MATH 403 (Winter 2010) Assignment 5 **Due: April 9, 2010**

Four questions will be randomly selected for marking. All questions carry equal marks.

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1. Let $A = \text{conv}\{(0,0,0), (4,0,0), (0,4,0), (0,0,4)\}$ and $B = [2,3] \times [2,3] \times [2,3]$. Prove that the hyperplane

$$H = H((3,4,5);24) = \{(x,y,z) : 3x + 4y + 5z = 24\}$$

separates A and B by verifying that $A \subseteq H^{\leq}$ and $B \subseteq H^{\geq}$. Does H strictly separate A and B? Why?

2. Find the point in the set $C = \left\{(x,y) : y \geq \frac{8}{x},\ x > 0\right\}$ that is closest to the point $(-4,1)$. Use this result to find a hyperplane in $\mathbf{R}^2$ that strictly separates C and $B = \{(-4,1)\}$. [Hint: Find the absolute minimum distance from $(-4,1)$ to the boundary of C. You may use the theorem about factorization of polynomials that says that the only possible candidates for an integer root of the polynomial $x^4 + px^3 + qx^2 + rx + s$ are $\pm$the factors of s, if p, q, r, s are integers. Note also that $x > 0$.]

3. Consider the cross-polytope $\dagger_4 = \text{conv}\{e_1, -e_1, e_2, -e_2, e_3, -e_3, e_4, -e_4\}$ in $\mathbf{R}^4$. (Notice that you cannot draw this object since it is in $\mathbf{R}^4$.) Verify that if α is a real number strictly between -1 and 1, i.e. $-1 < \alpha < 1$, then the hyperplane

$$H = H((1,-1,-1,\alpha);1)$$

is a supporting hyperplane for $\dagger_4$. At which point(s) does H support $\dagger_4$?

4. Consider the closed convex set $C = \{(x,y) : y \geq x^2\}$ and the set $B = [-1,1] \times [-4,-3]$. Find all supporting hyperplanes of C that separate B and C. [Hint: A sketch of B and C might help.]

5. Let $p_1 = (1,0)$, $p_2 = (2,1)$, $p_3 = (1,3)$, $p_4 = (0,4)$ and $K = \text{conv}\{p_1, p_2, p_3, p_4\}$. Express K as the intersection of a minimal number of closed half-planes whose boundaries are supporting hyperplanes to K. Express each of your closed half-planes in the form

$$H^{\leq}((a_1,a_2);1) \text{ or } H^{\geq}((a_1,a_2);1)$$

where (a_1, a_2) is some non-zero vector in $\mathbf{R}^2$.

6. Find all fixed points, if any, of the following functions:

   (a) $f : [0,1] \to [0,1]$ defined by $f(x) = 4x(1-x)$.

   (b) $g : Q_2 \to Q_2$ defined by $g(x,y) = (1-x-y, x)$.

   (c) $h : \mathbf{R}^3 \to \mathbf{R}^3$ defined by $h(x,y,z) = (y+1, z+2, x+3)$.

7. Explain why none of the situations below contradicts the Brouwer's Fixed Point Theorem. (You don't need to verify what is claimed in the given statement. You just need to explain why it does not contradict the Theorem.)

(a) Let C denote the unit circle $\{(x,y) : x^2 + y^2 = 1\}$. The function $f : C \to C$ defined by $f(x,y) = (-y, x)$ is continuous on C but does not have a fixed point.

(b) The function $f : [0,1] \to [0,1]$ defined by $f(x) = \begin{cases} 1, & 0 \le x \le \frac{1}{2}, \\ 0, & \frac{1}{2} < x \le 1 \end{cases}$ is not continuous on $[0,1]$ and does not have a fixed point.

(c) The function $f : [0,1] \to [0,1]$ defined by $f(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2}, \\ 1, & \frac{1}{2} < x \le 1 \end{cases}$ is not continuous on $[0,1]$ and yet has two fixed points.

(d) The function $f : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $f(x,y) = (-y, -x+1)$ is continuous on $\mathbf{R}^2$ but does not have a fixed point.

(e) The function $f : [0,1] \times [0,1] \to [0,1] \times [0,1]$ defined by $f(x,y) = (x^2, y^2)$ is continuous on $[0,1] \times [0,1]$ and has fixed points.

8. Consider the triangular region $\Delta = \text{conv}\{(0,0), (4,0), (0,4)\}$. We form two triangulations T_{-1} and T_1 of Δ as follows. Take all the *lattice points* in Δ, i.e. all the points in Δ with integer coordinates. The boundary of Δ is cut into line segments by the lattice points on it. To get T_{-1}, we connect the lattice points using all appropriate horizontal line segments, vertical line segments and line segments of slope -1. To get T_1, we connect the lattice points using all appropriate horizontal line segments, vertical line segments and line segments of slope 1. At each lattice point (m,n), divide $m(m+1) + n^2$ by 3 and then label the point as A, B or C according as the remainder is 0, 1 or 2 (respectively).

(a) Draw both T_{-1} and T_1, showing the labels for all the lattice points in Δ.

(b) Explain why our labelling is a Sperner's labelling.

(c) For each of T_{-1} and T_1, shade all small ABC triangles.
