RESEARCH STATEMENT

CRISTIAN RIOS

1. Degenerate Subelliptic Monge Ampère Equations

Introduction

On the past years I have worked in collaboration with Eric Sawyer from McMaster University and with Richard Wheeden from Rutgers University on problems related to the subelliptic Monge Ampère equation,

$$\det\left(\nabla^2 f(x)\right) = k(x) \geq 0, \qquad x \in \Omega \subset \mathbb{R}^n, \tag{1}$$

where f is convex in Ω, $\nabla^2 f$ is the Hessian matrix and Ω is a uniformly convex domain.

The uniformly elliptic case ($0 < c \leq k < \infty$) has been extensively studied and a rather complete theory of existence, uniqueness and a priori regularity for solutions to (1) as well for solutions to the Dirichlet problem

$$\begin{aligned} \det\left(D^2 f(x)\right) &= k(x), \qquad x \in \Omega \\ f(x) &= \varphi(x), \qquad x \in \partial\Omega \end{aligned} \tag{2}$$

is available [1], [26], [5], [4]. In [4] the authors prove that the Monge Ampère equation is hypoelliptic: weak solutions to (2) with smooth data are necessarily smooth.

On the other hand, the function $f_n(x) = c_n |x|^{2+\frac{2}{n}}$ satisfies

$$\det\left(\nabla^2 f_n(x)\right) = |x|^2, \qquad x \in \mathbb{R}^n, \tag{3}$$

for an appropriate constant c_n, and f is constant on any sphere centered at the origin. Even though the right hand side is smooth and *vanishes at a single point*, the solution f fails to be C^α for any $\alpha > 2 + \frac{2}{n}$.

In [13], Guan, Trudinger and Wang proved that if k is merely nonnegative and smooth, then there always exists a unique convex generalized solution to (2), which lies in $C^{1,1}\left(\overline{\Omega}\right)$. An explicit example due to Siboni (see [12]) show that $C^{1,1}$ cannot be improved.

The first hypoellipticity result for subelliptic Monge Ampère equations is a two dimensional result and it is due to P. Guan. In [12] the author proved that if k vanishes to a finite order at the origin, more precisely,

$$C^{-1}\left(x^{2m} + By^{2\ell}\right) \leq k(x,y) \leq C\left(x^{2m} + By^{2\ell}\right), \tag{4}$$

where $B \geq 0$, $\ell \geq m$ are positive integers and $C > 0$; then every convex solution to (1) f satisfying

$$\frac{\partial^2 f}{\partial y^2} \geq c > 0, \qquad \text{on a neighborhood of } (0,0), \tag{5}$$

is smooth around the origin. Note that the example f_n given above has a vanishing Hessian at the origin.Guan applies the classic partial Legendre transform

$$\left\{ \begin{array}{rcl} s &=& x, \\ t &=& f_x, \end{array} \right. \tag{6}$$

to write the two dimensional (1) as a quasilinear equation in divergence form

$$\partial_s^2 w + \partial_t k\left(s, w\right) \partial_t w = 0, \tag{7}$$

where $w\left(s,t\right) = y$. Then he establishes a hypoellipticity theorem for subelliptic quasilinear equations including (7), and applies it to obtain his result for solutions to the Monge Ampère equation.

Sawyer and Wheeden [25] obtained the first hypoellipticity result for *degenerate* subelliptic Monge Ampère equations. The authors followed Guan's approach through the transformation (6). Under the conditions

$$k\left(x,y\right) > 0, \qquad x \neq 0, \tag{8}$$

$$\left|\partial_y k\left(x,y\right)\right| \leq B\ \left(k\left(x,y\right)\right)^{\frac{3}{2}}, \qquad \text{for some } B > 0, \tag{9}$$

the Moser iteration technique is adapted to obtain an hypoellipticity theorem for weak solutions to the Dirichlet problem (2). The growth condition (9) says that as k vanishes it becomes a function of the x variable only, that is, as k degenerates, equation (7) becomes essentially more linear. In [25] the authors also established a priori estimates and regularity of solutions for the two dimensional degenerate *Gaussian curvature* problem.

The general Partial Legendre Transform and applications

In this section, we describe work in collaboration with E.T. Sawyer and R.L. Wheeden.

The Partial Legendre Transform (6), translates the *fully nonlinear* problem (1) into the *quasilinear* equation (7). Whereas the degeneracy and smoothness of the coefficients are preserved, quasilinearity has apparent advantages over nonlinearity in the study of solutions. The quest for an n-dimensional version of the Partial Legendre Transform might have its origin over half a century ago when D. Alexandrov first introduced this "Legendre-like" transformation in two dimensions. Although the n-dimensional Partial Legendre Transform we use was always available, the technique that allows us to use this transformation to treat n-dimensional Monge Ampère equations was not ready until now.

Given f and k smooth verifying (1), we consider the transformation

$$\begin{cases} s &= x_1, \\ t_i &= f_i, \qquad i = 2, \ldots, n, \end{cases}$$

and the complementary variables

$$\begin{cases} u &= f_1, \\ v_i &= x_i, \qquad i = 2, \ldots, n. \end{cases}$$

Then, the following "Cauchy Riemann" system holds:

$$\begin{aligned} \frac{\partial u}{\partial s} &= k \det\left(\frac{\partial \vec{v}}{\partial \vec{t}}\right), \\ \frac{\partial u}{\partial \vec{t}} &= -\left(\frac{\partial \vec{v}}{\partial s}\right)^{tr}. \end{aligned}$$

Then, the equality of second order partial derivatives $\dfrac{\partial^2 u}{\partial \vec{t} \partial s} = \dfrac{\partial^2 u}{\partial s \partial \vec{t}}$ gives the following system

$$\frac{\partial^2 v_\ell}{\partial s^2} + \frac{\partial}{\partial t_\ell}\left(k \det\left(\frac{\partial \vec{v}}{\partial \vec{t}}\right)\right) = 0, \qquad \ell = 2, \ldots, n. \tag{10}$$

Since $\mathbf{M}\cdot\left(\frac{\partial\vec{v}}{\partial\vec{t}}\right)=\det\left(\frac{\partial\vec{v}}{\partial\vec{t}}\right)\mathbb{I}$, where $\mathbf{M}=\operatorname{co}\left(\frac{\partial\vec{v}}{\partial\vec{t}}\right)$ is the matrix of cofactors of $\left(\frac{\partial\vec{v}}{\partial\vec{t}}\right)$, and $\mathbb{I}$ is the $(n-1)$-identity matrix, (10) can be written as

$$\left\{\frac{\partial^2}{\partial s^2}+\left(\frac{\partial}{\partial\vec{t}}\right)^t k\,\mathbf{M}\frac{\partial}{\partial\vec{t}}\right\}v_\ell=0,\qquad \ell=2,\ldots,n.$$

Note that when $n=2$ we recover equation (7), since in this case $\mathbf{M}=1$.

A remarkable fact about the system (10) is that it can also be written as a *nondivergence form* quasilinear system of the same order. This is due to the fact that since $\mathbf{M}$ is the cofactor matrix of a Jacobian, its columns have $\vec{t}$-divergence zero. We exploit this special feature to adapt techniques from [12], [25] and others to obtain a collection of results:

The n-dimensional subelliptic case. When $n>2$ (10) is fully nonlinear, but the biggest difference with the two dimensional case, is that the system (10) is not strongly elliptic when $n\geq 3$, *even if k is positive.* To overcome these structural obstacles, we differentiate the system once with respect to s and $\vec{t}$, to obtain an $(n-1)\times n$-system for the unknowns $p_\ell^j=\frac{\partial v_\ell}{\partial t_j}$, where $t_1=s$:

$$\left\{\frac{\partial^2}{\partial s^2}+\left(\frac{\partial}{\partial\vec{t}}\right)^t k\,\mathbf{M}\frac{\partial}{\partial\vec{t}}\right\}p_\ell^j=-\left(\frac{\partial}{\partial\vec{t}}\right)^t\left(\frac{\partial}{\partial_{t_j}}k\,\mathbf{M}\right)p_\ell^{2,\ldots,n},\tag{11}$$

where $2\leq\ell\leq n$ and $1\leq j\leq n$. At first glance it seems that the right hand side has *full order*, that is, it involves *second order* derivatives of p. Nevertheless, since $\left(\frac{\partial}{\partial\vec{t}}\right)^t\mathbf{M}=(0,\ldots,0)$, we have that (11) is a *quasilinear* system with right hand side depending only on ∇p, $(\nabla p)^2$ (linearly), and the variables s and $\vec{t}$. Moreover, the system (11) is *strongly elliptic.*

This transformation allowed us to generalize to n dimensions Guan's subelliptic result [12] for Monge Ampère equations. We allow the right hand side k to also depend on derivatives of f, including in this way the equation for prescribed Gaussian curvature. The subelliptic condition (4) takes the form

$$k(x,z,p)\approx\left(x_1^{2\ell}+\psi(x)\right)K(x,z,p),\qquad (x,z,p)\in\Omega\times\mathbb{R}\times\mathbb{R}^n,$$

where ℓ is a positive integer, $\psi\geq 0$, $\psi^{\frac{1}{2\ell}}$ is Lipschitz and $K\approx 1$. The condition (5) is now given by

$$\det\begin{pmatrix}\frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2\partial x_n}\\ \vdots & & \vdots\\ \frac{\partial^2 f}{\partial x_n\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2}\end{pmatrix}\geq c>0.\tag{12}$$

That is, the Hessian matrix of f has full rank on the last $n-1$ variables. Simple examples show that (12) can not be replaced by any rank less than $n-1$.

As a corollary we obtained an interesting geometric application.. If the Gaussian curvature k_n of a convex function f has a nondegenerate critical point, i.e. $k_n\approx|x|^2$, then f is smooth *if and only if* $k_{n-1}\neq 0$, where k_j denotes the j^{th} symmetric function on the principal curvatures of f.

These results are contained in [22], and have been submitted.. A preprint is available at the author's web-site.

The n-dimensional degenerate subelliptic case. Motivated by the results in [25], we considered an n-dimensional generalization for equation (7), that includes degenerate subelliptic cases. We considered the Dirichlet problem

$$\begin{aligned} \operatorname{div} \mathbf{A}(x,w)\nabla w &= f(x,w) + \vec{g}(x,w)\cdot\nabla w, && \text{in } \Omega \\ w &= \varphi, && \text{on } \partial\Omega \end{aligned} \tag{13}$$

where $\Omega \subset \mathbb{R}^n$, is a smooth domain with positively curved boundary, and $\mathbf{A}$ is a smooth symmetric matrix that satisfies

$$\xi^t \mathbf{A}(x,z)\,\xi \approx \sum_{i=1}^{n} k_i(x,z)\,\xi_i^2, \tag{14}$$

where the parameters k_i are nonnegative such that

$$\begin{aligned} k_1 &\approx 1 \\ (k_i)^N &\leq C_* k_j, \quad 2 \leq i,j \leq n. \end{aligned} \tag{15}$$

We proved that under the conditions (14), (15) and an equivalent of (9) for the matrix $\mathbf{A}$, all continuous weak solutions to (13) are C^∞, with C^∞ semi-norms controlled by the boundary data, f, $\vec{g}$, $\mathbf{A}$ and the constants involved in the hypotheses.

Next, we generalize this result to systems that include the following:

$$\begin{aligned} &\left\{\frac{\partial^2}{\partial s^2} + \left(\frac{\partial}{\partial\vec{t}}\right)^t k\,\mathbf{M}\frac{\partial}{\partial\vec{t}}\right\} q_\ell^{i,j} \\ = \; & -\left(\frac{\partial}{\partial\vec{t}}\right)^t \left(\frac{\partial}{t_i}k\,\mathbf{M}\right) q_\ell^{(2,\ldots,n),j} - \left(\frac{\partial}{\partial\vec{t}}\right)^t \left(\frac{\partial}{\partial_{t_j}}k\,\mathbf{M}\right) q_\ell^{i,(2,\ldots,n)} \\ & -\left(\frac{\partial}{\partial\vec{t}}\right)^t \left(\frac{\partial^2}{\partial t_i \partial_{t_j}}k\,\mathbf{M}\right) p_\ell^{2,\ldots,n}, \qquad 1 \leq i,j \leq n,\ 2 \leq \ell \leq n, \end{aligned} \tag{16}$$

which we obtain by differentiating (11) once more. We needed to take this extra derivative because the quasilinear theory we developed for (13) can only handle a right hand side which is at most *linear* on the first derivatives of the unknowns. Since the system (16) satisfies this restriction, we obtained a higher dimensional version of the result in [25], with some extra restrictions though. Indeed, we proved that if f is a $C^{3,1}$ convex solution to the Monge Ampère Dirichlet problem (2), such that f satisfies the rank condition (12), and $k(x)$ satisfies

$$|\partial_{x_i} k(x)| \leq B\,k(x)^{\frac{3}{2}}, \qquad 2 \leq i \leq n, \quad x \in \Omega,$$

then f is smooth in Ω. This is the first known hypoellipticity result for n-dimensional degenerate subelliptic Monge Ampère equations. A preprint [23] is available at the author's web site.

Open problems.

(1) Since we differentiated (10), in dimensions larger than two we need to require solutions to be $C^{2,1}$ in order to apply the quasilinear theory developed in [22]. This leaves an open gap with the $C^{1,1}$ case. The results in [13] show that convex weak solutions to (2) have Lipschitz first order derivatives, in this sense the $C^{1,1}$ case is sharp. On the same vein, to treat the degenerate subelliptic case in [23] we need solutions to be $C^{3,1}$, leaving and even larger gap with the $C^{1,1}$ case. The

"closing" of these gaps, or to the contrary, showing the necessity of them, are very interesting and challenging open problems to face.

(2) The function $f(x,y) = \frac{1}{18}\left(x^2+y^2\right)^{\frac{3}{2}}$ shows that even in the case $\det\left(\nabla^2 f\right) = x^2+y^2$, solutions to the Dirichlet problem (2) fail to be smooth. A fascinating question that is widely open is whether *all*, say, $C^{1,1}$ solutions to $\det\left(\nabla^2 f\right) = x^2+y^2$ fail to be smooth. That is, we ask wether smoothness of solutions is linked to *precise* shape of k. Notice that $g(x,y) = x^2\left(x^2+y^2\right)$ is smooth and satisfies $\det\left(\nabla^2 g\right) \approx x^2+y^2$ on a neighborhood of the origin. Also, we know that if $f_{yy}>0$ and $k(x,y) \approx x^2+y^2$ then f is necessarily smooth.

(3) The classic Monge Ampère theory requires "uniform ellipticity" in two different ways: the *right hand side* k must be positive, bounded and bounded away from zero; and the *domain* must have positive, bounded, and bounded away from zero principal curvatures, we call these domains *elliptic.* So far the literature has only addressed the right hand side k as a source of degeneracy, and nothing was known on the flatness of the domain as a source of singularity of solutions. When the domain is elliptic, then convex weak solutions to the Dirichlet problem (2) are $C^{1,1}$, [13]. In work done in collaboration with E. Sawyer, we have shown that a *necessary* condition for the existence of $C^{1,1}$ solutions to (2) in two dimensions, is that

$$k(P) \leq C\gamma(P), \qquad \text{for all } P \in \partial\Omega,$$

where $\gamma(P)$ is the curvature of $\partial\Omega$ at P, see [21]. The question wether this condition is sufficient is still open.

(4) The classic partial Legendre transform provides a very simple proof of a famous result due to Jörgens [16], if f is a convex C^2 function in $\mathbb{R}^2$ satisfying

$$f_{xx}f_{yy} - f_{xy} = 1,$$

on the whole plane, then u is a *quadratic polynomial.* Indeed, if (s,t) are given by (6) and $z = f_x$, $w = y$,then $z(s,t)$ and $w(s,t)$ satisfy the Cauchy Riemann equations, and therefore the function $g(s+it) = z(s,t) + i\,w(s,t)$, is *holomorphic* on the entire complex plane $\mathbb{C}$. A simple application of Liouville's theorem implies g' vanishes in $\mathbb{C}$, which in turn implies $\nabla^3 f \equiv 0$ in $\mathbb{R}^2$. Several generalizations of Jörgens' result have been obtained in higher dimensions [3], [27] by different methods. Now that a higher order partial Legendre transform is available to treat Monge Ampère equations, it is interesting to explore whether it would render a simpler proof of these type of results.

2. Elliptic Nondivergence Linear Equations

Introduction

On my doctoral research at the University of Minnesota, I studied the Dirichlet problem

$$\begin{aligned} \mathcal{L}u &= 0 \quad \text{in } D \\ u &= f \quad \text{on } \partial\Omega \end{aligned} \tag{17}$$

where Ω is a bounded, Lipschitz domain in $\mathbb{R}^n$, $\mathcal{L}u = a^{i,j}(x)\,\partial_{i,j}u(x)$ (the repeated indices summation convention is used), $\partial_{i,j} = \frac{\partial^2\cdot}{\partial x_i \partial x_j}$ and the matrix $A(x) = (a^{i,j}(x))_{i,j=1}^n$ is real, symmetric, uniformly bounded and verifies the ellipticity condition

$$0 < \lambda\,|\xi|^2 \leq a^{i,j}(x)\,\xi_i\,\xi_j \qquad \text{for all } x \in r,\ \xi \in \mathbb{R}^n \setminus \{0\}. \tag{18}$$

In [19], the authors constructed examples of operators $\mathcal{L}$ as above, with continuous coefficients, and for which solutions to (17) are not unique when the boundary data lies in $L^p(\partial\Omega)$, for any p, $1 \le p \le \infty$. This means that the harmonic measure induced by $\mathcal{L}$ on $\partial\Omega$, is *singular* with respect to *surface measure* σ (c.f. [17]). This arises the question of under which conditions on the coefficients A, we can assure uniqueness of solutions of (17), with proper control depending only on the L^p norm of the boundary data, and the geometry of the domain. This question is related to the existence of a "good definition" of solutions to $\mathcal{L}u = 0$ when the coefficients of $\mathcal{L}$ are not smooth, and it was formulated as an open problem in C. Kenig's book [17] (problem 3.3.9., see also 3.3.5–3.3.8).

Regularity of the Dirichlet Problem with L^p data

The remarkable results in [11] greatly motivated the study of the aforementioned problem in the non-divergence setting. In [11] the authors consider two operators with divergence structure, $\mathcal{M}_k u = \operatorname{div}(A_k \nabla u)$, $k = 0, 1$, and provide *sharp* conditions on the disagreement of the coefficients $\varepsilon(x) = A_1(x) - A_0(x)$, so that if we have "good" estimates for the Dirichlet problem for $\mathcal{M}_0$ then we can also guarantee "good" estimates for the Dirichlet problem for $\mathcal{M}_1$. In this context, "good" means that there exists p, $1 < p < \infty$ such that the L^p-Dirichlet problem is solvable, in a suitable sense, this is referred as "$\mathcal{M}_1$ verifies D_p", (see [17] for details). The specific condition on the disagreement of the coefficients is that

$$\begin{aligned} &d\nu(x) = \tfrac{a^2(x)}{\delta(x)}\,dx \quad \text{is a } \textit{Carleson measure} \text{ with respect to } \sigma \text{ on } \partial\Omega, \\ &\text{i.e. } \sup_{r>0,\, Q\in\partial\Omega} h^2(r,Q) < \infty, \qquad \text{with } h^2(r,Q) = \tfrac{\nu(B_r(Q)\bigcap D)}{\sigma(B_r(Q)\bigcap \partial\Omega)}, \end{aligned} \tag{19}$$

where

$$a(x) = \sup_{B(x)} |\varepsilon(y)|, \qquad B(x) = B_{\frac{\delta(x)}{2}}(x), \quad \delta(x) = \operatorname{dist}(x, \partial\Omega), \tag{20}$$

and $B_r(x)$ denotes the Euclidean ball of radius r entered at x. This result allows us to take any operator $\mathcal{M}_0$ verifying a D_p condition for some p, $1 < p < \infty$, and produce a family of perturbed operators $\mathcal{M}_1$ verifying some $D_{\tilde{p}}$ condition, $1 < \tilde{p} < \infty$. In [20], I established a similar result in the non-divergence setting, with the further assumption of coefficients in the space $\mathrm{VMO}(\mathbb{R}^n)$. Under condition (19), I showed that the harmonic measures of two non-divergence elliptic operators $\mathcal{L}_0$and $\mathcal{L}_1$are simultaneously in the weight class $A_\infty(\sigma)$ on $\partial\Omega$. As shown in [11], this is equivalent to prove that if $\mathcal{L}_0$ verifies a D_p condition for some p, $1 < p < \infty$, then there exists $\tilde{p}$, $1 < \tilde{p} < \infty$, such that $\mathcal{L}_1$ verifies $D_{\tilde{p}}$.

For ν as in (19), but with $A_0$and A_1 now corresponding to the non-divergence operators $\mathcal{L}_0$and $\mathcal{L}_1$, and if moreover

$$\lim_{r\to 0} \sup_{Q\in\partial\Omega} h(r,Q) = 0, \tag{21}$$

then we proved that the operators $\mathcal{L}_0$ and $\mathcal{L}_1$verify simultaneously D_p for *the same* p, $1 < p < \infty$, this was done in [7] for the divergence case.

1. Some related open questions

(1) There are a number of interesting open problems to be considered in this theory, even in the case of continuous coefficients. In [14], it is shown that in a star-like domain divergence operators with coefficients independent of the radial variable verify D_2. In [9] a related result is proved for divergence operators with uniformly continuous coefficients which verify a "square Dini" condition in a "radial

direction" near the boundary. I verified that if the coefficients of a non-divergence elliptic operator $\mathcal{L}$ are as in [9] then the same result holds, namely, D_2 holds for $\mathcal{L}$. The natural question is whether the radially independent case holds in the non-divergence setting. This is a challenging open problem (see problem 3.3.8. [17]).

(2) In view of the previous theory for divergence equations, it is interesting to ask whether we can remove the VMO assumption on the coefficients. This is a difficult question mainly because in the general case there is no regularity theory for the Dirichlet's problem (17). The most crucial interior estimate in our proofs is a weighted "Cacciopoli inequality" for the second derivatives due to L.Escauriaza and C.Kenig [8], this inequality is easily derived from the interior regularity theorem when the coefficients are in VMO [6]. Such inequalities might hold without the VMO assumption in an appropriate weighted sense, we thank C. Kenig for his suggestions on this topic. This problem is interesting in itself, and any progress in this matter would open the extent of the results obtained in [20] and would have many potential applications.

(3) In [11] a new characterization of the weight classes is given, and then used to construct examples which show the sharpness of the results there. The question whether the results obtained in the non-divergence case are sharp and in which sense remains an open problem. The counterexamples furnished in [19] constructed through a compactness approximation argument do not provide estimates for the *square Dini* module. Further analysis of these techniques in the frame of operators with VMO coefficients (or *even* continuous coefficients) could provide hints on the question of sharpness of our result (this question is also listed in [17] 3.3.7). M. Safonov suggested that the celebrated construction by N.S. Naridashvili (see [24]) can be refined to show the necessity of some of our hypothesis.

(4) A very interesting direction of study is the case of operators with a non-trivial drift term. In a recent work my C. Kenig and J. Pipher [18] the authors extend the results in [11] to the case of elliptic operators in divergence form with a singular drift. This suggests that positive results might also hold in the nondivergence case with a singular drift term (c.f.[2] for some partial results).

(5) Other natural directions of study are the corresponding problems in the parabolic (non-divergence) situation [10] and the extension of this theory to non-tangentially accessible domains [15].

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