Math 249 Lecture 08: Exponential and Logarithmic Functions

Objective: To review the algebraic properties of the exponents and logarithms.

Significance and Uses: The algebraic properties of exponents and logarithms are considered basic mathematical facts, and they will be used in examples in this course at will. The properties of logarithms become the key to a process called "logarithmic differentiation' [Four weeks from now.]

Assumed knowledge: It is assumed that you have already studied the following topics covered in the Alberta's Pure Math 10-20-30 curricula.

(1) Topic 5 in Pure Math 10: Exponents and Radicals: Use exact values, arithmetic operations and algebraic operations on real numbers to solve problems. Subtopic 5.5: Explain and apply the exponent laws for powers of numbers and for variables with rational exponents. Subtopic 5.6: Perform operations on irrational numbers of monomial and binomial form, using exact values.

(2) Topic 2 of Pure Math 30: Exponents, Logarithms and Geometric Series - Solve exponential, logarithmic and trigonometric equations and identities. Subtopic 2.3: Solve exponential equations having bases that are powers of one another. Subtopic 2.4: Use the laws of exponents and logarithms to - solve and verify exponential equations and identities; solve logarithmic equations; simplify logarithmic expressions.

(3) Topic 2 of Pure Math 30: Exponents, Logarithms and Geometric Series (continued) - Represent and analyze exponential and logarithmic functions, using technology as appropriate. Subtopic 2.5: Graph and analyze an exponential function, using technology. Subtopic 2.6: Model, graph and apply exponential functions to solve problems. Subtopic 2.7: Change functions from exponential form to logarithmic form, and vice versa. Subtopic 2.8: Use logarithms to model practical problems. Subtopic 2.9: Explain the relationship between the laws of logarithms and the laws of exponents. Subtopic 2.10: Graph and analyze logarithmic functions with and without technology.

For details, consult
http://www.education.gov.ab.ca/k_12/curriculum/bySubject/math/pure.pdf.

A Summary of Facts:

- Algebraic properties for exponents:

$$\begin{aligned} a^0 &= 1,\ a^1 = a \\ a^{x+y} &= a^x a^y \\ a^{-x} &= \frac{1}{a^x} \\ a^{x-y} &= \frac{a^x}{a^y} \\ (a^x)^y &= a^{xy} \\ (ab)^x &= a^x b^x \end{aligned}$$

- Definition of logarithm:

$$c = \log_a b \Leftrightarrow a^c = b.$$

  That is, $\log_a b$ is that number c so that when you calculate a^c you get b.

- Algebraic properties for logarithm:

$$\begin{aligned}
\log_a 1 &= 0,\ \log_a a = 1 \\
\log_a (xy) &= \log_a x + \log_a y,\ x > 0,\ y > 0 \\
\log_a \left(\frac{x}{y}\right) &= \log_a x - \log_a y,\ x > 0,\ y > 0 \\
\log_a (x^r) &= r \log_a x,\ x > 0
\end{aligned}$$

- Change of base:

$$\log_a x = \frac{\log_b x}{\log_b a},\ x > 0$$

- Cancellation equations

$$\begin{aligned}
\log_a (a^x) &= x,\ -\infty < x < \infty \\
a^{\log_a x} &= x,\ x > 0
\end{aligned}$$

- Remarks on notations: $\log_a x$ reads "the logarithm of x to the base a" and is to be seen as $\log_a (x)$, not as $\log ({}_a x)$. In it, $\log_a$ is the function symbol just like sin and $\sqrt{\ }$ are. It applies to whatever is written after it, and that the "${}_a x$" is NOT to be read as "a to the power of x". The subscript a goes with the "log" before it and not the x after it.

Something New:

- The number e

$$e = 2.7182818284590....$$

  Significance of e : Slope of tangent line. Area under graph.

- Natural logarithm:

$$\ln x = \log_e x.$$

Examples.

1. Simplify the following expressions.
   (a) $\left(a^{2x+y} \sqrt[3]{a^y}\right)^4 / a^{3x}$
   (b) $2 \log_2 6 + \log_4 (25) - \log_2 (45)$
   (c) $\log_{1/2} (32)$
   (d) $\ln (e^{-3}) + e^{\ln 5}$
2. If $\log_a 2 = p$ and $\log_b 3 = q$, express 6 in terms of $a,\ b,\ p$, and q.
3. Solve the following equations.
   (a) $2 \log_3 (x + 6) - 3 = \log_3 x$
   (b) $\log_5 (x - 3) + \log_5 (x + 1) = 1$
   (c) $4^x - 5 (2^x) + 6 = 0$
   (d) $2^{x+3} = 3^{4-x}$

4. Solve the following equations.

   (a) $2e^x - 3 = 0$
   (b) $\ln x + \ln(x-1) = 1$

Exercises.

1. Simplify the following expressions.

   (a) $\log_3 15 + \log_3\left(\frac{9}{20}\right) + 2\log_3 2$
   (b) $\log_2(2^x) + \log_3(3^{2x}) - \log_5(5^{3x})$
   (c) $\ln(9e) - 2\ln 3 + e^{\ln 5}$

2. If $\log_2 a = p$ and $\log_3 b = q$, express 6 in terms of a, b, p, and q.

3. Solve the following equations

   (a) $3^{x^2} = 9^x$
   (b) $\log_3(x-2) = 1 - \log_3(x-4)$
   (c) $3^x + 5(3^{-x}) = 6$

Answers:

1. (a) 3. (b) 0. (c) 6.

2. $a^{1/p}b^{1/q}$

3. (a) 0, 2. (b) 5. (c) 0, $\log_3 5$.
