Isospectral Families of High-order Systems*

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November 28, 2006

Keywords: Matrix Polynomials, Isospectral Systems, Structure Preserving Transformations, Palindromic Symmetries.

Abstract

Earlier work of the authors concerning the generation of isospectral families of second order (vibrating) systems is generalized to higher-order systems (with no spectrum at infinity). Results and techniques are developed first for systems without symmetries, then with Hermitian symmetry and, finally, with palindromic symmetry. The construction of linearizations which retain such symmetries is discussed. In both cases, the notion of *strictly isospectral* families of systems is introduced - implying that properties of both the spectrum and the sign-characteristic are preserved. Open questions remain in the case of strictly isospectral families of palindromic systems. Intimate connections between Hermitian and unitary systems are discussed in an Appendix.

*The authors are grateful to an anonymous reviewer for perceptive comments which led to significant improvements in exposition.

†The research of Peter Lancaster was partly supported by a grant from the Natural Sciences and Engineering Research Council of Canada

1 Introduction

By a high-order system we mean an $n \times n$ matrix polynomial $L(\lambda) = A_\ell \lambda^\ell + \cdots + A_1\lambda + A_0$ with coefficients in $\mathbb{C}^{n\times n}$. Such a polynomial with $A_\ell \neq 0$ is said to have *degree* ℓ, (sometimes referred to as "order" ℓ) and "high-order" implies that $\ell \geq 2$. It will be assumed throughout that the leading coefficient A_ℓ is nonsingular. (Inverse problems with A_ℓ singular are the subject of [GKR].)

One objective of this work is the generalization of results obtained recently by the authors in [PL] for second order systems to systems of higher order and, as far as is reasonably possible, notations and conventions will be consistent. *Palindromic* polynomials are not discussed in the earlier paper, but do appear here - as well as polynomials with Hermitian coefficients. In both cases the idea of a *sign characteristic* appears. The idea of "strictly isospectral" transformations is introduced, for which both the spectrum and the sign characteristic are preserved.

In preliminary sections the notions of standard pairs and structure preserving transformations are developed, and then employed in the construction of isospectral families of systems. The roles of standard pairs and triples, as well as the the role of the moments (Definition 4) in the analysis of high-order matrix polynomials are summarized. They are employed in the development of structure preserving equivalence transformations (SPE's) which also preserve the spectrum, i.e. they generate isospectral systems. For systems without symmetries the main results are Theorems 6 and 7 (formulated in terms of standard triples) and Theorem 8 (formulated in terms of SPE's).

A careful discussion of linearizations in Section 4 leads to constructive methods for generating isospectral families of systems with Hermitian or palindromic symmetries. A special class of linearizations is identified here (Theorem 11, for example). The central role played by these linearizations has also been recognized (very recently, and in a more general algebraic context) by Higham *et al.* in [hmt]. In Section 5 it is proved that, for Hermitian systems, *strictly isospectral* families can be generated by applying structure preserving *congruence* transformations (SPC's) to their linearizations (Theorems 13 and 14).

Section 6 concerns families of linearizations with block-symmetries which are useful for palindromic polynomials and lead to Theorem 17. They are discussed in Section 8 in the context of earlier work on "H-unitary" matrices. An Appendix displays the intimate connections between systems with various forms of symmetry or skew-symmetry and the four kinds of palindromic symmetries introduced in [M4].

2 Preliminaries

Definition 1 An $\ell n \times \ell n$ matrix pencil $\lambda R - S$ is a *linearization* of a system $L(\lambda)$ if

$$\begin{bmatrix} L(\lambda) & 0 \\ 0 & I_{(\ell-1)n} \end{bmatrix} = E(\lambda)(\lambda R - S)F(\lambda)$$

for some matrix $\ell n \times \ell n$ matrix polynomials $E(\lambda)$ and $F(\lambda)$ with constant nonzero determinants.

Two important and well-known examples of linearizations for $L(\lambda)$ are $\lambda A - B$ where

$$A := \begin{bmatrix} A_1 & A_2 & \cdots & A_\ell \\ A_2 & & \iddots & 0 \\ \vdots & \iddots & \iddots & \vdots \\ A_\ell & 0 & \cdots & 0 \end{bmatrix}, \quad B := \begin{bmatrix} -A_0 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & A_3 & \cdots & A_\ell \\ 0 & A_3 & & \iddots & 0 \\ \vdots & \vdots & \iddots & \iddots & \vdots \\ 0 & A_\ell & 0 & \cdots & 0 \end{bmatrix}, \tag{1}$$

and $\lambda I - C$ where $C = A^{-1}B$ is the *companion matrix* of $L(\lambda)$:

$$C := \begin{bmatrix} 0 & I_n & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & I_n \\ -A_\ell^{-1}A_0 & -A_\ell^{-1}A_1 & \cdots & -A_\ell^{-1}A_{\ell-2} & -A_\ell^{-1}A_{\ell-1} \end{bmatrix}. \tag{2}$$

We refer to the linearization $\lambda A - B$ as the *primary linearization* of $L(\lambda)$.

Definition 2 A pair of matrices, $U \in \mathbb{C}^{n\times\ell n}$ and $T \in \mathbb{C}^{\ell n\times\ell n}$ is a *standard pair* (or an *ℓn-standard pair*) if the matrix $\begin{bmatrix} U \\ UT \\ \vdots \\ UT^{\ell-1} \end{bmatrix}$ is nonsingular.

It is not difficult to see that, for a standard pair (U, T), the dimension of each eigenspace of T cannot exceed n. In particular, the structure of matrix C above is such that, for any system $L(\lambda)$, the corresponding companion matrix, C, satisfies this condition. Indeed, if we define $U = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}$ and $T = C$, then (U, C) is a standard pair.

Definition 3 Three matrices, $U \in \mathbb{C}^{n\times\ell n}$, $T \in \mathbb{C}^{\ell n\times\ell n}$ and $V \in \mathbb{C}^{\ell n\times n}$ form a *standard triple* (or an *ℓn -standard triple*) if:
(U, T) is a standard pair.
$UT^jV = 0$ for $j = 0, 1, \ldots, \ell - 2$.
$UT^{\ell-1}V$ is nonsingular.

A standard pair (U, T) can always be extended to a standard triple. Indeed, a standard triple (U, T, V) satisfies

$$\begin{bmatrix} U \\ UT \\ \vdots \\ UT^{\ell-1} \end{bmatrix} V = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ M^{-1} \end{bmatrix} \tag{3}$$

for some nonsingular matrix $M \in \mathbb{C}^{n\times n}$. The prime example of a standard triple consists of the standard pair $U = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}$, $T = C$ as introduced above, and $V = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ A_\ell^{-1} \end{bmatrix}$.

It is important to note that each standard triple generates an equivalence class (or *orbit*) of standard triples by similarity; i.e. if (U, T, V) is a standard triple then, for any nonsingular $R \in \mathbb{C}^{\ell n\times\ell n}$,

$$U_1 = UR^{-1}, \quad T_1 = RTR^{-1}, \quad V_1 = RV$$

is also a standard triple. Clearly, each such class contains triples for which the second (main) matrix is in Jordan canonical form. Such a triple is said to be a *Jordan triple*.

Definition 4 The *moments* generated by an ℓn-standard triple (U, T, V) are the $n \times n$ matrices defined by $\Gamma_r = UT^rV$ for all integers r for which T^r is defined.

The use of the word "moments" is natural if we consider the asymptotic expansion of the "resolvent" form appearing in the next statement:

Lemma 1 *If $(U,\, T,\, V)$ is a standard triple for a system $L(\lambda)$ then*

$$L(\lambda)^{-1} = U(\lambda I - T)^{-1}V. \tag{4}$$

Conversely, if $(U,\, T,\, V)$ is a triple of matrices (with sizes $n \times \ell n$, $\ell n \times \ell n$, and $\ell n \times n$, respectively) for which (4) *holds, then $(U,\, T,\, V)$ is a standard triple for $L(\lambda)$.*

(For the proof see Theorems 14.2.2 and 14.2.4 of [LT], for example)).

The definition of a standard triple ensures that, in a sequence of moments, $\Gamma_0 = \Gamma_1 = \cdots = \Gamma_{\ell-2} = 0$, and $\Gamma_{\ell-1}$ is nonsingular. It is important to note that the moments are invariant under the similarities defined above. Indeed, the moments are independent of the choice of standard triple from an equivalence class of triples.

Given a system $L(\lambda)$ of degree ℓ, an associated equivalence class of standard triples is easily generated. Take the standard pair (U, C) described above, assign the nonsingular $n \times n$ matrix $M = A_\ell$ in equation (3) and solve for V to complete a triple. Then the corresponding equivalence class can be generated from this triple by similarity transformations.

The point of view of our inverse problem is the converse statement: Given an equivalence class of standard triples, there is a corresponding uniquely defined system. This is a consequence of the following result, which is a mildly extended version of Theorem 2.4 of [GLR1].

Lemma 2 *Let (U, T, V) be an ℓn-standard triple and let $M = (UT^{\ell-1}V)^{-1}$. If $n \times n$ matrices $X_0, \ldots, X_{\ell-1}$ are defined by*

$$\begin{bmatrix} X_0 & \cdots & \cdots & X_{\ell-1} \end{bmatrix} = -UT^\ell \begin{bmatrix} U \\ UT \\ \vdots \\ UT^{\ell-1} \end{bmatrix}^{-1},$$

then $L(\lambda) = M(\lambda^\ell I_n + \cdots + \lambda X_1 + X_0)$ is an $n \times n$ matrix polynomial with nonsingular leading coefficient for which $(U,\, T,\, V)$ is an associated standard triple.

This discussion can be summarized as follows:

Theorem 3 *There is a one-to-one correspondence between the set $\mathcal{L}$ of $n \times n$ matrix polynomials of degree ℓ with nonsingular leading coefficient and the set of equivalence classes of ℓn-standard triples.*

The approach to the inverse problem adopted here is to begin by generating a standard triple, and using this to form the corresponding matrix polynomial. By using triples from different equivalence classes, but with a common Jordan form, classes of polynomials can be generated which are *isospectral*.

The following lemma concerning matrix A of (1) will also be useful:

Lemma 4 *If (U, T, V) is a standard pair for $L(\lambda)$, then*

$$A^{-1} = \begin{bmatrix} A_1 & A_2 & \cdots & A_\ell \\ A_2 & & \iddots & 0 \\ \vdots & \iddots & \iddots & \vdots \\ A_\ell & 0 & \cdots & 0 \end{bmatrix}^{-1} = \begin{bmatrix} U \\ UT \\ \vdots \\ UT^{\ell-1} \end{bmatrix} \begin{bmatrix} V & TV & \cdots & T^{\ell-1}V \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & \Gamma_{\ell-1} \\ \vdots & & \Gamma_{\ell-1} & \Gamma_\ell \\ 0 & & & \vdots \\ \Gamma_{\ell-1} & \Gamma_\ell & \cdots & \Gamma_{2(\ell-1)} \end{bmatrix}.$$

For the proof see Theorem 14.2.5 of [LT].

3 Standard pairs and structure preserving transformations

The first result concerns similarity transformations which preserve the structure of companion matrices. It will also play a role in the subsequent study of strict equivalence transformations[1].

Proposition 5 *Let C be as in (2) and $X \in \mathbb{C}^{n\times n\ell}$ be* **any** *matrix for which (X, C) form a standard pair. Then, with*

$$G := \begin{bmatrix} X \\ XC \\ \vdots \\ XC^{\ell-1} \end{bmatrix} \tag{5}$$

$C' := GCG^{-1}$ is also a companion matrix.

Conversely, if C and C' are $\ell n \times \ell n$ companion matrices and if there is a G such that $C'G = GC$, then there is an $X \in \mathbb{C}^{n\times n\ell}$ such that G has the form (5).

Proof: Since (X, C) form a standard pair G is nonsingular. Now write

$$GCG^{-1} = \begin{bmatrix} XC \\ \vdots \\ XC^{\ell-1} \\ XC^\ell \end{bmatrix} G^{-1}.$$

The first block row XC is the second block row of G and hence $XCG^{-1} = [0\, I_n\, 0 \cdots 0]$. Similarly the second block row XC^2 is the third block row of G and hence $XC^2G^{-1} = [0\, 0\, I_n\, 0 \cdots 0]$. This argument continues for $(\ell - 1)$ block rows and gives $XC^{\ell-1}G^{-1} = [0 \cdots 0\, I_n]$. Now the companion matrix structure is complete.

For the converse, given companion matrices C and C' and a $G \in \mathbb{C}^{\ell n\times \ell n}$ such that $C'G = GC$, partition G into $n \times \ell n$ blocks,

$$G = \begin{bmatrix} G_1 \\ \vdots \\ G_\ell \end{bmatrix},$$

and it is easily seen that, for $j = 2, \dots, \ell$, $G_j = G_{j-1}C$. It follows that, defining $X = G_1$, $G_j = XC^{j-1}$ for $j = 1, 2, \dots, \ell$. □

[1]A *strict equivalence* transformation of a matrix A is a matrix EAF where E and F are nonsingular.

Following the row-by-row procedure of the first part of the proof one step further, the last block row can be used to define another system L' with coefficients A'_j by writing

$$XC^{\ell}G^{-1} = -(A'_{\ell})^{-1}[A'_0\ A'_1 \cdots A'_{\ell-1}],$$

(and the system generated is unique to within a nonsingular left multiplier). Thus, Proposition 5 gives a method for mapping one system to another, and the similarity mapping the companion matrix of system $L(\lambda)$ to the companion matrix of system(s) $L'(\lambda)$ is called a *structure preserving similarity* or, more briefly, an SPS. As they are generated by similarities, all systems generated from $L(\lambda)$ in this way share the same Jordan canonical form, and are therefore said to be *isospectral*.

Matrices G of the form (5) have a long history. They play a central role in systems theory (where the pair (X, C) is said to be *observable*) and the study of rational matrix functions (see, for example, books of Lancaster [L2], Gohberg *et al.*[GLR1], [GLR2], [GKV], Kailath [K], and Wonham [W]). The point to be emphasized here is that the matrix G defines a similarity between companion matrices. Such similarity transformations are termed *structure preserving* because the block structure of the companion matrix is preserved (i.e. the positions of the zero blocks and the super-diagonal blocks containing the identity matrix I_n are preserved). For earlier related work concerning "structure preserving transformations" see [GFP1], [GFP2] and [GPF].

Since linearizations are defined in terms of *pairs* of matrices, it is natural to consider strict equivalence transformations (of the pair $\lambda R - S$ of Definition 1) and, in particular, those which are structure preserving in the above sense. Thus, the next results (which generalize Theorem 7 of [PL]) introduce the notion of a structure preserving equivalence (SPE). Notice the freedom implicit in the choice of matrix X in these results. Indeed, Theorem 7 shows how to generate a family of systems which, because they are all strictly equivalent to a linearization of $L(\lambda)$, will all be isospectral.

The important feature of the next two results is the freedom of choice allowed for matrices X for which (X, C) form a standard pair. The interesting choices are those X for which (X, C) is a standard pair, but *not* for the system $L(\lambda)$ (when $X = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}$), even though C is determined by $L(\lambda)$. Having fixed X, there is then a family of candidates for the matrix Y of a standard triple (X, C, Y) generated by the choice of M in (3).

Observe first that *all* systems isospectral with $L(\lambda)$ can be generated in this way.

Theorem 6 *Let $L(\lambda)$ be $n \times n$ with degree ℓ, $\det A_\ell \neq 0$, and companion matrix C. Then every system $\hat{L}(\lambda)$ of degree ℓ which is isospectral with $L(\lambda)$ has an ℓn-standard triple (X, C, Y) for some choice of X and Y.*

Proof: Let $\hat{L}(\lambda)$ have a Jordan triple $(\hat{X}, J, \hat{Y})$. Since $L(\lambda)$ and $\hat{L}(\lambda)$ are isospectral J must be also be a Jordan form for $L(\lambda)$, and hence C. So there is a T such that $C = TJT^{-1}$. Then, since standard triples are preserved under similarity, $\hat{L}$ has a standard triple

$$(\hat{X}T^{-1},\, TJT^{-1},\, T\hat{Y}) = (\hat{X}T^{-1},\, C,\, T\hat{Y}).$$

The proof is completed by taking $X = \hat{X}T^{-1}$, $Y = T\hat{Y}$. $\square$

Theorem 7 *Given an $n \times n$ system $L(\lambda)$ of degree ℓ with $\det A_\ell \neq 0$, let A, B, C be defined as in* (1) *and* (2)*. Let X be any matrix for which (X, C) form an ℓn-standard pair and let (X, C, Y) be a corresponding standard triple. Then (with G defined by* (5)*) the pair*

$$E := [Y\ CY \cdots C^{\ell-1}Y]^{-1}A^{-1}, \quad F = G^{-1}, \tag{6}$$

determine a structure preserving equivalence in the sense that $A' := EAF$ and $B' := EBF$ have the block structures of (1) *and, furthermore, the matrix A'_ℓ is nonsingular.*

Proof: In this calculation we use the moments generated by the triple (X, C, Y) as in Definition 4; they implicitly define a transformed set of matrix coefficients. We have

$$\begin{aligned} A' = EAF &= \begin{bmatrix} Y & CY & \dots & C^{\ell-1}Y \end{bmatrix}^{-1} (A^{-1}A)G^{-1} \\ &= (G \begin{bmatrix} Y & CY & \dots & C^{\ell-1}Y \end{bmatrix})^{-1} \\ &= \begin{bmatrix} 0 & \cdots & 0 & \Gamma_{\ell-1} \\ \vdots & \iddots & \iddots & \Gamma_\ell \\ 0 & \iddots & & \vdots \\ \Gamma_{l-1} & \Gamma_\ell & \cdots & \Gamma_{2(l-1)} \end{bmatrix}^{-1} . \end{aligned}$$

Now apply Lemma 4 and it is found that A' has the block Hankel structure required, and that A'_ℓ is nonsingular.

To investigate the structure of B', we have

$$(A')^{-1}B' = (F^{-1}A^{-1}E^{-1})EBF = F^{-1}A^{-1}BF = GCG^{-1} = C',$$

by Proposition 5. Thus $B' = A'C'$ and, since A' and C' have the required structure, so does B'. $\square$

Theorem 8 *Two $n \times n$ systems of degree ℓ are isospectral if and only if their primary linearizations are related by an SPE.*

Proof: If two linear pencils are related by an SPE, they are certainly strictly equivalent, and they necessarily share the same Jordan form. So we have only to show that, if two systems are isospectral, then their linearizations $\lambda A - B$ and $\lambda A' - B'$ are related by an SPE.

If systems L and L' are isospectral, then their respective companion matrices C and C' are similar; say $C' = GCG^{-1}$ for some nonsingular G. The converse statement of Proposition 5 implies that G has the form appearing in (5). Since $C' = (A')^{-1}B'$,

$$\begin{aligned} \lambda A' - B' &= A'(\lambda I - C') = A'G(\lambda I - C)G^{-1} \\ &= A'G(\lambda I - A^{-1}B)G^{-1} = (A'GA^{-1})(\lambda A - B)G^{-1} \\ &= E(\lambda A - B)F \qquad \text{where} \qquad E := A'GA^{-1}, \;\; F := G^{-1}. \end{aligned} \tag{7}$$

Thus, $\lambda A' - B'$ and $\lambda A - B$ are strictly equivalent. We use Theorem 7 to show that this strict equivalence is structure preserving.

Checking equations (6) we immediately see that F has the desired form. Also, Lemma 4 gives

$$A' = \begin{bmatrix} Y & CY & \cdots & C^{\ell-1}Y \end{bmatrix} G^{-1}.$$

Thus, from (7),

$$E = A'GA^{-1} = \begin{bmatrix} Y & CY & \cdots & C^{\ell-1}Y \end{bmatrix} A^{-1},$$

which also has the form required by Theorem 7 and implies that the SPE is structure preserving. $\square$

In what follows, the results of this section will be applied to the study of systems with symmetries. In the following section we revisit linearizations and discuss a wide class of these linear pencils which have useful symmetries.

4 Linearizations

In this section an easily generated class of linearizations will be discussed. This class will be important in our later discussions of systems with symmetries. The reader should be aware of other recent works in this direction including [AV], [GKL], [hmt], [M4], and [MMT].

First consider the matrices[2] $S_i = AC^i$; defined for all integers i for which C^i is well-defined. For any i we may define a sequence beginning with $S_0 = A$ and $S_1 = B$ (see equation (1)). The first members of the sequence (for $i = 0, 1, \ldots, \ell$) have the form

$$S_i := \begin{bmatrix} 0 & \cdots & 0 & -A_0 & 0 & \cdots & \cdots & 0 \\ \vdots & \iddots & \iddots & \vdots & \vdots & & & \vdots \\ 0 & \iddots & & \vdots & \vdots & & & \vdots \\ -A_0 & \cdots & \cdots & -A_{i-1} & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & A_{i+1} & \cdots & \cdots & A_\ell \\ \vdots & & & \vdots & \vdots & & \iddots & 0 \\ \vdots & & & \vdots & \vdots & \iddots & \iddots & \vdots \\ 0 & \cdots & \cdots & 0 & A_\ell & 0 & \cdots & 0 \end{bmatrix}. \tag{8}$$

In particular,

$$S_{\ell-1} = \begin{bmatrix} 0 & \cdots & 0 & -A_0 & 0 \\ \vdots & \iddots & \iddots & \vdots & \vdots \\ 0 & \iddots & & \vdots & \vdots \\ -A_0 & \cdots & \cdots & -A_{\ell-2} & 0 \\ 0 & \cdots & \cdots & 0 & A_\ell \end{bmatrix}, \quad S_\ell = \begin{bmatrix} 0 & \cdots & 0 & -A_0 \\ \vdots & \iddots & \iddots & -A_1 \\ 0 & \iddots & & \vdots \\ -A_0 & -A_1 & \cdots & -A_{\ell-1} \end{bmatrix}.$$

Note that, for $i = 0, 1, \ldots, \ell$, the matrix S_i does not depend on A_i.

It is easily seen (and has been noted by Garvey *et al.* [GPF]) that all S_i are nonsingular if and only if A_ℓ and A_0 are nonsingular. In this case C is nonsingular and, for all k for which C^k is defined,

$$\lambda S_{k-1} - S_k = AC^{k-1}(\lambda I - C) = S_{k-1}(\lambda I - C).$$

This strict equivalence shows that, when *both* A_ℓ and A_0 are nonsingular, $\lambda S_{k-1} - S_k$ is a linearization of $L(\lambda)$ for each k.

The following lemma will be useful later.

Lemma 9 *Let $\lambda A - B$ be the primary linearization of the system $L(\lambda)$ and $\lambda A' - B'$ be obtained from $\lambda A - B$ by the strict equivalence of Theorem 7. Then the matrices S_i and $S'_i := A'(C')^i$, $i = 0, 1, \ldots, \ell$ satisfy the same strict equivalence: $ES_iF = S'_i$, $i = 0, 1, \ldots, \ell$.*

Proof: Since $A' = EAF$ and $B' = EBF$,

$$ES_iF = E(AC^i)F = A'F^{-1}C^iF = A'(F^{-1}CF)^i.$$

[2]These block-symmetric matrices have a long history including Lancaster [L1] for the case $n = 1$, and have recently found interest in the context of isospectral flows in a paper by Garvey *et al.* [GPF] where they are termed LAMs (Lancaster Augmented Matrices) for $i = 0, \cdots, \ell$.

But, by Proposition 5, $C' = F^{-1}CF$, so $ES_iF = A'(C')^i = S'_i$. $\square$

To define a large class of linearizations note first of all that the set of all $\ell n \times \ell n$ matrix pencils $\lambda M - N$ form a vector space, say $\mathcal{V}$. Then:

Lemma 10 *If $A_\ell \neq 0$ then the pencils $L_j(\lambda) := \lambda S_{j-1} - S_j$, $j = 1, 2, \ldots, \ell$ generate an ℓ-dimensional subspace* [3] *$\mathcal{V}_0$ of $\mathcal{V}$.*

Proof: It is easily verified that, if $A_l \neq 0$, then a linear combination $\sum_{j=1}^{\ell} c_j L_j(\lambda)$ is the zero pencil if and only if all the c_j are zero. Hence $L_1(\lambda), \ldots L_\ell(\lambda)$ are linearly independent in $\mathcal{V}$ and the result follows. $\square$

Theorem 11 *Assume that* $\det A_\ell \neq 0$. *Then a pencil $L_c(\lambda) = \sum_{j=1}^{\ell} c_j L_j(\lambda) \in \mathcal{V}_0$ is a linearization of $L(\lambda)$ if and only if the polynomial $p_c(\lambda) = \sum_{j=1}^{\ell} c_j \lambda^{j-1}$ is nonzero at all the eigenvalues of $L(\lambda)$.*

Proof: (This theorem should be compared with Theorem 5.7 of [M4].) Since $S_j = AC^j$,

$$L_c(\lambda) = \sum_{j=1}^{\ell} c_j(\lambda S_{j-1} - S_j) = A(\lambda I - C)(\sum_{j=1}^{l} c_j C^{j-1}).$$

Since A is nonsingular and $\lambda I - C$ is a linearization, $L_c(\lambda)$ is also a linearization if and only if $p_c(C) = \sum_{j=1}^{l} c_j C^{j-1}$ is nonsingular.

Now the eigenvalues of C are just the eigenvalues of $L(\lambda)$, say $\lambda_1, \ldots, \lambda_{\ell n}$, so the eigenvalues of $p_c(C)$ are $p_c(\lambda_j)$ for $j = 1, 2, \ldots, \ell n$. Since $\det(p_c(C))$ is just the product of the eigenvalues of $p_c(C)$, the result follows. $\square$

The following special cases of the theorem are worthy of note:

Corollary 12 *If* $\det A_\ell \neq 0$ *then:*
(a) *The primary linearization $L_1(\lambda) = \lambda A - B$ is associated with $p_c(\lambda) \equiv 1$.*
(b) *A pencil $L_j(\lambda)$ with $j \geq 2$ is a linearization for $L(\lambda)$ if and only if* $\det A_0 \neq 0$.
(c) *If α is* not *an eigenvalue of $L(\lambda)$ then*

$$\lambda(S_1 - \alpha S_0) - (S_2 - \alpha S_1)$$

is a linearization of $L(\lambda)$.

Proof: (a) Apply the theorem with $p_c(\lambda) \equiv 1$ to obtain $L_c(\lambda) = L_1(\lambda)$.
(b) Apply the theorem with $p_c(\lambda) = \lambda^j$ to see that, when $j \geq 1$, $P(\lambda)$ must not have a zero eigenvalue. But this is equivalent to the condition $\det A_0 \neq 0$.
(c) Apply the theorem with $p(c) = \lambda - \alpha$. $\square$

In connection with item (c) above, let us also recall that, if an α is known for which $\det L(\alpha) \neq 0$, then there is the possibility of removing the hypothesis det $A_\ell \neq 0$ by making the parameter transformation: $\mu = (\lambda - \alpha)^{-1}$. For example, when $\ell = 2$ the eigenvalue problem for $L(\lambda) = A_2\lambda^2 + A_1\lambda + A_0$ with A_2 singular is transformed to that for

$$(A_2\alpha^2 + A_1\alpha + A_0)\mu^2 + (A_2(2\alpha) + A_1)\mu + A_2$$

which has a nonsingular leading coefficient.

[3] The subspace $\mathcal{V}_0$ is just the space of linearizations $\mathbb{L}_1$ introduced in Definition 3.1 of [M4a]. In the terminology of that paper, $L_j(\lambda)$ is generated by the "right ansatz" vector e_j for $j = 1, 2, \ldots, \ell$. See also [hmt].

5 Hermitian Systems

Suppose the coefficient matrices of $L(\lambda)$ are Hermitian. Then, if C is the corresponding companion matrix, a general choice of X in a standard pair (X, C) (as in Theorem 4) will not necessarily generate an isospectral matrix polynomial with Hermitian coefficient matrices. It is the goal of this section to investigate structure preserving transformations which retain the Hermitian property of system coefficients.

It is obvious that A and B of (1) and, indeed, all of the S_i are Hermitian for Hermitian $A_0, A_1, \cdots, A_\ell$. Hence, in Theorem 7, the condition $F = E^*$ is sufficient to ensure the preservation of symmetry. The following theorem is deduced from that theorem and introduces the notions of *selfadjoint standard triple* and *structure preserving congruence* (SPC).

Theorem 13 *Let $L(\lambda)$ be an $n \times n$ Hermitian matrix polynomial of degree ℓ with* $\det A_\ell \neq 0$. *Then there exist ℓn-standard triples of the (self-adjoint) form $(X, C, A^{-1}X^*)$. For such self-adjoint triples, $F = E^*$ in (6), and the strict equivalence of Theorem 7 reduces to a structure preserving congruence transformation between the primary linearizations.*

Proof: For the existence statement see Theorem 1.9 of [GLR0] (or Theorem 10.6 of [GLR1] for the monic case).

If we are given a standard triple (X, C, Y) with $X = Y^*A$ observe first that, since $A^* = A$, $B^* = B$ and $C = A^{-1}B$, we also have $C^* = ACA^{-1}$. Indeed, we have

$$(C^*)^r = AC^rA^{-1}, \quad \text{for} \quad r = 0, 1, 2, \ldots. \tag{9}$$

Then, from (6),

$$F^{-1} = \begin{bmatrix} X \\ XC \\ \vdots \\ XC^{\ell-1} \end{bmatrix} = \begin{bmatrix} Y^*A \\ Y^*AC \\ \vdots \\ Y^*AC^{\ell-1} \end{bmatrix} = \begin{bmatrix} Y^* \\ Y^*ACA^{-1} \\ \vdots \\ Y^*AC^{\ell-1}A^{-1} \end{bmatrix} A = \begin{bmatrix} Y^* \\ Y^*C^* \\ \vdots \\ Y^*(C^*)^{\ell-1} \end{bmatrix} A^* = (E^*)^{-1}$$

and we have used (9) at the last step. Thus $F = E^*$ and the strict equivalence defined by E and F becomes a congruence. □

Here, the freedom in the choice of X provides a class of isospectral Hermitian systems. But more is true. It is well-known that simultaneous congruence of Hermitian pencils preserves other invariants.[4] They are included in the *sign characteristic* of the Hermitian pencil $\lambda A - B$. In brief, there is a canonical Hermitian matrix P determined by a block structure like that of J, together with a set of multipliers ± 1 associated with the real eigenvalues. Then $\lambda A - B$ is congruent to $\lambda P - PJ$ (see [LR] or Chapter 12 of [GLR2] for details). When both J and the sign characteristic are preserved a transformation is said to be *strictly isospectral.* This property is ensured by the congruence transformations of Theorem 13. On the other hand, it is certainly possible to devise strict equivalence transformations between Hermitian systems which, in general, will not be *strictly* isospectral although they will certainly be isospectral.

[4]The sign characteristic is also required in defining generalized orthogonality properties for eigenvectors. It is an intrinsic property of Hermitian $L(\lambda)$. In particular, it plays an important part in any discussion of perturbations of the coefficients of $L(\lambda)$ which preserve the Hermitian property (see [GLR], for example).

Concerning the freedom of choice for X: since we require $Y = A^{-1}X^*$, we may also think in terms of finding Y. Now, (using the definitions of the S_j) the equations (3) defining Y take the form

$$Y^* S_0 Y = Y^* S_1 Y = \cdots = Y^* S_{\ell-2} Y = 0, \qquad Y^* S_{\ell-1} Y = A_\ell^{-1}. \tag{10}$$

Thus, the subspace $\operatorname{Im} Y$ is to be neutral with respect to $S_0, \ldots, S_{\ell-2}$ and $S_{\ell-1}$-positive. Constructive methods for solving these equations are a central theme of earlier papers concerned primarily with the quadratic case, $\ell = 2$ (see [L3], [LP], [PL], [L4]).

The following statement is a modification of Theorem 8 for Hermitian systems and ensures that the construction of Theorem 13 is capable of generating *all* Hermitian systems related to $P(\lambda)$ by the strictly isospectral property.

Theorem 14 *Two $n \times n$ Hermitian systems of degree ℓ with primary linearizations $\lambda A - B$, $\lambda A' - B'$ are strictly isospectral if and only if there is an ℓn-standard pair (X, C) (with $C = A^{-1}B$) for the system $\lambda A - B$ such that $F^*(\lambda A - B)F = \lambda A' - B'$ (i.e. an SPC) where*

$$F = \begin{bmatrix} X \\ XC \\ \vdots \\ XC^{\ell-1} \end{bmatrix}^{-1}.$$

Proof: If the primary linearizations of two Hermitian systems are congruent, they certainly have the same canonical structures and this implies that the underlying systems are strictly isospectral. So we have only to show that, for two strictly isospectral Hermitian systems with linearizations $\lambda A - B$ and $\lambda A' - B'$ of the form (1), there is a nonsingular $F(= G^{-1})$ with G of the form (5) for which $F^*AF = A'$ and $F^*BF = B'$.

To see this, let (X, J, PX^*) be a *self-adjoint* Jordan triple for the Hermitian system $L(\lambda)$ with associated canonical matrix P and define

$$Q = \begin{bmatrix} X \\ XJ \\ \vdots \\ XJ^{\ell-1} \end{bmatrix}$$

(see [GLR0] for the existence of such triples). Setting $(U, T, V) = (X,\ J,\ PX^*)$ in Lemma 4, it is easily seen that

$$\begin{bmatrix} PX^* & JPX^* & \cdots & J^{\ell-1}PX^* \end{bmatrix} AQ = I_{\ell n} \tag{11}$$

and, since $J^r P = P(J^*)^r$ for $r = 0, 1, \ldots, \ell - 1$ and $P^2 = I$, we obtain $Q^*AQ = P$.

Also, because $B = AC$ and $CQ = QJ$,

$$Q^*BQ = (PR)(AC)Q = (PR)AQJ = P(RAQ)J = PJ.$$

Consequently, $Q^*(\lambda A - B)Q = \lambda P - PJ$.

However, applying the same argument to a second system $L'(\lambda)$ which is strictly isospectral we obtain

$$Q^*(\lambda A - B)Q = \lambda P - PJ = (Q')^*(\lambda A' - B')Q',$$

whence,

$$(Q(Q')^{-1})^*(\lambda A - B)Q(Q')^{-1} = \lambda A' - B'.$$

An SPC is obtained by defining $F = Q(Q')^{-1}$. Reference to the proof of Theorem 8 shows that F^{-1} does, indeed, have the required structure. □

Example 1: A primitive solution of equations (10) is $Y_0^* = \begin{bmatrix} 0 & \cdots & 0 & A_\ell^{-1} \end{bmatrix}$, in which case $X_0 = [I\ 0\ \cdots 0]$ and $F = I$. The congruence of Theorem 14 becomes the identity transformation.

Example 2: Consider the 2×2 monic Hermitian matrix polynomial $L(\lambda)$ of degree three with coefficients

$$A_3 = I_2, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.5 & -0.5 \\ -0.5 & -0.5 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This system has six (truncated) eigenvalues:

$$\lambda_1 = -1.2106, \quad \lambda_2 = +1.2106, \qquad \lambda_{3,4} = 0.5446 \pm i(0.7276), \qquad \lambda_{5,6} = -0.5446 \pm i(0.7276).$$

The Jordan canonical form is $J = \mathrm{diag}[\lambda_1, \cdots, \lambda_6]$ and the canonical Hermitian matrix is

$$P = \mathrm{diag}[\,-1,\ +1, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}].$$

It can be verified that the matrix Y with $Y^* = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$ is a solution of equations (10). Then defining

$$X = Y^*A = \begin{bmatrix} 0 & -1 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

we obtain a selfadjoint triple (X, C, Y) in the sense of Theorem 13. Now compute the moments $\Gamma_0, \ldots, \Gamma_5$ for this triple, and the coefficients of a strictly isospectral system are determined by the moments. In fact, the transformed Hermitian system has (exact) coefficients

$$A_3' = I_2, \quad A_2' = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, \quad A_1' = \begin{bmatrix} 1.5 & -2.5 \\ -2.5 & 1.5 \end{bmatrix}, \quad A_0' = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the isospectral property can be checked numerically. □

6 Systems with palindromic symmetries

We adopt notations introduced by Mackey *et al.* in [M4] to capture palindromic symmetries. The *reverse* of a matrix polynomial $L(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ is denoted by $\mathrm{rev}L(\lambda) := \sum_{j=0}^{\ell} \lambda^j A_{\ell-j}$. Also, we let $\otimes$ denote the operation of *either* conjugate transposition (denoted by an index $*$) *or* transposition (denoted by an index T). With either interpretation, when applied to square matrices, the $\otimes$-operation is an *involution*, i.e. $(A^{\otimes})^{\otimes} = A$. When applied to a matrix polynomial the $\otimes$ operation acts only on the coefficients, and not on the argument. Thus, $L(\lambda)^{\otimes} = \sum_{j=0}^{l} \lambda^j A_j^{\otimes}$. Notice also that $\mathrm{rev}(L(\lambda)^{\otimes}) = (\mathrm{rev}L(\lambda))^{\otimes}$, so we may write $\mathrm{rev}L(\lambda)^{\otimes}$ without ambiguity.

Definition 5 A system $L(\lambda)$ with the property

$$\text{rev}L(\lambda)^{\otimes} = \mu L(\lambda), \quad \mu = \pm 1$$

is said to be $\otimes$-*palindromic* if $\mu = 1$ or $\otimes$-*anti-palindromic* if $\mu = -1$.

More generally, $L(\lambda)$ is said to be *palindromic* if $A_{l-j}^{\otimes} = \mu A_j$ for $j = 0, 1, \ldots, \ell$, and this includes four categories of problems when $\otimes$ and μ are each given their two interpretations. When attention is to be confined to just one of the two options for $\otimes$, we use the terms $*$-palindromic, or T-palindromic with the obvious meanings.

Matrix polynomials with these (and more) palindromic symmetries have been the subject of recent investigations and have significant practical applications (see [MV], [HMM], [M4]). It is argued persuasively that there are numerical advantages in using linearizations which reflect any palindromic symmetry which the underlying matrix polynomial may have. In this section we will develop structure preserving transformations which, in addition, preserve palindromic symmetries of these four varieties.

The reader is referred to [M4] for specific examples of palindromic systems. Difference equations giving rise to palindromic systems also arise naturally when differential systems are approximated by replacing derivatives with symmetric difference quotients. (See Sections II.2.4 and III.2.4 of [GLR2] for an algebraic investigation.) Note that a *linear* pencil $\lambda A - B$ is $\otimes$-palindromic or $\otimes$-anti-palindromic, if and only if $B^{\otimes} = -\mu A$, and then the pencil has the form $\lambda A + \mu A^{\otimes}$ for some $\ell n \times \ell n$ matrix A.

In the sequel attention will be confined to problems defined over $\mathbb{C}$ in which case the symmetry of spectra with respect to the unit circle will be seen to play a vital role. If, *in addition* to the palindromic symmetry, a system has only *real* coefficients, then the spectrum is also symmetric with respect to the real line, and "symplectic spectral symmetry" arises (see [M4], for example). We do not consider problems of this type explicitly.

To derive linearizations which preserve the palindromic symmetry of $L(\lambda)$ we need some preparation. First let us define for all $n \in \mathbb{N}$:

$$R_{\ell,n} := \begin{bmatrix} 0 & & I_n \\ & \iddots & \\ I_n & & 0 \end{bmatrix} \in \mathbb{R}^{\ell n \times \ell n}. \tag{12}$$

To emphasize the dependence of the matrices S_i of (8) on the underlying matrix polynomial $L(\lambda)$ (and at the risk of some confusion) we will write $S_i = S_i(L)$. The following properties will be important for the derivation of linearizations with palindromic symmetries:

Lemma 15 (a) *For a system $L(\lambda)$ of degree ℓ*

$$S_{\ell-i}(L) = -R_{\ell,n} S_i(\text{rev}L) R_{\ell,n}, \quad i = 0, 1, \ldots, \ell. \tag{13}$$

(b) *Let A_0 be nonsingular (as well as A_ℓ), and let C_L and C_{revL} be the companion matrices of L and* rev L. *Then*

$$C_L R_{\ell,n} C_{\text{revL}} = R_{\ell,n}, \tag{14}$$

and (13) *holds for all integers i.*

Proof: These results follow from direct calculations with the definitions and (8). $\square$

In preparation for the next lemma, define

$$\mathbb{P} := \{c \in \mathbb{C}^\ell \,:\, c^T = c^\otimes R_{\ell,1}\}. \tag{15}$$

Observe that $c \in \mathbb{P}$ if and only if $c_j = c^\otimes_{\ell-j+1}$ for $j = 1, 2, \ldots, \ell$ and $\mathbb{P}$ is a linear space[5] over $\mathbb{R}$. Also, for some $c \in \mathbb{C}^\ell$ and any matrix polynomial $L(\lambda)$, let

$$A_c(L) := R_{\ell,n} \sum_{i=1}^{\ell} c_i S_{i-1}(L), \quad B_c(L) = R_{\ell,n} \sum_{i=1}^{\ell} c_i S_i(L) = A_c(L)C.$$

Lemma 16 *Let $L(\lambda)$ have palindromic symmetry. Then the linear matrix polynomial*

$$L_{c,L}(\lambda) := \lambda A_c(L) - B_c(L) = A_c(L)(\lambda I - C). \tag{16}$$

has palindromic symmetry if and only if $c \in \mathbb{P}$. In this case,

$$L_{c,L}(\lambda) := \lambda A_c(L) + \mu A_c(L)^\otimes. \tag{17}$$

Proof: The block-symmetry of the matrices S_i implies that $S_i(L)^\otimes = S_i(L^\otimes)$ and then Lemma 15 gives

$$S_i(L)^\otimes = S_i(L^\otimes) = -R_{\ell,n} S_{\ell-i}(\mathrm{rev} L^\otimes) R_{\ell,n}. \tag{18}$$

Now we have,

$$B_c(L)^\otimes = (R_{\ell,n} \sum_{i=1}^{\ell} c_i S_i(L))^\otimes = (\sum_{i=1}^{\ell} c_i^\otimes S_i(L)^\otimes) R_{\ell,n}, \tag{19}$$

and using (18),

$$B_c(L)^\otimes = -R_{\ell,n} \sum_{i=1}^{\ell} c_i^\otimes S_{\ell-i}(\mathrm{rev} L^\otimes) = -R_{\ell,n} \sum_{j=1}^{\ell} c^\otimes_{\ell-j+1} S_{j-1}(\mathrm{rev} L^\otimes). \tag{20}$$

But $\mathrm{rev} L^\otimes = \mu L$ and, when $c \in \mathbb{P}$, $\quad c^\otimes_{\ell-j+1} = c_j$. Thus

$$B_c(L)^\otimes = -R_{\ell,n} \sum_{j=1}^{\ell} c_j S_{j-1}(\mu L) = -\mu A_c(L) \tag{21}$$

as required. For the converse, it follows from the last two displayed equations and Lemma 10 that, if $B_c(L)^\otimes = -\mu A_c(L)$, then $c \in \mathbb{P}$. $\square$

Although the condition $c \in \mathbb{P}$ ensures the palindromic symmetry of L_c, not every such linear matrix polynomial will be a linearization for $L(\lambda)$. However, by definition, and using the notation of Lemma 10,

$$L_{c,L}(\lambda) = R_{\ell,n} \sum_{i=1}^{\ell} c_i(\lambda S_{i-1} - S_i) = R_{\ell,n} \sum_{i=1}^{\ell} c_i L_i(\lambda). \tag{22}$$

Recalling that $p_c(\lambda) := \sum_{j=1}^{\ell} c_j \lambda^{j-1}$, Theorem 11 can be used to determine palindromic linearizations:

[5]For the operation $\otimes = *$, the dimension of $\mathbb{P}$ is always ℓ. For the operation $\otimes = T$ the dimension is ℓ or $\ell + 1$ according as ℓ is even or odd.

Theorem 17 *Let $L(\lambda)$ be a μ-palindromic matrix polynomial, $c \in \mathbb{P}$, and assume that $p_c(\lambda)$ is nonzero at the eigenvalues of $L(\lambda)$. Then the linear matrix polynomial of* (16), (17) *is a μ-palindromic linearization for $L(\lambda)$.*

Notice that linearizations constructed in this way have two symmetries: they have both palindromic symmetry and the block symmetry of the matrices S_i.

Example 3: Let $L(\lambda)$ be palindromic with $\ell = 4$. Then $p_c(\lambda) = c_1 + c_2\lambda + c_3\lambda^2 + c_4\lambda^3$, and $c \in \mathbb{P}$ implies

$$c^T = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} = \begin{bmatrix} \overline{c_4} & \overline{c_3} & \overline{c_2} & \overline{c_1} \end{bmatrix}.$$

If we take $c = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}$ then $p_c(\lambda) = 1 + \lambda^3$, and $p_c(\lambda)$ has its zeros at the cube-roots of -1. If $L(\lambda)$ is nonsingular at these three points, then a palindromic linearization is $L_{c,L}(\lambda)$ where

$$R_{\ell,n}L_{c,L}(\lambda) = \lambda(S_0 + S_3) - (S_1 + S_4).$$

This can be verified by direct calculation. In other words, $\lambda A_c + \mu A_c^{\otimes}$ is a palindromic linearization, where $A_c = R_{\ell,n}(S_0 + S_3)$.

Clearly, the choice $c = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}$ is also possible. Then $p_c(\lambda) = \lambda^2 + \lambda^3$ with just two distinct zeros. Since this puts a weaker constraint on $L(\lambda)$, it will generally be the better choice and gives $A_c = R_{\ell,n}(S_1 + S_2)$. □

The properties illustrated in Example 3 suggest that the *primary (palindromic) linearization* for palindromic systems be defined in terms of the single matrix A_c where:

$$A_c = \begin{cases} R_{\ell,n}(S_{k-1} + S_k) & \text{when } \ell = 2k, \\ R_{\ell,n}S_k & \text{when } \ell = 2k+1. \end{cases} \tag{23}$$

Example 4. When $\ell = 2$ we choose $p_c(\lambda) = 1 + \lambda$, and the primary linearization is determined by

$$A_c = R_{2,n}(S_0 + S_1) = \begin{bmatrix} A_2 & A_2 \\ A_1 - A_0 & A_2 \end{bmatrix} = \begin{bmatrix} \mu A_0^{\otimes} & \mu A_0^{\otimes} \\ -A_0 + A_1 & \mu A_0^{\otimes} \end{bmatrix},$$

provided that -1 is not in the spectrum of $L(\lambda)$.

When $\ell = 3$ we choose $p_c(\lambda) = \lambda$, and the primary linearization is determined by

$$A_c = R_{3,n}S_1 = \begin{bmatrix} 0 & A_3 & 0 \\ 0 & A_2 & A_3 \\ -A_0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mu A_0^{\otimes} & 0 \\ 0 & \mu A_1^{\otimes} & \mu A_0^{\otimes} \\ -A_0 & 0 & 0 \end{bmatrix},$$

provided only that 0 is not an eigenvalue of $L(\lambda)$. □

7 SPE's for palindromic systems

Now we can define structure preserving equivalence transformations between systems which preserve palindromic symmetries. Thus:

Theorem 18 *Let $L(\lambda)$ be a palindromic system with a palindromic linearization of the form* (16), (17). *For every pair (E,F) defining an SPE for $L_{c,L}(\lambda)$ as in Theorem 7 (so that $EA_cF = A'_c$ and $EB_cF = B'_c$ are structured), the pair $(R_{\ell,n}ER_{\ell,n}, F)$ defines an SPE from $L_{c,L}(\lambda)$ to $\lambda A'_c - B'_c$, and this pencil is palindromic.*

Proof: Using the definition (16) and (22) followed by Lemma 9,

$$(R_{\ell,n}ER_{\ell,n})L_{c,L}(\lambda)F = R_{\ell,n}E\sum_{i=1}^{\ell} c_j(\lambda S_{j-1} - S_j)F = R_{\ell,n}\sum_{j=1}^{\ell} c_j(\lambda S'_{j-1} - S'_j).$$

But $R_{\ell,n}\sum_{j=1}^{\ell} S'_{j-1} = A'_c(L)$ and $R_{\ell,n}\sum_{j=1}^{\ell} S'_j = B'_c(L)$, so that A'_c and B'_c have the required structure and the strict equivalence $(R_{\ell,n}ER_{\ell,n})L_{c,L}(\lambda)F = \lambda A'_c - B'_c$ holds. Furthermore, the fact that $c \in \mathbb{P}$ implies that $\lambda A'_c - B'_c$ is palindromic. □

Example 5: Consider the third order matrix polynomial given by $A_0 = A_3 = I_2$, $A_1 = A_2^* = \begin{bmatrix} 1 & i \\ 1 & 1 \end{bmatrix}$ which has $*-$palindromic symmetry and $\mu = 1$. The eigenvalues are $\lambda_1 = i$, $\lambda_2 = 1/\overline{\lambda}_1 = i$ and

$$\begin{aligned}
\lambda_3 &= \frac{i-1}{2\sqrt{2}}\left[1+\sqrt{2}+\sqrt{(\sqrt{2}-1)(\sqrt{2}+3)}\right] = 1.3316(i-1) \\
\lambda_4 &= \frac{i-1}{2\sqrt{2}}\left[1+\sqrt{2}-\sqrt{(\sqrt{2}-1)(\sqrt{2}+3)}\right] = 0.3755(i-1) \\
\lambda_5 &= \frac{i-1}{2\sqrt{2}}\left[\sqrt{2}-1+i\sqrt{(\sqrt{2}+1)(3-\sqrt{2})}\right] = -0.8382-0.5453i \\
\lambda_6 &= \frac{i-1}{2\sqrt{2}}\left[\sqrt{2}-1-i\sqrt{(\sqrt{2}+1)(3-\sqrt{2})}\right] = -0.5453-0.8382i
\end{aligned}$$

Note that $\lambda_4 = 1/\overline{\lambda}_3$ and that $|\lambda_5| = |\lambda_6| = 1$. If we choose

$$\begin{aligned}
X &= \begin{bmatrix} 0.1501 & -1.0552 & -0.7774 & 0.1485 & 0.6387 & 0.8127 \\ 0.6391 & -0.5000 & -0.9427 & 0.5056 & 0.2785 & 1.5156 \end{bmatrix} \\
&+ i\begin{bmatrix} -0.0296 & -1.4770 & -1.6868 & -2.2177 & -0.1802 & -1.2522 \\ -0.0821 & -0.8192 & -1.5736 & -2.4305 & -0.1494 & -0.9187 \end{bmatrix}
\end{aligned}$$

then (X, C) form a standard pair and, with reference to Theorem 7, $F = G^{-1}$ can be calculated. The choice $A'_3 = I_2$ implies that

$$Y = F\begin{bmatrix} 0 \\ 0 \\ I_2 \end{bmatrix}$$

and hence (X, C, Y) form a standard triple. We can now compute E as given in Theorem 7.

To obtain a $*$-palindromic linearization $L_{c,L}$ for L we note that $c \in \mathbb{P}$ requires that

$$c^T = \begin{bmatrix} \alpha & r & \overline{\alpha} \end{bmatrix}, \quad \alpha \in \mathbb{C}, \quad r \in \mathbb{R}.$$

Thus $L_{c,P}(\lambda) = \lambda A_c - B_c$ where

$$A_c = R_2(\alpha S_0 + rS_1 + \overline{\alpha} S_2) = \begin{bmatrix} \alpha I_2 & rI_2 & \overline{\alpha} I_2 \\ \alpha A_1^* - \overline{\alpha} I_2 & \alpha I_2 + rA_1^* - \overline{\alpha} A_1 & rI_2 \\ \alpha A_1 - rI_2 & \alpha A_1^* - \overline{\alpha} I_2 & \alpha I_2 \end{bmatrix}$$

and $B_c = R_2(\alpha S_1 + rS_2 + \overline{\alpha} S_3) = -A_c^*$.

But there is a second condition, namely that $p_c(\lambda_i) = \overline{\alpha}\lambda_i^2 + \lambda_i r + \alpha \neq 0$ for $i = 1, \cdots, 6$. To obtain a sparse form for A_c and B_c we may choose $\alpha = 0$ and $r = 1$ which means that $p_c(\lambda)$ has the repeated eigenvalue zero and this is not an eigenvalue of $L(\lambda)$. Thus for $c = [0, 1, 0]^T$ we have

$$L_c(\lambda) = \lambda R_2 S_1 - R_2 S_2 = \lambda \begin{bmatrix} 0 & I_2 & 0 \\ 0 & A_1^* & I_2 \\ -I_2 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & I_2 \\ -I_2 & -A_1 & 0 \\ 0 & -I_2 & 0 \end{bmatrix}.$$

Applying the SPE as described in Theorem 18 gives $R_2 E R_2 L_c(\lambda) F = L_c'(\lambda) = \lambda A_c' - B_c'$ where

$$A_c' = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.2929 + 0.7071i & 0 & 1 & 0 \\ 0 & 0 & 0 & 1.7071 - 0.7071i & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $B_c' = -(A_c')^*$. The resulting strictly isospectral system is, indeed, $*$-palindromic but also diagonal. In more detail we have $A_0' = A_3' = I_2$ and $A_1' = \mathrm{diag}(1 - \phi, 1 + \phi) = (A_2')^*$, where $\phi := (i + 1)/\sqrt{2}$. $\square$

Theorem 18 is an analogue of Theorem 7 with the additional property that the palindromic symmetry of $L(\lambda)$ is inherited by its linearization. So the spectrum is preserved, but the question of *strictly* isospectral systems arises once more, together with interest in SPC's (rather than SPE's) which preserve a sign characteristic as well as the spectrum (cf. Theorem 14). The next section provides an introduction to these ideas in the context of $*$-palindromic systems.

8 Unitary properties of $*$-palindromic polynomials

Keep in mind that, because we confine attention to systems $L(\lambda)$ with $\det(A_\ell) \neq 0$, any system with one of the four palindromic symmetries also has $\det(A_0) \neq 0$, and hence $L(\lambda)$ has no zero eigenvalue; $0 \notin \sigma(L)$. Then, by applying the operation $*$ or T to $L(\lambda)x = 0$, it is easily verified that:

$$\begin{array}{llcl} \text{For} & * -\text{palindromic systems (with } \mu = \pm 1), & \lambda \in \sigma(L) \;\Leftrightarrow\; & (\overline{\lambda})^{-1} \in \sigma(L), \\ \text{For} & T - \text{palindromic systems (with } \mu = \pm 1), & \lambda \in \sigma(L) \;\Leftrightarrow\; & \lambda^{-1} \in \sigma(L). \end{array}$$

The first of these statements implies that the spectrum of a $*$-palindromic system is symmetric with respect to the unit circle: either the eigenvalues satisfy $|\lambda_i| = 1$, or they occur in pairs $\lambda_j \neq \lambda_k$ where $\lambda_j \overline{\lambda_k} = 1$. This symmetry implies that the linearizations must be unitary in a suitable indefinite inner product (see Section 12.6 of [GLR2]). Let us confirm this for the palindromic linearizations of Theorem 17. With $\otimes = *$ we have $L_c(\lambda) = \lambda A_c + \mu A_c^*$, and since A_c is nonsingular,

$$L_c(\lambda) = A_c(\lambda I + \mu U) \tag{24}$$

where $U = A_c^{-1} A_c^*$. But it is known that *any* matrix of the form $\pm M^{-1} M^*$ is H-unitary for some nonsingular H (see Section 4.4 of [GLR2] for details, and also Johnson and DiPrima [JD]). Indeed there is a family of indefinite inner products to choose from, namely, matrices of the form

$$H_z := \overline{z} A_c + z A_c^* \tag{25}$$

where $|z| = 1$. Notice that H_z now depends on $c \in \mathbb{P}$ (of (15)), and also on z.

Since $\lambda I + \mu U$ is strictly equivalent to $L_c(\lambda)$ it is also a linearization (but not palindromic). Thus the canonical structures of H-unitary matrices (and not just the spectrum) are inherited by the palindromic linearization (24). A detailed investigation of these unitary properties would take us too far afield, and is left for further investigation. Let us simply note that bilinear transformations can be used to transform palindromic systems (some of which have unitary properties) to Hermitian systems which are often more easily treated. In particular, a simple bilinear transformation maps a palindromic linearization into a Hermitian linear system *and preserves the sign characteristic* in the process.

This could be useful for situations where the Hermitian symmetries are seen to be more tractable. To be specific, note that the transformation $\lambda(\zeta) = (\zeta - i)/(\zeta + i)$ maps the upper half of the ζ-plane one-to-one onto the interior of the unit circle in the λ plane. Then define the Hermitian pencil

$$H(\zeta) := \begin{cases} (\zeta + i)L_c(\lambda(\zeta)) = & \zeta(A_c + A_c^*) + i(A_c^* - A_c) \quad \text{when } \mu = 1, \\ i(\zeta + i)L_c(\lambda(\zeta)) = & \zeta i(A_c - A_c^*) + (A_c^* + A_c) \quad \text{when } \mu = -1. \end{cases}$$

Now the unimodular eigenvalues of $L_c(\lambda)$ are transformed to the real eigenvalues of $H(\zeta)$ and the *sign characteristic is preserved*. This can be established as in the proof of Theorem 5.15.1 of [GLR2]. (See a recent paper of Rodman [R] for a detailed analysis of linear palindromic systems.)

More generally, there are simple bilinear transformations mapping higher order palindromic systems of all the types described in Definition 5 into (suitably) symmetric systems, and conversely. These transformations are summarized in an appendix to this paper.

9 Conclusions

Isospectral families of matrix polynomials (or "systems") have been examined by making intensive use of methods developed by Gohberg, Lancaster and Rodman in [GLR0], [GLR1], and [GLR2]. Isospectral families of systems are identified with structure preserving strict equivalence transformations (SPE's) of their linearizations (Theorem 8). (A technique introduced by Garvey, Friswell and Prells in [GFP1], [GFP2] and [GPF].)

A particularly useful class of linearizations is identified (Theorem 11), and plays an important role in the analysis of Hermitian systems. Their symmetries are reflected in their linearizations, and it has been shown how *strictly* isospectral Hermitian systems can be identified with structure preserving congruence (SPC) transformations of their linearizations (Theorem 13).

A start has been made on an analogous treatment of isospectral families of systems with palindromic symmetry. SPE's of linearizations for palindromic systems have been developed which preserve the palindromic symmetry (Theorem 18). However, the association of *strictly* isospectral palindromic systems with SPC transformations of their linearizations remains open.

APPENDIX: Bilinear transformations (See also Theorem 4.5 of [M4].)

First recall the definitions introduced early in Section 6, and let $L(\lambda) = \sum_{j=0}^{\ell} A_i \lambda^j$. We suppose that the coefficients of L satisfy $A_j^{\otimes} = \varepsilon_j A_j$ where $\varepsilon_j = \pm 1$, i.e. $L(\lambda)$ has a "symmetric" property and is to be transformed to a system with a palindromic property.

Let $\alpha = \pm 1$. Define the bilinear transformations $\lambda(\zeta) = (\zeta\overline{w} - \alpha w)/(\zeta - \alpha)$, and then

$$K(\zeta) := (\zeta - \alpha)^{\ell} L(\lambda(\zeta)) = \sum_{i=0}^{\ell} \varepsilon_j A_j (\zeta - \alpha)^{\ell - j} (\zeta\overline{w} - \alpha w)^i. \tag{26}$$

We claim that $K(\zeta)$ is palindromic in the sense that $\zeta^{\ell}(K(1/\zeta))^{\otimes} = \mu K(\zeta)$ for $\mu = 1$ or $\mu = -1$. To verify this claim, a calculation gives

$$\mathrm{rev}(K(\zeta))^{\otimes} = \zeta^{\ell}(K(1/\zeta)^{\otimes} = \zeta \sum_{j=0}^{\ell} \varepsilon_j A_j (1/\zeta - \alpha)^{\ell - j} (\bar{w}^{\otimes}/\zeta - \alpha w^{\otimes})^j.$$

Writing $\zeta^{\ell} = (\zeta^{\ell - j})(\zeta^j)$ and using the fact that $\alpha^{-1} = \alpha$ it is found that

$$\mathrm{rev}(K(\zeta))^{\otimes} = (-\alpha)^{\ell} \sum_{j=1}^{\ell} \varepsilon_j A_j (\zeta - \alpha)^{\ell - j} (\zeta w^{\otimes} - \alpha \bar{w}^{\otimes})^j.$$

Comparing this expression with (26) it is found that:

Case I: $\mathrm{rev}(K(\zeta))^* = \mu K(\zeta)$ for $\mu = (-\alpha)^{\ell}$, and $\varepsilon_j = 1$ for $j = 0, 1, \ldots, \ell$.

Case II: $\mathrm{rev}(K(\zeta))^* = \mu K(\zeta)$ for $\mu = (-\alpha)^{\ell+1}$ and $\varepsilon_j = -1$ for $j = 0, 1, \ldots, \ell$.

Case III: $\mathrm{rev}(K(\zeta))^T = \mu K(\zeta)$ for $\mu = (-\alpha)^{\ell}$ and $\varepsilon_j = (-1)^j$ for $j = 0, 1, \ldots, \ell$, and $w = jp,\ p > 0$.

Case IV: $\mathrm{rev}(K(\zeta))^T = \mu K(\zeta)$ for $\mu = (-\alpha)^{\ell+1}$ and $\varepsilon_j = (-1)^{j+1}$, for $j = 0, 1, \ldots, \ell$, and $w = jp,\ p > 0$.

It is easily seen that Cases III and IV include the "T-odd" and "T-even" matrix polynomials of [M4].

For the inverse transformations let $K(\zeta) = \sum_{j=1}^{\ell} \zeta^j B_j$ where $\mathrm{rev}(K(\zeta))^{\otimes} = \mu K(\zeta)$ or, what is equivalent, $B_j^{\otimes} = \mu B_{\ell - j}$ for $\mu = \pm 1$. Using the transformation $\zeta = \alpha(\lambda - w)/(\lambda - \bar{w})$ we obtain

$$L(\lambda) = \sum_{j=0}^{\ell} A_j \lambda^j = (\lambda - \bar{w})^{\ell} K(\zeta(\lambda)) = \sum_{j=1}^{\ell} B_j \alpha^j (\lambda - w^{\otimes})^j (\lambda - \bar{w}^{\otimes})^{\ell - j}, \tag{27}$$

and it is claimed that, for $\alpha = \pm 1$ has symmetry properties. In fact, $A_j^{\otimes} = \varepsilon_j A_j$ where $\varepsilon_0, \ldots, \varepsilon_{\ell}$ are each equal to ± 1. Indeed, a little calculation will serve to verify that

$$(L(\lambda))^{\otimes} = \mu \alpha^{\ell} \sum_{j=1}^{\ell} \mu B_j \alpha^j (\lambda - \bar{w}^{\otimes})^j (\lambda - w^{\otimes})^{\ell - j}. \tag{28}$$

Comparison with (27) yields:

Case 1: $(L(\lambda))^* = \varepsilon L(\lambda)$ with $\varepsilon = \mu\alpha^\ell$ and $\varepsilon_j = 1$ for $j = 0, 1, \dots, \ell$.

Case 2: $(L(\lambda))^* = \varepsilon L(\lambda)$ with $\varepsilon = -\mu\alpha^\ell$ and $\varepsilon_j = -1$ for $j = 0, 1, \dots, \ell$.

Case 3: $(L(\lambda))^T = \varepsilon L(\lambda)$ with $\varepsilon = \mu(-\alpha)^{\ell+1}$, $w = ip,\ p > 0$ and $\varepsilon_j = (-1)^j$ for $j = 0, 1, \dots, \ell$.

Case 4: $(L(\lambda))^T = \varepsilon L(\lambda)$ with $\varepsilon = \mu(-\alpha)^\ell$, $w = ip,\ p > 0$ and $\varepsilon_j = (-1)^{j+1}$ for $j = 0, 1, \dots, \ell$.

In particular, notice that in Case 1 $L(\lambda)$ is selfadjoint if parameters are chosen so that $\varepsilon = \mu\alpha^\ell = 1$, and Case 2 admits skew-Hermitian coefficients. The last two cases concern "T-odd" and "T-even" polynomials.

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