PMAT 435 Analysis I (Short Notes)

Convergence

A sequence (s_n) is said to be (*monotonic*) *increasing* if $s_n \leq s_{n+1}$ for all $n \in N$, i.e., $s_1 \leq s_2 \leq s_3 \leq ...$; It is said to be (*monotonic*) *decreasing* if $s_n \geq s_{n+1}$ for all $n \in N$, i.e., $s_1 \geq s_2 \geq s_3 \geq ...$; It is said to be *monotonic* if it is either monotonic increasing or monotonic decreasing.

Exercise. Are the following sequences increasing? decreasing? neither?
(1) $s_n = 100 - n^2$.
(2) $s_n = 100n - n^2$.
(3) $s_n = x^n, \ x \in \mathbf{R}$.

Monotone Convergence Theorem. Every monotonic increasing sequence that is bounded above is convergent.

Exercise. Prove that
(1) every decreasing sequence that is bounded below is convergent.
(2) every increasing sequence that is not bounded above diverges to ∞.
(3) every decreasing sequence that is not bounded below diverges to $-\infty$.

Exercise. Show that the following sequences converge, and find their limits.
(1) $s_1 = 1, \ s_{n+1} = \frac{1}{4}(2s_n + 5), \ n \in \mathbf{N}$.
(2) $\sqrt{6}, \ \sqrt{6+\sqrt{6}}, \ \sqrt{6+\sqrt{6+\sqrt{6}}}, \ ...$
(3) Let $a > 0$. Let $s_1 > \sqrt{a}$, and let $s_{n+1} = \frac{1}{2}\left(s_n + \frac{a}{s_n}\right)$ for $n \in \mathbf{N}$.

Exercise. Show that $s_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$ diverges to ∞.

Let (s_n) be a sequence. If $n_1 < n_2 < n_3 < ...$ are positive integers, the sequence $s_{n_1}, \ s_{n_2}, \ s_{n_3}, \ ...$ is called a *subsequence* of (s_n), and is denoted by (s_{n_k}). (Here, k is the index.)

Example. The following are subsequences of (s_n).
(1) $s_2, \ s_5, \ s_6, \ s_7, \ s_{13}, \ s_{100}$, ...
(2) (s_{2n}), i.e., $s_2, \ s_4, \ s_6, \ s_8, \ ...$
(3) (s_{2n-1}), i.e., $s_1, \ s_3, \ s_5, \ s_7, \ ...$
(4) (s_{2^k}), i.e., $s_2, \ s_4, \ s_8, \ s_{16}, \ s_{32}, \ ...$

Theorem. If (s_n) converges to s, so do all its subsequences.

Theorem. Every sequence has a monotonic subsequence.

Theorem. Every bounded sequence has a convergent subsequence.

Exercise. Show that if (s_{2n}) and (s_{2n-1}) converge to the same limit s, then (s_n) converges to s.

Exercise. Let $a > 0$. Let $s_1 > \sqrt{a}$, and let $s_{n+1} = \frac{a + s_n}{1 + s_n}$ for all $n \in N$. Show that (s_{2n}) and (s_{2n-1}) converge to the same limit. Hence find the limit of (s_n).

A sequence (s_n) is said to be a *Cauchy sequence* if it satisfies the following condition.

(CS) For every $\varepsilon > 0$ there exists N such that for all $m, n \geq N$, $|s_m - s_n| < \varepsilon$.

(CS) says that given any $\varepsilon > 0$, no matter how small, there is a stage N in the sequence beyond which any two terms will differ by an amount less than ε. So, the terms in the sequence are eventually arbitrarily close to one another. We also note that the meaning of (CS) can be expressed in an alternative form:

(CS′) For every $\varepsilon > 0$ there exists N such that for all $n \geq N$ and $p \in \mathbf{N}$, $|s_{n+p} - s_n| < \varepsilon$.

Theorem. Every Cauchy sequence is bounded.

Theorem. Every convergent sequence is a Cauchy sequence.

Cauchy Convergence Criterion. Every Cauchy sequence converges.

Thus, a sequence is a Cauchy sequence if and only if it is convergent. The notion of a Cauchy sequence provides us with a characterization of convergence in terms of just the terms in the sequence without explicit reference to the limit.
