ, $\lambda^2, \ldots$). As in the monomial case, matrices of "companion" form play an important part in theory and numerical practice. In particular, they are used here to construct "strong linearizations" of $P(\lambda)$. The paper contains three theorems concerning linearizations constructed for representations in (a) a general class of "degree graded" polynomials, (b) Bernstein polynomials, (c) Lagrange polynomials.

1 Introduction

An $s \times s$ matrix polynomial $P(\lambda)$ of degree n has s^2 entries, each of which is a scalar (complex) polynomial in λ with degree not exceeding n. Grouping like powers of λ together determines the representation $P(\lambda) = \sum_{j=0}^{n} \lambda^j A_j$, where the coefficients $A_j \in \mathbb{C}^{s \times s}$. Clearly, the polynomial could also be uniquely determined by $n+1$ samples of the function: $P_j := P(z_j)$, where the points $z_0, z_1, \ldots, z_n \in \mathbb{C}$ are distinct.

The process of gathering the $n+1$ matrices of coefficients of the successive powers of λ could be described as "interpolation by monomials". Indeed, the matrices $P_0, P_1, \ldots, P_n$ may be samples of a function $\hat{P}(\lambda)$ of a more general type; analytic, for example, and one may be interested in how the interpolant $P(\lambda)$ approximates $\hat{P}(\lambda)$. In this case classical approximation theory suggests that representation of $P(\lambda)$ in a basis other than the monomials would be advantageous.

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We consider only matrix polynomials which are *regular* in the sense that the determinant, $\det P(\lambda)$, does not vanish identically. It will be convenient to change the convention that $A_n \neq 0$ and admit zero matrices in the leading positions. Thus the *degree* is fixed by convenience and to say that $P(\lambda)$ has degree n does not imply $A_n \neq 0$. Practical and algorithmic concerns with such polynomials frequently involve the determination of eigenvalues; namely, those $\lambda_0 \in \mathbb{C}$ for which the rank of $P(\lambda_0)$ is less than s. Thus, the eigenvalue multiplicity properties (geometric and algebraic) have a role to play. The set of all eigenvalues of $P(\lambda)$ form the *spectrum* of $P(\lambda)$ and is denoted by $\sigma(P)$.

It is natural to study spectral properties of the polynomial via the associated pencil $\lambda C_1 - C_0$, where (when $n = 4$, for example)

$$C_1 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & A_4 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 0 & 0 & 0 & -A_0 \\ I & 0 & 0 & -A_1 \\ 0 & I & 0 & -A_2 \\ 0 & 0 & I & -A_3 \end{bmatrix}. \tag{1}$$

This has been extensively used and recognised; see [9], [10], [16], for example, among many other sources. The vital property of this pencil is that it forms a "strong" linearization of $P(\lambda)$ in the sense that it reproduces the multiplicity structures of the eigenvalues of both $P(\lambda)$ and its *reverse* polynomial $P^{\#}(\lambda) := \lambda^n P(1/\lambda)$.

A major objective of this paper is the study of corresponding structures which arise when the scalar entries of $P(\lambda)$ are represented in various classical (and useful) polynomial bases (i.e. other than monomials). For example, matrix polynomials expressed in other bases occur in computer-aided geometric design where Bernstein-Bézier bases and the Lagrange basis occur (see e.g. [3], [7], and [8]). Also, there are problems in partial differential equations with symmetries in the boundary conditions where Legendre polynomials are the most natural. Lagrange polynomial interpolation is traditionally viewed as a tool for theoretical analysis; however, recent work reveals several advantages to computation in the Lagrange basis (see e.g. [5, 11]). Finally, in approximation theory, Chebyshev polynomials have a special place due to their minimum-norm property (see e.g. [17]). In the present paper, corresponding analogues and extensions of (1) are to be formulated, and the property of strong linearization is to be investigated.

The strategy adopted is to apply reducing equivalence transformations to linearizations (such as $\lambda C_1 - C_0$). These transformations are constructed using λ-dependent block-triangular LU-decompositions of the linearizations in question (as developed in [1] and [2]). In addition to our use of these decompositions, they play a role in the formulation of iterative algorithms for eigenvalue computation, and in computing frequency response functions. In contrast to [2], the emphasis here is on a deeper analysis of spectral properties of the linearizations.

The details of this program depend on a particular property of the polynomial basis employed: whether it is *degree-graded* (consists of polynomials of degrees $0, 1, 2, \ldots, n$ (like the monomials)), or whether all polynomials have the same degree (as with the Lagrange interpolating polynomials). The paper is organised accordingly: Section 2 contains some preliminaries on linearization, Section 3 is concerned with degree-graded bases. Linearizations for representations in Bernstein and Lagrange bases are the subjects of Sections 4 and 5, respectively.

2 Linearization

Our use of the term "linearization" of an $s \times s$ regular matrix polynomial $P(\lambda) = \sum_{j=0}^{n} A_j \lambda^j$ will be consistent with the discussion of [14]. Since matrix polynomials with an "eigenvalue at infinity" are endemic in this work we first define the reverse polynomial $P^{\#}(\lambda) := \lambda^n P(\lambda^{-1})$ and then:

Definition: An $sn \times sn$ linear matrix pencil $\lambda A - B$ is a *strong linearization* of the regular matrix polynomial $P(\lambda)$ if there are unimodular matrix polynomials $E(\lambda),\ F(\lambda)$ such that

$$\begin{bmatrix} P(\lambda) & 0 \\ 0 & I_{s(n-1)} \end{bmatrix} = E(\lambda)(\lambda A - B)F(\lambda), \tag{2}$$

and there are unimodular matrix polynomials $H(\lambda),\ K(\lambda)$ such that

$$\begin{bmatrix} P^{\#}(\lambda) & 0 \\ 0 & I_{s(n-1)} \end{bmatrix} = H(\lambda)(A - \lambda B)K(\lambda). \tag{3}$$

It will be convenient for us to use Theorem 4 of [14]. Thus:

Theorem 1 *Let $P(\lambda)$ be an $s \times s$ regular matrix polynomial with leading coefficient A_n (possibly zero) and let $\lambda A - B$ be an $sn \times sn$ linear matrix function. Assume that, for each distinct finite eigenvalue λ_j there exist functions $E_j(\lambda)$ and $F_j(\lambda)$ which are unimodular and analytic on a neighbourhood of λ_j and for which*

$$\begin{bmatrix} P(\lambda) & 0 \\ 0 & I_{s(n-1)} \end{bmatrix} = E_j(\lambda)(\lambda A - B)F_j(\lambda), \tag{4}$$

If A_n is singular (or zero) assume also that there are functions $E_0(\lambda)$ and $F_0(\lambda)$ which are unimodular and analytic on a neighbourhood of $\lambda = 0$ and for which

$$\begin{bmatrix} P^{\#}(\lambda) & 0 \\ 0 & I_{s(n-1)} \end{bmatrix} = E_0(\lambda)(A - \lambda B)F_0(\lambda). \tag{5}$$

Then $\lambda A - B$ is a strong linearization of $P(\lambda)$.

If it is known only that (2) holds (or (4)) and there is no reference to an eigenvalue at infinity (if any) then we may call $\lambda A - B$ a *weak linearization*.

3 Degree-graded polynomial bases

3.1 Orthogonal polynomials

Consider a sequence of real polynomials $\{\phi_n(\lambda)\}_{n=0}^{\infty}$ with $\phi_n(\lambda)$ of degree n and $\phi_{)}(\lambda) \equiv 1$. If they are orthonormal on an interval of the real line (with respect to some nonnegative weight function) then they necessarily satisfy a three-term recurrence relation (see Chapter 10 of [6], for example). These relations can be written in the form

$$\lambda\phi_j(\lambda) = \alpha_j\phi_{j+1}(\lambda) + \beta_j\phi_j(\lambda) + \gamma_j\phi_{j-1}(\lambda), \qquad j = 1, 2, \ldots, \tag{6}$$

where the $\alpha_j,\ \beta_j,\ \gamma_j$ are real, $\phi_{-1}(\lambda) \equiv 0$, $\phi_0(\lambda) \equiv 1$, and, if k_j is the leading coefficient of $\phi_j(\lambda)$,

$$0 \neq \alpha_j = \frac{k_j}{k_{j+1}}, \qquad j = 0, 1, 2, \ldots. \tag{7}$$

The choices of coefficients $\alpha_j,\ \beta_j,\ \gamma_j$ defining three well-known sets of orthogonal polynomials (asociated with the names of Chebyshev and Legendre) are summarised in Table 1. More generally, any sequence of polynomials $\{\phi_j(\lambda)\}_{j=0}^{\infty}$ with $\phi_j(\lambda)$ of degree j is said to be *degree-graded* and obviously forms a linearly independent set.

Table 1: Three well-known orthogonal polynomials

Polynomial	$T_n(x)$	$P_n(x)$	$C_n(x)$
Name of polynomial	Chebyshev(1st kind)	Legendre(Spherical)	Chebyshev(2nd kind)
Weight function	$(1-x^2)^{-\frac{1}{2}}$	1	$(1-x^2)^{-\frac{1}{2}}$
Orthogonality interval	$[-1,1]$	$[-1,1]$	$[-1,1]$
Leading coefficient k_n	2^{n-1}	$\frac{(2n)!}{2^n(n!)^2}$	2^n
α_n	$\frac{1}{2}$	$\frac{n+1}{2n+1}$	$\frac{1}{2}$
β_n	0	0	0
γ_n	1	$\frac{n}{2n+1}$	1

An $s \times s$ matrix polynomial $P(\lambda)$ of degree n can now be expressed in terms of a set of degree-graded polynomials in the form

$$P(\lambda) = A_n\phi_n(\lambda) + A_{n-1}\phi_{n-1}(\lambda) + \cdots + A_1\phi_1(\lambda) + A_0\phi_0(\lambda). \tag{8}$$

where $A_0,\ A_1, \ldots, A_n \in \mathbb{C}^{n\times n}$. We define associated $sn \times sn$ block-matrices

$$C_0 = \begin{bmatrix} \beta_0 I_s & \gamma_1 I_s & 0 & & \cdots & 0 & -k_{n-1}A_0 \\ \alpha_0 I_s & \beta_1 I_s & \gamma_2 I_s & 0 & \cdots & 0 & -k_{n-1}A_1 \\ 0 & \alpha_1 I_s & \beta_2 I_s & \gamma_3 I_s & \cdots & 0 & -k_{n-1}A_2 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & 0 & \alpha_{n-4}I_s & \beta_{n-3}I_s & \gamma_{n-2}I_s & -k_{n-1}A_{n-3} \\ 0 & & \cdots & 0 & \alpha_{n-3}I_s & \beta_{n-2}I_s & -k_{n-1}A_{n-2} + k_n\gamma_{n-1}A_n \\ 0 & & \cdots & & 0 & \alpha_{n-2}I_s & -k_{n-1}A_{n-1} + k_n\beta_{n-1}A_n \end{bmatrix}, \tag{9}$$

$$C_1 = \operatorname{diag}\begin{bmatrix} I_s & \cdots & \cdots & I_s & k_nA_n \end{bmatrix}, \tag{10}$$

(and observe how the matrices of (1) fit into this scheme). This construction is essentially that of a "comrade" matrix introduced by Barnett (see Chapter 5 of [4]).

A little computation shows that

$$\begin{bmatrix} \phi_0(\lambda)I_s & \phi_1(\lambda)I_s & \phi_2(\lambda)I_s & \phi_3(\lambda)I_s & \phi_4(\lambda)I_s \end{bmatrix}(\lambda C_1 - C_0) = \begin{bmatrix} 0 & 0 & 0 & 0 & k_4P(\lambda) \end{bmatrix}. \tag{11}$$

The first $n-1$ row-into-column products simply reproduce some of the relations (6). For the last such product use equations (6), (7), and (8). In the notation of [16] this equation reads: [2]

$$\left(\mathbf{\Phi}^T(\lambda) \otimes I\right)(\lambda C_1 - C_0) = k_{n-1}e_n^T \otimes \mathbf{P}(\lambda)$$

[2] The authors thank a reviewer for pointing out the connections with work of [12] and [16]. This equation shows a clear connection with the 'left ansatz' of [16, eq. (3.9)]. This analogy suggests that, as in [16], for each polynomial basis $\mathbf{\Phi}(\lambda)$ two vector spaces of linearizations may be defined, and that, as in [12], these vector spaces may be explored for linearizations that preserve structure, or are particularly well-suited for the task at hand. These considerations warrant further study.

where $\mathbf{\Phi}^T(\lambda) = [\phi_0(\lambda), \phi_1(\lambda), \ldots, \phi_{n-1}(\lambda)]$.

Now suppose that λ_0 is an eigenvalue of $P(\lambda)$ with left eigenvector y, i.e. $y^H P(\lambda_0) = 0$ (where the superscript $()^H$ denotes the Hermitian (complex-conjugate) transpose of a matrix or vector). Then evaluating (11) at λ_0 and premultiplying by y^H gives:

$$\left[\begin{array}{ccccc} \phi_0(\lambda_0)y^H & \phi_1(\lambda_0)y^H & \phi_2(\lambda_0)y^H & \phi_3(\lambda_0)y^H & \phi_4(\lambda_0)y^H \end{array}\right](\lambda_0 C_1 - C_0) = 0\,. \quad (12)$$

This shows that every finite eigenvalue of $P(\lambda)$ is also an eigenvalue of $\lambda C_1 - C_0$ and also shows how left eigenvectors of $\lambda C_1 - C_0$ can be generated from those of $P(\lambda)$. [3] A similar explicit characterization of the relationship of a right eigenvector w of $P(\lambda)$ corresponding to finite eigenvalue λ with a right eigenvector of the pencil $\lambda C_1 - C_0$ can be made (see [1]). This argument shows that $P(\lambda)$ and $\lambda C_1 - C_0$ have the same spectrum, but more is true.

Theorem 2 *Let $P(\lambda)$ be a matrix polynomial of degree n and $\{\phi_n(\lambda)\}_{n=0}^{\infty}$ be a degree-graded system of polynomials satisfying the recurrence relation* (6). *Then the pencil $\lambda C_1 - C_0$ defined by* (9) *and* (10) *is a strong linearization of $P(\lambda)$.*

Proof: To take advantage of the first part of Theorem 1, we pre-multiply $\lambda C_1 - C_0$ by the $sn \times sn$ block permutation matrix

$$S := \begin{bmatrix} 0 & I_s & 0 & \cdots & 0 \\ 0 & 0 & I_s & & \\ & & & \ddots & \\ 0 & \cdots & & 0 & I_s \\ I_s & 0 & \cdots & & 0 \end{bmatrix}, \quad (13)$$

and note that $\lambda C_1 - C_0$ is a strong linearization if and only if the same is true of $S(\lambda C_1 - C_0)$. In Section 2.3 of [2] the λ-dependent block LU factors of $S(\lambda C_1 - C_0)$ are obtained. Indeed, it is easily verified that $S(\lambda C_1 - C_0) = L(\lambda)U(\lambda)$ where

$$L(\lambda) = \begin{bmatrix} I_s & & & \\ & I_s & & \\ & & \ddots & \\ -\frac{\phi_1(\lambda)}{\phi_0(\lambda)} & \cdots & -\frac{\phi_{n-1}(\lambda)}{\phi_0(\lambda)} I_s & I_s \end{bmatrix}. \quad (14)$$

$$U(\lambda) = \begin{bmatrix} -\alpha_0 I_s & (\lambda-\beta_1)I_s & -\gamma_2 I_s & & & U_{1,n}(\lambda) \\ & \ddots & \ddots & \ddots & & \vdots \\ & & -\alpha_{n-4}I_s & (\lambda-\beta_{n-3})I_s & -\gamma_{n-2}I_s & U_{n-3,n}(\lambda) \\ & & & -\alpha_{n-3}I_s & (\lambda-\beta_{n-2})I_s & U_{n-2,n}(\lambda) \\ & & & & -\alpha_{n-2}I_s & U_{n-1,n}(\lambda) \\ & & & & & U_{n,n}(\lambda) \end{bmatrix}, \quad (15)$$

and

$$U_{i,n}(\lambda) = \begin{cases} k_{n-1}A_i, & i = 1{:}(n-3), \\ k_{n-1}A_{n-2} - k_n\gamma_{n-1}A_n, & i = n-2, \\ k_{n-1}A_{n-1} + k_n(\lambda - \beta_{n-1})A_n, & i = n-1, \\ \frac{1}{\alpha_0\cdots\alpha_{n-2}}P(\lambda). & i = n. \end{cases} \quad (16)$$

[3] This is a generalization of part(ii) of Theorem 5.2 of [4]; special cases have appeared in [1].

Clearly, $L(\lambda)$ is well-defined and nonsingular for all λ. Consequently, $U(\lambda)$ is singular at the eigenvalues of $P(\lambda)$ and, recalling condition (7), all of these eigenvalues are associated with $U_{n,n}(\lambda)$. If we define $\widetilde{U}(\lambda)$ to be the same as $U(\lambda)$ except for this last block entry which is replaced by

$$\widetilde{U}_{n,n}(\lambda) = \frac{1}{\alpha_0 \cdots \alpha_{n-2}} I_s, \tag{17}$$

then we have $\det L(\lambda) \equiv \det(\widetilde{U}(\lambda)) \equiv \pm 1$ and

$$S(\lambda C_1 - C_0) = L(\lambda)U(\lambda) = L(\lambda) \begin{bmatrix} I_{n(s-1)} & 0 \\ 0 & P(\lambda) \end{bmatrix} \widetilde{U}(\lambda).$$

It follows that

$$\begin{bmatrix} I_{(n-1)s} & 0 \\ 0 & P(\lambda) \end{bmatrix} = E(\lambda)(\lambda C_1 - C_0)F(\lambda), \tag{18}$$

where $E(\lambda) := L^{-1}(\lambda)S$ and $F(\lambda) := \widetilde{U}^{-1}(\lambda)$ are analytic and invertible at all the finite eigenvalues of $P(\lambda)$. It follows from Theorem 1 that $\lambda C_1 - C_0$ is a (weak) linearization of $P(\lambda)$. (The explicit forms for $E(\lambda)$ and $F(\lambda)$ may be useful elsewhere and are relegated to an Appendix to this paper.)

Now consider the reverse polynomial $P^{\#}(\lambda) = \lambda^n P(\lambda^{-1})$. Referring to Theorem 1, we are to show that there are matrix functions $H(\lambda)$ and $K(\lambda)$ which are analytic and nonsingular in a neighbourhood of $\lambda = 0$ and for which

$$\begin{bmatrix} I_{(n-1)s} & 0 \\ 0 & P^{\#}(\lambda) \end{bmatrix} = H(\lambda)(C_1 - \lambda C_0)K(\lambda). \tag{19}$$

First consider the λ-dependent block LU factors of $\lambda C_1 - C_0$ as obtained in [2]. In fact, it is easily verified that $\lambda C_1 - C_0 = L(\lambda)U(\lambda)$ where

$$L(\lambda) = \begin{bmatrix} I_s & & & \\ -\frac{\phi_0(\lambda)}{\phi_1(\lambda)} I_s & I_s & & \\ & \ddots & \ddots & \\ & & -\frac{\phi_{n-2}(\lambda)}{\phi_{n-1}(\lambda)} I_s & I_s \end{bmatrix}, \tag{20}$$

$$U(\lambda) = \begin{bmatrix} \alpha_0 \frac{\phi_1(\lambda)}{\phi_0(\lambda)} I_s & -\gamma_1 I_s & & & U_{1,n}(\lambda) \\ & \ddots & \ddots & & \vdots \\ & & \alpha_{n-3} \frac{\phi_{n-2}(\lambda)}{\phi_{n-3}(\lambda)} I_s & -\gamma_{n-2} I_s & U_{n-2,n}(\lambda) \\ & & & \alpha_{n-2} \frac{\phi_{n-1}(\lambda)}{\phi_{n-2}(\lambda)} I_s & U_{n-1,n}(\lambda) \\ & & & & U_{n,n}(\lambda) \end{bmatrix}, \tag{21}$$

and

$$U_{i,n}(\lambda) = \begin{cases} k_{n-1} A_0, & i = 1, \\ k_{n-1} A_{i-1} + \frac{\phi_{i-2}(\lambda)}{\phi_{i-1}(\lambda)} U_{i-1,n}(\lambda), & i = 2{:}(n-2), \\ k_{n-1} A_{n-2} + \frac{\phi_{n-3}(\lambda)}{\phi_{n-2}(\lambda)} U_{n-2,n}(\lambda) - k_n \gamma_{n-1} A_n, & i = n-1, \\ \frac{\phi_0(\lambda)}{(\alpha_0 \cdots \alpha_{n-2})\phi_{n-1}(\lambda)} P(\lambda). & i = n. \end{cases} \tag{22}$$

Making the transformation $\lambda \to \lambda^{-1}$ in the decomposition $\lambda C_1 - C_0 = L(\lambda)U(\lambda)$ we obtain

$$C_1 - \lambda C_0 = \lambda L(\lambda^{-1})U(\lambda^{-1}).$$

So we obtain block LU-factors for the reverse pencil: $C_1 - \lambda C_0 = L_1(\lambda)U_1(\lambda)$, where

$$L_1(\lambda) := L(\lambda^{-1}), \qquad U_1(\lambda) := \lambda U(\lambda^{-1}).$$

To examine the behaviour of $C_1 - \lambda C_0$ near $\lambda = 0$, we first define $H(\lambda) = L_1^{-1}(\lambda)$. Then change the "last" (n, n) block entry of $U_1(\lambda)$ to

$$\frac{\phi_0(\frac{1}{\lambda})}{(\alpha_0 \cdots \alpha_{n-2})\lambda^{n-1}\phi_{n-1}(\frac{1}{\lambda})} I_s \tag{23}$$

to obtain a matrix function, $\widetilde{U}_1(\lambda)$, whose determinant is a nonzero constant. Then set $K(\lambda) := \widetilde{U}_1^{-1}(\lambda)$, and it can be verified that (19) holds.

Now consider the properties of $H(0)$ and $K(0)$. Since $\phi_j(\lambda)/\phi_{j+1}(\lambda) \to 0$ as $\lambda \to \infty$, it follows from (20) that $\lim_{\lambda\to 0} L(\lambda^{-1}) = I$. But we define $H(\lambda) = L^{-1}(\lambda^{-1})$ and so $\lim_{\lambda\to 0} L^{-1}(\lambda^{-1}) = I$. It follows that $H(\lambda)$ is analytic and invertible at $\lambda = 0$.

Then observe that, from (21) and (22), as $\lambda \to \infty$, the orders of magnitude of entries of $U(\lambda)$ are $O(\lambda)$ for diagonal terms and $0(1)$ for off-diagonal terms. Thus, for $U_1(\lambda) = \lambda U(\lambda^{-1})$, (and keeping in mind the exceptional (n, n) term of (23)) the corresponding orders of magnitude as $\lambda \to 0$ are $0(1)$ and $0(\lambda)$, respectively. Consequently, both $H(\lambda)$ and $K(\lambda)$ are analytic and invertible in a neighbourhood of $\lambda = 0$. This completes the proof. □

3.2 Symmetrizing the linearization

If the data matrices $A_0, A_1, \ldots, A_n$ are Hermitian, then the resulting polynomial $P(\lambda)$ is Hermitian for real λ. Although the symmetry appears to be lost in the pencil $\lambda C_1 - C_0$, it can be recovered in the case of the monomial basis (and when A_n is nonsingular) on postmultiplication of the companion matrix

$$C_0 C_1^{-1} = \begin{bmatrix} 0 & 0 & 0 & -A_0 A_4^{-1} \\ I & 0 & 0 & -A_1 A_4^{-1} \\ 0 & I & 0 & -A_2 A_4^{-1} \\ 0 & 0 & I & -A_3 A_4^{-1} \end{bmatrix}.$$

by the Hermitian "symmetrizer",

$$H_0 := \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ A_2 & A_3 & A_4 & 0 \\ A_3 & A_4 & 0 & 0 \\ A_4 & 0 & 0 & 0 \end{bmatrix} \tag{24}$$

(and we take $n = 4$ for convenience). In this way the eigenvalue problem for the Hermitian matrix polynomial $P(\lambda)$ can be examined in terms of the Hermitian pencil $\lambda H_0 - (C_0 C_1^{-1})H_0$. There is an extensive theory for problems of this kind developed in [10]. This symmetrization also works if the data matrices are not Hermitian but rather complex symmetric ($A_j^T = A_j$ for each j). In either case, the block symmetries of such a pencil can provide computational advantages.

It turns out that, in some cases, this symmetrizing property extends to the pencils generated by other bases. Indeed, the following proposition is easily verified:

Proposition 3 *Let $\{\phi_n(\lambda)\}_{n=0}^{\infty}$ be a degree-graded system of polynomials satisfying a recurrence relation* (6) *in which $\alpha_j = \alpha \neq 0$, $\beta_j = \beta$, and $\gamma_j = \gamma$ for all j. Moreover, let $P(\lambda)$ be a Hermitian matrix polynomial defined in that basis with A_n nonsingular. Then, when the generalized companion matrix $C_0 C_1^{-1}$ of $P(\lambda)$ (formed by* (9) *and* (10)*) is multiplied on the right by the Hermitian symmetrizer* (24)*, the result is also Hermitian. A similar result holds in the complex symmetric case.*

Clearly, under the hypotheses of the theorem $\lambda H_0 - (C_0 C_1^{-1})H_0$ is a Hermitian linearization of $P(\lambda)$. For cases when A_n is singular Hermitian linearizations can be found in [12].

3.3 Special degree-graded bases

As mentioned above, the family of degree-graded polynomials with recurrence relations of the form (6) include all the orthogonal bases, but is not limited to them. Here, we illustrate with some well-known non-orthogonal bases for which Theorem 2 holds and, consequently, for which the linearization $\lambda C_1 - C_0$ of (9) and (10) is strong.

- The monomial basis.
  Put $\alpha_j = 1$ and $\beta_j = \gamma_j = 0$ in (6) to generate the monomial basis.
- The Newton basis.
  Let an $s \times s$ matrix polynomial $P(\lambda)$ be specified by the data $\{(z_j, P_j)\}_{j=0}^{n}$ where the z_j's are distinct. If the "Newton polynomials" are defined by setting $N_0(\lambda) = 1$ and, for $k = 1, \cdots, n$,
  $$N_k(\lambda) = \prod_{j=0}^{k} (\lambda - z_j), \tag{25}$$
  then $P(\lambda) = \sum_{j=0}^{n} A_j N_j(\lambda)$. The A_j's can be found either by divided differences or, equivalently, by solving the block-triangular system:
  $$\begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ I & N_1(z_1)I & 0 & & \\ I & N_1(z_2)I & N_2(z_2)I & & \\ \vdots & \vdots & \vdots & \ddots & \\ I & N_1(z_n)I & N_2(z_n)I & \cdots & N_n(z_n)I \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} = \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix}. \tag{26}$$
  See [2] for more details. The Newton polynomials are generated by (6) if we set $\alpha_j = 1$, $\beta_j = z_j$ and $\gamma_j = 0$.
- The Pochhammer basis.
  The Pochhammer basis is just a special Newton basis with nodes $z_j = -(a + j)$, $j = 0, \ldots, n-1$. It is generated by setting $\alpha_j = 1$, $\beta_j = -(a+j)$ and $\gamma_j = 0$ in (6). It has been used in combinatorial applications and in the solution of difference equations. Also, sparse polynomial interpolation algorithms have been developed using this basis (see [15], for example).

4 Linearization in the Bernstein basis

Bernstein polynomials are defined on a finite interval $[a,\, b]$ and have the form:

$$b_{j,n}(\lambda; a, b) = \frac{1}{(b-a)^n}\binom{n}{j}(\lambda - a)^j (b-\lambda)^{n-j} \tag{27}$$

for $n = 1, 2, \cdots$ and $j = 0, 1, \cdots, n$. Clearly, they are not degree-graded. An $s \times s$ matrix polynomial $P(\lambda)$ of degree n can be written as a linear combination of Bernstein polynomials. Thus,

$$P(\lambda) = \sum_{j=0}^{n} A_j b_{j,n}(\lambda; a, b), \tag{28}$$

where $A_0,\, A_1, \ldots, A_n \in \mathbb{C}^{n\times n}$. Now there are natural analogues of equations (9) and (10) of the form (with $n = 5$, for example)

$$C_0 = \begin{bmatrix} \frac{5a}{b-a}I_s & 0 & 0 & 0 & -\frac{b}{b-a}A_0 \\ \frac{b}{b-a}I_s & \frac{4a}{2(b-a)}I_s & 0 & 0 & -\frac{b}{b-a}A_1 \\ 0 & \frac{b}{b-a}I_s & \frac{3a}{3(b-a)}I_s & 0 & -\frac{b}{b-a}A_2 \\ 0 & 0 & \frac{b}{b-a}I_s & \frac{2a}{4(b-a)}I_s & -\frac{b}{b-a}A_3 \\ 0 & 0 & 0 & \frac{b}{b-a}I_s & \frac{a}{5(b-a)}A_5 - \frac{b}{b-a}A_4 \end{bmatrix}, \tag{29}$$

$$C_1 = \begin{bmatrix} \frac{5}{b-a}I_s & 0 & 0 & 0 & -\frac{1}{b-a}4A_0 \\ \frac{1}{b-a}I_s & \frac{4}{2(b-a)}I_s & 0 & 0 & -\frac{1}{b-a}A_1 \\ 0 & \frac{1}{b-a}I_s & \frac{3}{3(b-a)}I_s & 0 & -\frac{1}{b-a}A_2 \\ 0 & 0 & \frac{1}{b-a}I_s & \frac{2}{4(b-a)}I_s & -\frac{1}{b-a}A_3 \\ 0 & 0 & 0 & \frac{1}{b-a}I_s & \frac{1}{5(b-a)}A_5 - \frac{1}{b-a}A_4 \end{bmatrix}. \tag{30}$$

Block matrices of this form have been used in [1, 13, 18], for example.

A little computation shows that, in contrast to equation (11) (for degree-graded polynomials),

$$[b_{0,5}(\lambda; a, b)I_s \quad b_{1,5}(\lambda; a, b)I_s \quad \ldots \quad b_{4,5}(\lambda; a, b)I_s](\lambda C_1 - C_0) = [0 \quad 0 \quad 0 \quad 0 \quad \frac{b-\lambda}{b-a}P(\lambda)].$$

As in the degree-graded case, it can be seen that $\lambda C_1 - C_0$ and $P(\lambda)$ have the same eigenvalues. There is also an analogue of Theorem 2.

Theorem 4 *Let $P(\lambda)$ be a matrix polynomial of degree n, $\{b_{i,n}(\lambda; a, b)\}_{i=0}^{n}$ be a system of Bernstein polynomials and write $P(\lambda)$ in the form* (28)*. Then the pencil $\lambda C_1 - C_0$ defined as in* (29) *and* (30) *is a strong linearization of $P(\lambda)$.*

Proof: The proof is in three parts. Weak linearization is established in parts (a) and (b). Part (a) concerns any neighbourhood of b which does not include a. Part (b) concerns any neighbourhood of a which does not include b. Part (c) concludes the proof by showing that the linearization is strong. As in preceding arguments, we use block LU factorizations of $\lambda C_1 - C_0$, or of $S(\lambda C_1 - C_0)$, depending on the location of λ relative to a and b.

(a) As long as $\lambda \neq b$, the λ-dependent block LU factors of $S(\lambda C_1 - C_0)$ corresponding to a pencil of the form (29)– (30) and of degree n are given explicitly in [2] as follows:

$$L(\lambda) = \begin{bmatrix} I_s & 0 & \cdots & & 0 \\ 0 & I_s & 0 & & 0 \\ \vdots & & & \ddots & \vdots \\ & & & I_s & 0 \\ -\frac{\binom{n}{1}(\lambda-a)}{(b-\lambda)} I_s & \cdots & & -\frac{\binom{n}{n-1}(\lambda-a)^{n-1}}{(b-\lambda)^{n-1}} & I_s \end{bmatrix}, \tag{31}$$

$$U(\lambda) = \begin{bmatrix} -\frac{(b-\lambda)}{(b-a)} I_s & \frac{(n-1)(\lambda-a)}{2(b-a)} I_s & 0 & \cdots & 0 & U_{1,n}(\lambda) \\ 0 & -\frac{(b-\lambda)}{(b-a)} I_s & \frac{(n-2)(\lambda-a)}{3(b-a)} I_s & & & U_{2,n}(\lambda) \\ & & \ddots & \ddots & & \vdots \\ \vdots & & & -\frac{(b-\lambda)}{(b-a)} I_s & \frac{2(\lambda-a)}{(n-1)(b-a)} I_s & U_{n-2,n}(\lambda) \\ & & & & -\frac{(b-\lambda)}{(b-a)} I_s & U_{n-1,n}(\lambda) \\ 0 & & \cdots & & 0 & U_{n,n}(\lambda) \end{bmatrix}, \tag{32}$$

where

$$U_{i,n}(\lambda) = \begin{cases} \frac{b-\lambda}{b-a} A_0, & i = 1{:}(n-2), \\ \frac{b-\lambda}{b-a} A_{n-1} + \frac{(\lambda-a)}{n(b-a)} A_n, & i = n-1, \\ \frac{(b-a)^{n-1}}{(b-\lambda)^{n-1}} P(\lambda). & i = n. \end{cases} \tag{33}$$

Thus, $S(\lambda C_1 - C_0) = L(\lambda) U(\lambda)$.

As in the degree-graded case (Theorem 2), we determine a $\widetilde{U}(\lambda)$ by replacing the last block entry of (32) with

$$\widetilde{U}_{n,n}(\lambda) = \frac{(b-a)^{n-1}}{(b-\lambda)^{n-1}} I_s. \tag{34}$$

Now define $E(\lambda) = L^{-1}(\lambda) S$ and $F(\lambda) = \widetilde{U}^{-1}(\lambda)$. With the possible exception of the point $\lambda = b$, $E(\lambda)$ and $F(\lambda)$ are analytic and invertible at the finite eigenvalues of $P(\lambda)$.

(b) Part (a) of the proof shows that, with the construction of equivalence transformations which are well-defined everywhere except at b, all partial multiplicities of all finite eigenvalues of $P(\lambda)$ are reproduced in $\lambda C_1 - C_0$ - with the possible exception of an eigenvalue at b. On the other hand, it is clear that there is a similar construction of an equivalence transformation having a as the exceptional point which can be used to show that the partial multiplicities of all eigenvalues of $P(\lambda)$ are reproduced in $\lambda C_1 - C_0$ - with the possible exception of an eigenvalue at a. (For more details see Section 4.2 of [2].). Consequently, **all** finite eigenvalues of $P(\lambda)$ re-appear in $\lambda C_1 - C_0$, together with their partial multiplicities. It follows from Theorem 1 that $\lambda C_1 - C_0$ is a weak linearization of $P(\lambda)$.

(c) For the reverse polynomial $P^{\#}(\lambda)$, we define (as in Theorem 2) $U_1(\lambda) = \lambda U(\lambda^{-1})$. Then change the "last" (n,n) block entry of $U_1(\lambda)$ to

$$\frac{(b-a)^{n-1}}{(\lambda b - 1)^{n-1}} I_s,$$

to obtain a matrix function, $\widetilde{U}_1(\lambda)$, whose determinant is a nonzero constant in a neighbourhood of $\lambda = 0$. The definition of matrix functions $H(\lambda)$ and $K(\lambda)$ and the rest of the proof are now as in Theorem 2. □

4.1 Symmetrizing the linearization

The idea discussed in Section 3.2 applies to the Bernstein case as well. Indeed, the following proposition is easily verified:

Proposition 5 *Let $\{b_{i,n}(\lambda; a, b)\}_{i=0}^{n}$ be a system of Bernstein polynomials as in* (27)*, and let $P(\lambda)$ be a Hermitian matrix polynomial represented in that basis. Then, when the generalized companion matrix $C_0 C_1^{-1}$ of $P(\lambda)$ (formed by* (29) *and* (30)*) is multiplied on the right by the Hermitian symmetrizer* (24)*, the result is also Hermitian.*

5 Linearization in the Lagrange basis

5.1 Linearization

As above, suppose that an $s \times s$ matrix polynomial $P(\lambda)$ of degree n is sampled at $n+1$ nodes, i.e. distinct (finite) points $z_0, z_1, \ldots, z_n$. We write $P_j := P(z_j)$. Lagrange polynomials are defined in terms of the nodes by

$$\ell_j(\lambda) = w_j \prod_{k=0,\, k\neq j}^{n} (\lambda - z_k), \qquad j = 0, 1, \ldots, n \tag{35}$$

where the "weights" w_j are

$$w_j = \prod_{k=0,\, k\neq j}^{n} \frac{1}{z_j - z_k}. \tag{36}$$

Then $P(\lambda)$ can be expressed in terms of its samples in the form $P(\lambda) = \sum_{j=0}^{n} \ell_j(\lambda) P_j$.

The companion pencil $\lambda C_1 - C_0$ as formulated in Section 3.2 of [1] is (when $n = 3$):

$$\lambda C_1 - C_0 = \begin{bmatrix} (\lambda - z_0)I & 0 & 0 & 0 & -P_0 \\ 0 & (\lambda - z_1)I & 0 & 0 & -P_1 \\ 0 & 0 & (\lambda - z_2)I & 0 & -P_2 \\ 0 & 0 & 0 & (\lambda - z_3)I & -P_3 \\ w_0 I & w_1 I & w_2 I & w_3 I & 0 \end{bmatrix}. \tag{37}$$

The extension to general n is obvious.

The singular coefficient C_1 suggests that the multiplicity of the eigenvalue at infinity of $\lambda C_1 - C_0$ will be higher than that of $P(\lambda)$. This is, indeed, the case and we show that $\lambda C_1 - C_0$ is a linearization, not of $P(\lambda)$, but of the polynomial

$$\hat{P}(\lambda) := \lambda^{n+2} 0_s + \lambda^{n+1} 0_s + P(\lambda), \tag{38}$$

obtained from $P(\lambda)$ by the (apparently) trivial device of adding terms in λ^{n+1} and λ^{n+2} with zero matrix coefficients to $P(\lambda)$ (see [14]). This ensures a defective eigenvalue at infinity. The following general result then determines the nature of the infinite eigenvalue of $P(\lambda)$ via that of the zero eigenvalue of $\hat{P}^{\#}(\lambda)$.

Proposition 6 *Let $P(\lambda) = \sum_{j=0}^{n} A_j \lambda_j$ with $\det(A_n) = 0$, $A_n \neq 0$, so that $P(\lambda)$ has an infinite eigenvalue. If this infinite eigenvalue of $P(\lambda)$ has partial multiplicities $m_1 \geq \cdots \geq m_t > 0$ then $t = s - \textbf{rank}(A_n)$ and $\hat{P}(\lambda)$ has an infinite eigenvalue with partial multiplicities $m_1 + 2, \cdots, m_t + 2,\ 2, \cdots, 2$ (the "2" being repeated $s - t$ times).*

Proof: The partial multiplicities of the eigenvalues of $P(\lambda)$ at infinity coincide with those of the zero eigenvalue of $P^{\#}(\lambda) = \lambda^n P(\frac{1}{\lambda})$. Applying the "local Smith form" (Lemma 3 of [14], for example) to examine this zero eigenvalue, we have

$$P^{\#}(\lambda) = E_0(\lambda)\ \texttt{diag}\left[\ \lambda^{m_1}, \quad \cdots, \quad \lambda^{m_t}, \quad 1, \quad \cdots, \quad 1\ \right] F_0(\lambda) \tag{39}$$

for matrix polynomials $E_0(\lambda), F_0(\lambda)$ invertible at $\lambda = 0$ and, since $P^{\#}(0) = A_n$, it follows that $s - t = \texttt{rank}(A_n)$, or $t = s - \texttt{rank}(A_n)$.

For the reverse polynomial $\hat{P}(\lambda)$ of (38), we have

$$\begin{aligned} \hat{P}^{\#}(\lambda) &= \lambda^{n+2}\hat{P}(\lambda^{-1}) = \lambda^{n+2}\{\lambda^{-(n+2)}0 + \lambda^{-(n+1)}0 + P(\lambda^{-1})\} \\ &= \lambda^{n+2}P(\lambda^{-1}) = \lambda^2 P^{\#}(\lambda). \end{aligned} \tag{40}$$

Now it follows from (39) that

$$\hat{P}^{\#}(\lambda) = E_0(\lambda)\ \texttt{diag}\left[\ \lambda^{m_1+2}, \quad \cdots, \quad \lambda^{m_t+2}, \quad \lambda^2, \quad \cdots, \quad \lambda^2\ \right] F_0(\lambda). \tag{41}$$

But this is just a local Smith form for $\hat{P}^{\#}(\lambda)$ and shows that $\hat{P}(\lambda)$ itself has an infinite eigenvalue with the multiplicities claimed. □

Theorem 7 *The pencil $\lambda C_1 - C_0$ (defined as in equation (37)) is a strong linearization of $\hat{P}(\lambda)$.*

Proof: Again, the proof is in three parts. Part (a) concerns eigenvalues of $P(\lambda)$ which are not equal to a node z_j for any j. Part (b) concerns eigenvalues which happen to coincide with a node, and completes the proof of the weak linearization property. Part (c) shows that the linearization is strong.

(a) Consider the λ-dependent block LU factors for $\lambda C_1 - C_0$ of (37) (and $P(\lambda)$ of degree n) as formulated in [2]:

$$L(\lambda) = \begin{bmatrix} I_s & 0 & \cdots & & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & & & \\ 0 & & & & 0 \\ \frac{w_0}{\lambda - z_0} I_s & \cdots & & \frac{w_n}{\lambda - z_n} I_s & I_s \end{bmatrix}, \tag{42}$$

$$U(\lambda) = \begin{bmatrix} (\lambda - z_0) I_s & 0 & \cdots & 0 & -P_0 \\ 0 & \ddots & & & \vdots \\ \vdots & & & (\lambda - z_n) I_s & -P_n \\ 0 & & \cdots & 0 & \frac{1}{(\lambda - z_0)\cdots(\lambda - z_n)} P(\lambda) \end{bmatrix}. \tag{43}$$

Define $\widetilde{U}(\lambda)$ by replacing the last block entry of $U(\lambda)$ with

$$\widetilde{U}_{n+2,n+2}(\lambda) = \frac{1}{(\lambda - z_0)\cdots(\lambda - z_n)} I_s. \tag{44}$$

Now define $E(\lambda) = L^{-1}(\lambda)$ and $F(\lambda) = \widetilde{U}^{-1}(\lambda)$. Clearly, E and F are analytic and invertible at those eigenvalues which are not coincidemt with a node. (The explicit forms for $E(\lambda)$ and $F(\lambda)$ appear in the Appendix.)

(b) Part (a) of the proof shows that (with the construction of equivalence transformations which are well-defined everywhere except at the nodes), the partial multiplicities of all finite eigenvalues of $P(\lambda)$ are reproduced in $\lambda C_1 - C_0$ - with the possible exception of an eigenvalue at a node z_i, $i = 0, 1, \cdots, n$.

Now suppose that z_j is an eigenvalue of $P(\lambda)$ and also a node. Without loss of generality, we can re-order the nodes so that this node becomes z_0. We are to show that the partial multiplicities of z_0 in $P(\lambda)$ and $\lambda C_1 - C_0$ are the same. Once again, we make use of a suitable block LU factorization.

Observe that, with S as in (13), $S(\lambda C_1 - C_0) = L(\lambda)U(\lambda)$ where

$$L(\lambda) = \begin{bmatrix} I_s & 0 & 0 & \cdots & & 0 \\ 0 & I_s & 0 & & \cdots & 0 \\ 0 & 0 & \ddots & & & \vdots \\ \vdots & & & & & \\ 0 & & & & & 0 \\ \frac{\lambda - z_0}{w_0} I_s & -\frac{w_1(\lambda - z_0)}{w_0(\lambda - z_1)} I_s & -\frac{w_2(\lambda - z_0)}{w_0(\lambda - z_2)} I_s & \cdots & -\frac{w_n(\lambda - z_0)}{w_0(\lambda - z_n)} I_s & I_s \end{bmatrix}, \tag{45}$$

$$U(\lambda) = \begin{bmatrix} w_0 I_s & w_1 I_s & \cdots & w_n I_s & 0 \\ 0 & (\lambda - z_1) I_s & & & -P_1 \\ 0 & & \ddots & & \vdots \\ \vdots & & & (\lambda - z_n) I_s & -P_n \\ 0 & \cdots & & 0 & -\frac{1}{w_0(\lambda - z_1)(\lambda - z_2)\cdots(\lambda - z_n)} P(\lambda) \end{bmatrix}. \tag{46}$$

Define $\widetilde{U}(\lambda)$ by replacing the last block entry of $U(\lambda)$ with

$$\widetilde{U}_{n+2,n+2}(\lambda) = -\frac{1}{w_0(\lambda - z_1)(\lambda - z_2)\cdots(\lambda - z_n)} I_s. \tag{47}$$

Now define $E(\lambda) = L^{-1}(\lambda)$ and $F(\lambda) = \widetilde{U}^{-1}(\lambda)$ and observe that they are analytic in some neighbourhood of z_0. Hence the partial multiplicities of such an eigenvalue (i.e. at a node) are the same in $P(\lambda)$ and $\lambda C_1 - C_0$.

Consequently, **all** finite eigenvalues of $P(\lambda)$ re-appear in $\lambda C_1 - C_0$, together with their partial multiplicites. Hence, by Theorem 1, $\lambda C_1 - C_0$ is a (possibly weak) linearization of $P(\lambda)$.

(c) For the reverse polynomial, change the "last" $(n+2, n+2)$ block entry of $U_1(\lambda) := \lambda U(\lambda^{-1})$ to

$$\frac{1}{(1 - \lambda z_0)\cdots(1 - \lambda z_n)} I_s$$

(which is well-defined for $\lambda \neq 1/z_j$, $j = 0, 1, \cdots, n$). As in the proof of Theorem 2, we can now define functions $H(\lambda)$ and $K(\lambda)$ which are analytic and invertible at $\lambda = 0$, and the rest of the proof follows as before. □

5.2 Symmetrizing the Lagrangian companion pencil

Multiplying $\lambda C_1 - C_0$ of (37) on the right by the block-diagonal

$$A := \begin{bmatrix} w_0^{-1}P_0 & 0 & 0 & 0 \\ 0 & w_1^{-1}P_1 & 0 & 0 \\ 0 & 0 & w_2^{-1}P_2 & 0 \\ 0 & 0 & 0 & -I \end{bmatrix}$$

we obtain

$$(\lambda C_1 - C_0)A = \begin{bmatrix} \frac{\lambda - z_0}{w_0}P_0 & 0 & 0 & P_0 \\ 0 & \frac{\lambda - z_1}{w_1}P_1 & 0 & P_1 \\ 0 & 0 & \frac{\lambda - z_2}{w_2}P_2 & P_2 \\ P_0 & P_1 & P_2 & 0 \end{bmatrix}.$$

The block-symmetry of this product can provide computational advantages, especially when the z_j (and hence w_j) are real and $P_0, \ldots, P_n$ are Hermitian ($P_j^H = P_j$), or when they are complex symmetric ($P_j^T = P_j$).

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A APPENDIX: The inverses of block LU factors.

We record the explicit formulae for $E(\lambda) := L^{-1}(\lambda)$ and $F(\lambda) := \widetilde{U}^{-1}(\lambda)$ in the cases of degree-graded polynomial bases (see Theorem 2) and Lagrange bases (see Theorem 7).

A.1 Degree-graded polynomial bases

$$E(\lambda) = \begin{bmatrix} I_s & 0 & \cdots & & 0 \\ 0 & I_s & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & I_s & 0 \\ \frac{\phi_1(\lambda)}{\phi_0(\lambda)} & \cdots & & \frac{\phi_{n-1}(\lambda)}{\phi_0(\lambda)}I_s & I_s \end{bmatrix}. \quad (48)$$

$$F_{i,j}(\lambda) = \begin{cases} -\frac{1}{\alpha_{i-1}}I_s, & i = j = 1{:}(n-1), \\ \alpha_0\alpha_1\cdots\alpha_{n-2}I_s, & i = j = n, \\ \frac{1}{\alpha_{j-1}}((\lambda - \beta_{j-1})F_{i,j-1} - \gamma_{j-1}F_{i,j-2}, & i = 1{:}(n-1); j = (i+1){:}n, \\ (\alpha_0\alpha_1\cdots\alpha_{n-3})U_{i,j}, & i = n-1; j = n, \\ \frac{1}{\alpha_{i-1}}((\lambda - \beta_i)F_{i+1,j} + (\alpha_0\alpha_1\cdots\alpha_{n-2})U_{i,j}), & i = n-2; j = n, \\ \frac{1}{\alpha_{i-1}}((\lambda - \beta_i)F_{i+1,j} - \gamma_{i+1}F_{i+2,j} + (\alpha_0\alpha_1\cdots\alpha_{n-2})U_{i,j}), & i = n-3{:}1; j = n, \end{cases} \quad (49)$$

where $U_{i,j}$ are the corresponding block entries of (15) given by (16).

A.2 Lagrange basis

$$E(\lambda) = \begin{bmatrix} I_s & 0 & \cdots & & 0 \\ 0 & I_s & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & I_s & 0 \\ -\frac{w_0}{\lambda - z_0} I_s & -\frac{w_1}{\lambda - z_1} I_s & \cdots & -\frac{w_n}{\lambda - z_n} I_s & I_s \end{bmatrix}, \qquad (50)$$

$$F_{i,j}(\lambda) = \begin{cases} \frac{1}{\lambda - z_{i-1}} I_s, & i = j = 1{:}(n+1), \\ (\lambda - z_0)\cdots(\lambda - z_n) I_s, & i = j = n+2, \\ (\lambda - z_0)\cdots(\lambda - z_{i-2})(\lambda - z_i)\cdots(\lambda - z_n)\hat{P}_{i-1}, & i = 1{:}(n+1); j = n+2, \\ 0_s, & \textit{otherwise}, \end{cases} \qquad (51)$$

where matrices $\hat{P}_i$ are the values of $\hat{P}(\lambda)$ evaluated at the nodes.

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