

# Graphs that Admit Right Angle Crossing Drawings

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## Abstract

We consider *right angle crossing (RAC) drawings* of graphs in which the edges are represented by polygonal arcs and any two edges can cross only at a right angle. We show that if a graph with  $n$  vertices admits a RAC drawing with at most 1 bend or 2 bends per edge, then the number of edges is at most  $6.5n$  and  $74.2n$ , respectively. This is a strengthening of a recent result of Didimo *et al.*

## 1 Introduction

The core problem in graph drawing is finding good and easily readable drawings of graphs. Recent cognitive experiments [10, 11] show that poly-line graph drawings with orthogonal crossings and a small number of bends per edge are just as readable as planar drawings. Motivated by these findings, Didimo *et al.* [7] studied the class of graphs which have a polyline drawing where crossing edges meet at a right angle. Such a drawing is called a *right angle crossing* drawing, or *RAC drawing*, for short.

The interior vertices of a polygonal arc are called *bends*. We say that a planar representation of a graph is an *RAC<sub>b</sub> drawing*, for some  $b \in \mathbb{N}_0$ , if the vertices are drawn as points, the edges are drawn as polygonal arcs with at most  $b$  bends joining the corresponding vertices, and any two polygonal arcs cross at a right angle (and not at a bend). Let  $R_b$ ,  $b \in \mathbb{N}_0$ , be the class of graphs that admit a *RAC<sub>b</sub> drawing*. It is clear that  $R_b \subseteq R_{b+1}$  for all  $b \in \mathbb{N}_0$ . Didimo *et al.* [7] showed that every graph is in  $R_3$ , hence  $R_3 = R_b$  for all  $b \geq 3$ . They proved that every graph with  $n \geq 4$  vertices in  $R_0$  has at most  $4n - 10$  edges, and this bound is best possible. They also showed that a graph with  $n$  vertices in the classes  $R_1$  and  $R_2$  has at most  $O(n^{4/3})$  and  $O(n^{7/4})$  edges, respectively.

**Results.** We significantly strengthen the above results, and show that every graph with  $n$  vertices in  $R_1$  and  $R_2$  has at most  $O(n)$  edges, and that the classes  $R_0$ ,  $R_1$  and  $R_2$  are pairwise distinct.

**Theorem 1.** *A graph  $G$  with  $n$  vertices that admits a  $RAC_1$  drawing has at most  $6.5n - 13$  edges.*

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**Theorem 2.** *A graph  $G$  with  $n$  vertices that admits a  $RAC_2$  drawing has less than  $74.2n$  edges.*

We use two quite different methods to prove our main results. In Section 2, we use the so-called discharging method to prove Theorem 1. In Section 3, we define *block graphs* on the crossing edges, and use the Crossing Lemma to prove Theorem 2. Each method gives a linear bound for the number of edges for graphs in both  $R_1$  and  $R_2$ , however, they would each give weaker constant coefficients for the other case (i.e., for  $R_2$  and  $R_1$ , respectively).

We complement our upper bounds with lower bound constructions in Section 4. We construct graphs with  $n$  vertices in the classes  $R_1$  and  $R_2$  with  $4.5n - O(\sqrt{n})$  and  $7.83n - O(\sqrt{n})$  edges, respectively. Combined with Theorems 1 and 2, they show that  $R_0 \neq R_1$  and  $R_1 \neq R_2$ .

**Related Work.** Angelini *et al.* [4] proved that every graph of maximum degree 3 admits a  $RAC_1$  drawing, and every graph of maximum degree 6 admits a  $RAC_2$  drawing. They also show that some planar directed graphs do not admit straight line *upward* RAC drawings.

A natural generalization of RAC drawings with straight line edges is given by Dujmović *et al.* [8]. They define  $\alpha$ -*angle crossing* ( $\alpha AC$ ) drawings to be straight line graph drawings where every pair of crossing edges intersect at an angle at least  $\alpha$ . In line with the results by Didimo *et al.* [7] on RAC drawings, they prove upper bounds on the number of edges for  $\alpha AC$  graphs and give lower bound constructions. Specifically, they prove that the number of edges in an  $\alpha AC$  graph is at most  $(\pi/\alpha)(3n - 6)$  for  $0 < \alpha < \pi/2$  and at most  $6n - 12$  for  $2\pi/5 < \alpha < \pi/2$ . In addition, they give lower bound constructions based on the square and hexagonal lattices for  $\alpha = \pi/k$ ,  $k = 2, 3, 4, 6$ . Di Giacomo *et al.* [6] also generalize RAC drawings in this way and call the minimum angle of any crossing the *crossing resolution*.

**Preliminaries.** The *crossing number* of a graph  $G$ , denoted  $cr(G)$ , is the minimum number of edge crossings in a drawing of  $G$  in the plane. The Crossing Lemma, due to Ajtai *et al.* [3] and Leighton [12], establishes a lower bound for  $cr(G)$  in terms of the number of vertices and edges. The strongest known version is due to Pach *et al.* [13].

**Lemma 1.** [13] *Let  $G$  be a graph with  $n$  vertices and  $m$  edges. If  $m \geq \frac{103}{6}n \approx 17.167n$ , then*

$$cr(G) \geq c \cdot \frac{m^3}{n^2}, \text{ where } c = \frac{1024}{31827} \approx 0.032. \quad (1)$$

Let  $G$  be a graph with  $n$  vertices and  $m$  edges, and let  $D$  be a  $RAC_b$  drawing of  $G$ . If there is no confusion, we make no distinction between the vertices (edges) of  $G$  and the corresponding points (polylines) of  $D$ .

A *plane (multi-)graph* is a (multi-)graph drawn in the plane without any edge crossings. The *faces* of a plane (multi-)graph are the connected components of the complement of  $G$ . Let  $D$  be a drawing of a graph  $G = (V, E)$ . A *rotation system* at a vertex  $v \in V$  in drawing  $D$  is the (clockwise) circular order in which the edges leave  $v$ . A *wedge* at a vertex  $v$  in  $D$  is an ordered pair of edges  $(e, e')$  incident to  $v$  that are consecutive in its rotation system. A face  $f$  in  $D$  is *adjacent* to a wedge  $(e, e')$  if  $e, v$ , and  $e'$  are consecutive in a counterclockwise traversal of the boundary of  $f$ . Every wedge is adjacent to a unique face in  $D$ . The *size* of a face is the number of edges (counted with multiplicity) on the boundary of  $f$ .

## 2 RAC Drawings with One Bend per Edge

**Discharging.** We apply a discharging method reminiscent to that of Ackerman and Tardos [2] to prove Theorem 1. This method was apparently introduced by Wernicke [14], but it gained considerable attention only after it was extensively used in the first valid proof of the famous Four Color Theorem [5]. Since then, it was instrumental in deriving various types of results in structural graph theory, see e.g. [9]. Dujmović *et al.* [8] applied the discharging method for an alternative proof for the upper bound of  $4n - 10$  on the number of edges in a graph on  $n$  vertices that admits a straight line RAC (i.e. RAC<sub>0</sub>) drawing, originally due to Didimo *et al.* [7].

*Proof of Theorem 1.* Let  $G = (V, E)$  be a graph in  $R_1$ . Fix a RAC<sub>1</sub> drawing  $D$  of  $G$  that minimizes the number of edge crossings. Partition  $G$  into two subgraphs  $G_0 = (V, E_0)$  and  $G_1 = (V, E_1)$ , where  $E_0 \subseteq E$  is the subset of crossing free edges and  $E_1 \subseteq E$  is the subset of edges with at least one crossing. Since  $G_0$  is planar, it has at most  $|E_0| \leq 3n - 6$  edges.

Let  $C$  be the set of crossing points in  $D$ . We construct a plane multigraph  $G' = (V', E')$  as follows: the vertices  $V' = V \cup C$  are the vertices in  $V$  and all crossings in  $C$ ; the edges are polygonal arcs between two consecutive vertices along the edges in  $E_1$ . That is, the edges in  $E'$  are obtained by subdividing the edges in  $E_1$  at crossing points. Since the bends of edges in  $E_1$  are not vertices in  $G'$ , they are bends of some edges in  $E'$ . Denote by  $F'$  the set of faces of  $G'$ . A bend of an edge determines two angles: a *convex* and a *reflex* angle. We say that face  $f \in F'$  is *adjacent* to a convex (resp. reflex) bend, if it has a convex (resp. reflex) interior angle at a bend point. A bounded face of size two is called a *lens*, and is adjacent to two parallel edges. A bounded face of size 3 is called a *triangle*.

**Lemma 2.** *Every lens  $f \in F'$  is adjacent to a convex bend. If it is adjacent to exactly one convex bend, then it is incident to one vertex in  $C$  and  $V$  each, and adjacent to one convex bend and one reflex bend.*

*Proof.* Every lens  $f \in F'$  is drawn as a simple polygon whose vertices are the incident vertices in  $V'$  and adjacent bends. Every simple polygon has at least 3 convex interior angles. A lens is incident to exactly two vertices in  $V'$ , so it must have a convex interior angle at an adjacent bend.

Let  $f \in F'$  be a lens adjacent to exactly one convex bend. Since every edge in  $E_1$  crosses some other edges, no two adjacent vertices in  $V'$  are in  $V$ . At each vertex in  $C$ , the incident faces have  $90^\circ$  interior angles since  $D$  is a RAC drawing. If both vertices of lens  $f$  are in  $C$  with  $90^\circ$  interior angles, then  $f$  must have two convex bends. So,  $f$  is incident to one vertex in  $C$  and  $V$  each. If  $f$  has only one bend (see Fig. 1(a)), then we can redraw the edge  $e \in E$  containing this bend in  $D$  with one fewer crossings (eliminating the crossing incident to  $f$ ), which contradicts the choice of the RAC<sub>1</sub> drawing  $D$ . So  $f$  must be adjacent to a reflex bend (see Fig. 1(b)) as well.  $\square$

**Lemma 3.** *Every triangle  $f \in F'$ , which is not the outerface, is adjacent to a convex bend.*

*Proof.* A triangle  $f \in F'$  has three vertices in  $V' = V \cup C$ , and each of its three edges is a polygonal arc with 0 or 1 bends. Since every edge in  $E_1$  crosses some other edges, no two adjacent vertices in  $V'$  are in  $V$ . That is, at least two vertices of  $f$  are in  $C$ , with an inner angle of  $90^\circ$ . If  $f$  is adjacent to  $k \in \{0, 1, 2, 3\}$  bends (at most one bend per edge), then  $f$  is a simple polygon with  $k + 3$  vertices, and so the sum of its interior angles is  $(k + 1)180^\circ$ . If all  $k$  bends are reflex, then the sum of interior angles would be more than  $90^\circ + 90^\circ + k \cdot 180^\circ = (k + 1)180^\circ$ .  $\square$

**Lemma 4.** We have  $|E_1| \leq 4n - 8$ .

*Proof.* Assume without loss of generality that  $G' = (V', E')$  is connected. For a face  $f \in F'$ , let  $s_f$  be the size of  $f$ . For a vertex  $v \in V' = V \cup C$ , let  $d_v$  denote the degree of  $v$  in  $G'$ . We put a charge  $\text{ch}(v) = d_v - 4$  on each vertex  $v \in V'$ , and a charge  $\text{ch}(f) = s_f - 4$  on each face  $f \in F'$ . By Euler's formula the sum of all charges is

$$\sum_{v \in V'} \text{ch}(v) + \sum_{f \in F'} \text{ch}(f) = -8. \quad (2)$$

Indeed,  $\sum_{v \in V'} (d_v - 4) + \sum_{f \in F'} (s_f - 4) = 2|E'| - 4|V'| + 2|E'| - 4|F'| = -8$ .

Since the charge at a vertex  $v \in C$  is 0, we have

$$\sum_{v \in V} \text{ch}(v) + \sum_{f \in F'} \text{ch}(f) = -8. \quad (3)$$

In what follows, we redistribute the charges in  $G'$  such that the total charge of all vertices and faces remains the same. The redistribution is done in two steps. In step 1, we move charges from some vertices to some faces; and in step 2 we move charges from some faces to some other faces. Our goal is to ensure that all faces have non-negative charges after the second step.

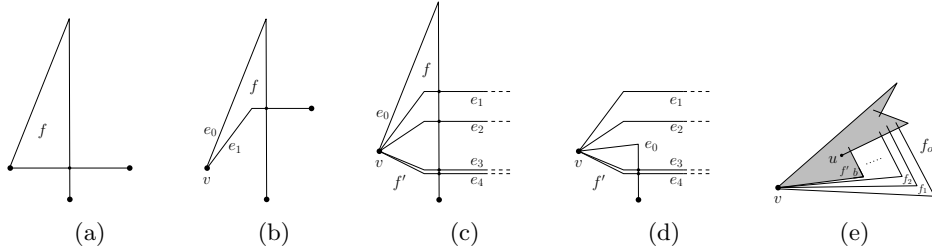


Figure 1: (a) lens  $f$  that can be redrawn, (b) lens  $f$  having only one convex bend on its boundary, (c) situation when  $G$  could be redrawn with fewer crossings, (d) its redrawing ( $i=4$ ), and (e) triangular outerface

**Step 1.** For every edge  $e \in E_1$  with one bend, we discharge  $\frac{1}{2}$  unit from each of the two endpoints of  $e$  to the face adjacent to the convex bend of  $e$ . The new charge at every vertex  $v \in V$  is  $\text{ch}'(v) \geq \frac{1}{2}d_v - 4$ . Since every face in  $F'$  of size at least 4 receives a non-negative charge already at the beginning, it is enough to take care of the triangles and lenses (bounded faces of size 3 and 2), whose initial charge was  $-1$  and  $-2$ , respectively.

By Lemma 3, each triangle  $f \in F'$  except the outerface is adjacent to a convex bend, and so its charge has increased by at least 1 in step 1. Its new charge  $\text{ch}'(f)$  is at least 0. Similarly, if a lens  $f \in F'$  is adjacent to two convex bends, then its charge after step 1 is 0. Hence, the only possible faces whose new charge is still negative are the outerface and the lenses adjacent to exactly one convex bend.

**Step 2.** In order to increase the charge of the outerface and lenses with exactly one convex bend from  $-1$  to 0, we perform the second discharge step. Note that in the first step we have increased the charge of some faces of size 4 or higher (which was unnecessary), so we can now divert the “wasted” charge to faces with negative charge.

Let  $f$  be a lens with exactly one convex bend. By Lemma 2,  $f$  is incident to one vertex  $v \in V$  and one in  $C$ , and it is adjacent to one convex bend and one reflex bend. Let  $e_0, e_1, \dots, e_{d_v-1}$  (see Fig 1(c)) denote the edges in  $E_1$  incident to  $v$  listed according to the rotation system at  $v$  (clockwise) such that the wedge  $(e_0, e_1)$  is adjacent to face  $f$ . We may assume without loss of generality that  $e_0$  has a convex bend and  $e_1$  has a reflex bend adjacent to  $f$ . Let  $i \in \mathbb{N}$  be the smallest integer such that the wedge  $(e_i, e_{i+1})$  is adjacent to the face, let us denote it by  $f'$ , of size at least 4. It is easy to see that  $i$  is well-defined, since every edge in  $E_1$  participates in a crossing.

We show that  $f'$  is adjacent to the convex bend of edge  $e_i$ . Any wedge  $(e_j, e_{j+1})$ ,  $1 \leq j \leq i-1$ , must be adjacent to a triangle bounded by parts of the edges  $e_j, e_{j+1}$ , and  $e_0$ . Since the (convex) bend of  $e_0$  is adjacent to  $f$ , all these triangles are adjacent to a straight line portion of  $e_0$ . If any of these triangles is adjacent to the convex bend of  $e_j$  and a convex bend or no bend of  $e_{j+1}$ , then we can redraw edge  $e_0$  to obtain a RAC<sub>1</sub> drawing of  $G$  with fewer crossings, eliminating the crossing incident to  $f$  (Figs. 1(c) and 1(d)). So the triangle at any wedge  $(e_j, e_{j+1})$ ,  $1 \leq j \leq i-1$ , is adjacent to the reflex bend of  $e_{j+1}$ . Hence  $f$  is adjacent to the convex bend of  $e_i$ .

Move 1 unit of charge (corresponding to the convex bend of  $e_i$ ) from  $f'$  to  $f$ . This increases the charge of  $f$  to 0. Since the size of  $f'$  is at least 4, its charge remains non-negative. It is also clear that the charge corresponding to the convex bend of  $e_i$  is diverted to exactly one lens from  $f'$ .

It remains to make sure that the outerface  $f_o$  gets non-negative charge in the end as well. If  $f_o$  has a negative charge after Step 2, then it is triangle. It must have exactly one vertex  $v$  from  $V$ , otherwise three of its vertices are crossings each contributing  $\frac{3}{2}\pi$  to the sum of the inner angles of the polygon which is the complement of the interior of  $f_o$ . Thus,  $f_o$  would have at least four bends, which is impossible. Moreover,  $f_o$  must be adjacent to three reflex bends, i.e. it looks like  $f_o$  on Figure 1(e). Then at least one of the inner faces adjacent to bends of  $f_o$  on edges incident to  $v$  is not a lens. Let  $f_1$  denote such a face. We inductively define  $f_{i+1}$  for  $i > 1$ : If  $f_i$  is not a triangle  $f_{i+1} = f_i$ . If  $f_i$  is a triangle we define  $f_{i+1}$  as follows. Let  $f_{i+1}$  denote the face on the opposite side of the reflex bend of  $f_i$ . The definition of  $f_{i+1}$  is correct, since the sum of the interior angles in the grey polygon in Figure 1(e) is  $4\pi$ . Eventually some  $f_i = f'$  has at least four vertices and one unit of charge of the bend between  $f_i$  and  $f_{i-1}$  can be diverted to the outerface. The charge at  $b$  has not been moved in Step 2.

After the second step of redistribution, every face in  $D'$  has a non-negative charge. Let  $\text{ch}''(v)$  and  $\text{ch}''(f)$  denote the charge at each vertex  $v \in V$  and  $f \in F'$  after step 2. We have

$$|E_1| - 4n = \sum_{v \in V} \left( \frac{1}{2}d_v - 4 \right) \leq \sum_{v \in V} \text{ch}''(v) \leq \sum_{v \in V} \text{ch}''(v) + \underbrace{\sum_{f \in F'} \text{ch}''(f)}_{\geq 0} = -8.$$

By reordering the terms in the above inequality, we have  $|E_1| \leq 4n - 8$ , as required □

At this point we have already proved that the number of edges in  $G$  is no more than  $|E_0| + |E_1| \leq (3n - 6) + (4n - 8) = 7n - 14$ .

We can improve this bound by applying Lemma 4 independently in each face of the plane graph  $G_0 = (V, E_0)$ , whose edges are the crossing-free edges in  $E$ . Notice that each edge in  $E_1$  is fully contained in exactly one face of  $G_0$ . Let  $F_0$  be the set of faces of  $G_0$ , and let  $d_f$  denote the number of vertices of a face  $f \in F_0$ . By Lemma 4, each face  $f \in F_0$  contains at most  $4d_f - 8$  edges of  $E_1$ , and it obviously contains no edges of  $E_1$  if  $f$  is a triangle (i.e.,  $d_f = 3$ ). Summing this upper bound

over all faces of  $G_0$ , we have

$$|E_1| \leq \sum_{f \in F_0, d_f > 3} (4d_f - 8). \quad (4)$$

**Lemma 5.** *If a plane graph  $G_0 = (V, E_0)$  has  $n$  vertices and  $3n - 6 - k$  edges, then*

$$\sum_{f \in F_0, d_f > 3} (4d_f - 8) \leq 8k. \quad (5)$$

*Proof.* Denote by  $\tau(G_0)$  the sum on the left hand side of (5). We proceed by induction on  $k$ . For  $k = 0$ , the plane graph  $G_0$  is a triangulation and  $\tau(G_0) = 0$ .

Assuming that the lemma holds for  $k \geq 0$ , we show that it holds for  $k' = k + 1$ . Let  $G_0$  be a plane graph with  $n$  vertices and  $3n - 6 - k'$  edges.  $G'_0$  can be obtained by removing an edge  $e$  from a plane graph  $G_0$  with  $3n - 6 - k$  edges, for which  $\tau(G_0) \leq 8k$  by induction. If edge  $e$  is a bridge, then we have  $\tau(G'_0) = \tau(G_0) \leq 8k < 8k'$ . Otherwise the removal of  $e$  merges two adjacent faces of  $G_0$ , say  $f_1$  and  $f_2$ . If none of  $f_1$  and  $f_2$  is a triangle, then  $4d_f - 8 = 4(d_{f_1} + d_{f_2} - 2) - 8 = (4d_{f_1} - 8) + (4d_{f_2} - 8)$ , and so  $\tau(G'_0) = \tau(G_0) \leq 8k < 8k'$ . If  $f_1$  is a triangle and  $f_2$  is a face of size more than three, then  $4d_f - 8 = (4(d_{f_2} + 1) - 8) + 4$ , and so  $\tau(G'_0) \leq \tau(G_0) + 4 \leq 8k + 4 < 8k'$ . If both  $f_1$  and  $f_2$  are triangles, then  $4d_f - 8 = 4 \cdot 4 - 8 = 8$ , and  $\tau(G'_0) \leq \tau(G_0) + 8 \leq 8k + 8 = 8k'$ . This completes the induction step, hence the proof of Lemma 5.  $\square$

We have two upper bounds for  $m$ , the number of edges in  $G$ . Lemma 4 gives  $m \leq |E_0| + |E_1| \leq (3n - 6 - k) + (4n - 8) = 7n - k - 14$ , and Lemma 5 gives  $m \leq |E_0| + |E_1| \leq (3n - 6 - k) + 8k = 3n + 7k - 6$ . Therefore, we have  $m \leq \max_{k \in \mathbb{N}_0} \min(7n - k - 14, 3n + 7k - 6) = 6.5n - 13$ , which is attained for  $k = n/2 - 1$ . This completes the proof of Theorem 1.  $\square$

### 3 RAC Drawings with Two Bends per Edge

**Block graphs.** The main tool in the proof of Theorem 2 is the block graph of a RAC drawing and the Crossing Lemma. Note that the block graph we define here is different than block graphs defined in the context of 2-connectivity. Let  $D$  be a  $\text{RAC}_2$  drawing of a graph  $G = (V, E)$ . Every edge is a polygonal arc that consists of line segments. Without loss of generality, we assume that every edge has two bends so that each edge has two *end segments* and one *middle segment*. A *block* of  $D$  is a connected component in the union of pairwise parallel or orthogonal segments in  $D$ . Formally, we define a binary relation on the segments in the polygonal arcs in the drawing  $D$ : two segments are related if and only if they cross. The transitive closure of this relation is an equivalence relation. We define a block of  $D$  as the union of all segments in an equivalence class. Since the union of crossing edges is connected, every block is a connected set in the plane. Furthermore, all segments in a block have at most two different (and orthogonal) orientations.

By Lemma 1, if  $m \geq \frac{103}{6}n$ , then the average number of crossings per segment is at least

$$\frac{2c}{3} \cdot \frac{m^2}{n^2},$$

where  $c = 1024/31827 \approx 0.032$ . We say a segment is *heavy* if it crosses at least  $\beta c \frac{m^2}{n^2}$  other segments, where  $0 < \beta < 2/3$  is the *heaviness parameter* specified later. A block is *heavy* if it contains a heavy segment.

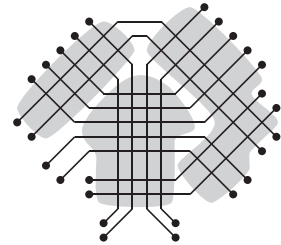


Figure 2: A  $\text{RAC}_2$  drawing of a graph and its heavy blocks

We define the *block graph*  $B(D)$  as a bipartite multi-graph whose two vertex classes are the vertices in  $V$  and the heavy blocks in  $D$ . The block graph has an edge between a vertex  $v \in V$  and a heavy block for every segment incident to  $v$  and contained in the heavy block (Fig. 2). Note that if a heavy block consists entirely of middle segments, it is not adjacent to any vertex in  $B(D)$ .

**Lemma 6.** *If  $D$  is a RAC drawing of a graph, then the block graph  $B(D)$  is planar.*

*Proof.* Recall that a heavy block  $u$  is a connected set which is incident to all vertices of  $G$  that are adjacent to  $u$  in  $B(D)$ . For every heavy block  $u$ , replace each crossing with a dummy vertex, and let  $T_u \subseteq u$  be a spanning tree of the vertices of  $G$  incident to heavy block  $u$ . We can construct a planar embedding of  $B(D)$ . The vertices of  $G$  are represented by the same point as in  $D$ . Each heavy block  $u$  is represented by an arbitrary point  $r_u$  in the relative interior of  $T_u$ . If vertex  $v$  of  $G$  is adjacent to a heavy block  $u$ , then connect  $v$  and  $r_u$  by a Jordan arc that closely follows the shortest path between  $v$  and  $r_u$  in the tree  $T_u \subseteq u$ . Since shortest paths in a tree do not cross, we can draw the edges successively without crossings.  $\square$

Denote by  $H$  the number of heavy blocks in  $D$ . The block graph  $B(D)$  is bipartite and planar, with  $H + n$  vertices. If it is *simple*, then it has at most  $2(H + n) - 4$  edges. However,  $B(D)$  is not necessarily simple: up to four segments of a heavy block may be incident to a vertex  $v$  in  $D$ .

**Lemma 7.** *The block graph  $B(D)$  has less than  $2H + 5n$  edges.*

*Proof.* Assume that two segments in a heavy block are incident to the same vertex  $v$ . Since the block is connected, there is a closed curve  $\gamma$  passing through  $v$  and the two segments such that all other blocks lie either in the interior or in the exterior of  $\gamma$ . Hence, multiple edges cannot interleave in the rotation order of a vertex  $v$ . Note also that segments in a block are pairwise parallel or orthogonal. It follows that  $B(D)$  becomes a *simple* bipartite plane graph after removing at most 3 duplicate edges at each vertex of  $D$ . That is, after removing up to  $3n$  edges, the remaining simple bipartite plane graph has at most  $2(H + n) - 4$  edges.  $\square$

Let  $S$  denote the number of segments that participate in some heavy block of  $D$ . Every heavy block contains at least one heavy segment and all other segments it crosses. That is, a heavy block contains more than  $\beta cm^2/n^2$  segments. Since every segment belongs to a unique block, we have

$$H < \frac{S}{\beta cm^2/n^2} = \frac{Sn^2}{\beta cm^2}. \quad (6)$$

The following lemma reformulates the Crossing Lemma for heavy segments in  $\text{RAC}_2$  drawings. We show that if a graph  $G$  has sufficiently many edges, then a constant fraction of edges must have a segment in some heavy block in a  $\text{RAC}_2$  drawing of  $G$ .

**Lemma 8.** *Let  $D$  be a  $\text{RAC}_2$  drawing of graph  $G$  with  $m \geq \frac{103}{6} \sqrt{2/(3\beta)}n$  edges. If one can delete  $xm$  edges from  $D$ , for some  $0 < x < 1$ , such that every remaining edge segment crosses less than  $\beta cm^2/n^2$  others, then  $x > 1 - \sqrt{3\beta/2}$ .*

*Proof.* Suppose  $xm$  edges were deleted from  $D$  to obtain  $D'$ , a drawing such that every edge segment crosses less than  $\beta cm^2/n^2$  other segments. Let  $G'$  be the graph associated with  $D'$ . The number of

remaining edges is  $|E(G')| = m - xm = (1 - x)m$ . If  $(1 - x)m \geq \frac{103}{6}n$ , then the Crossing Lemma gives  $\text{cr}(G') \geq c \cdot \frac{(1-x)^3 m^3}{n^2}$ , so the average number of crossings per segment in  $G'$  is at least

$$\frac{2\text{cr}(G')}{3(1-x)m} \geq \frac{2c}{3} \cdot \frac{(1-x)^2 m^2}{n^2}.$$

Every segment in  $D'$  crosses less than  $\beta cm^2/n^2$  others in  $D'$ . Comparing the upper and lower bounds for the average number of crossings per segment, we have

$$\frac{2c}{3} \cdot \frac{(1-x)^2 m^2}{n^2} < \beta c \frac{m^2}{n^2} \Rightarrow (1-x)^2 < 3\beta/2 \Rightarrow 1 < x + \sqrt{3\beta/2}.$$

If, however,  $(1-x)m < \frac{103}{6}n$  but  $m \geq \frac{103}{6}\sqrt{2/(3\beta)}n$ , then we have again  $x > 1 - \sqrt{3\beta/2}$ .  $\square$

Lemma 8 immediately gives a lower bound on  $S$ , the number of segments participating in heavy blocks.

**Lemma 9.** *Let  $D$  be a  $\text{RAC}_2$  drawing of graph  $G$ . If  $m \geq \frac{103}{6}\sqrt{2/(3\beta)}n$ , then  $S > (1 - \sqrt{3\beta/2})m$ .*

*Proof.* Let  $E_1$  be the set of edges containing a segment that participate in some heavy block in  $D$ . Clearly, we have  $|E_1| \leq S$ . If all edges of  $E_1$  are deleted from  $D$ , then every remaining segment crosses less than  $\beta cm^2/n^2$  others. By Lemma 8, we have  $S \geq |E_1| > (1 - \sqrt{3\beta/2})m$ .  $\square$

*Proof of Theorem 2.* We set the heaviness parameter to  $\beta = 0.062$ . If  $m \geq \frac{103}{6}\sqrt{2/(3\beta)}n > 56n$ , then we can use Lemmas 8 and 9, otherwise  $m \leq 56n$  and our proof is complete. Let  $D$  be a  $\text{RAC}_2$  drawing of  $G$ . Recall that every edge has two *end segments* and one *middle segment*. Let  $\alpha S$  be the number of end segments that participate in heavy blocks, where  $0 < \alpha < 1$ . The number of middle segments is  $m$ , which is a trivial upper bound on the middle segments that participate in heavy blocks. So the total number of segments in heavy blocks is at most  $S \leq m + \alpha S$ , which gives  $S \leq \frac{1}{1-\alpha}m$ .

In each heavy block, the segments can be partitioned into two sets of pairwise parallel segments. If we delete all edges that contain some segment in the smaller set of each heavy block, then the remaining segments are not heavy anymore. That is, by deleting at most  $\frac{S}{2} \leq \frac{1}{2(1-\alpha)}m$  edges, we obtain a  $\text{RAC}_2$  drawing with no heavy edge segment. By Lemma 8, we have

$$\frac{1}{2(1-\alpha)} > 1 - \sqrt{\frac{3\beta}{2}} \Rightarrow \frac{1}{1 - \sqrt{3\beta/2}} > 2(1-\alpha),$$

which implies

$$\alpha > 1 - \frac{1}{2(1 - \sqrt{3\beta/2})} \tag{7}$$

The block graph  $B(D)$  has  $\alpha S$  edges, since an edge in  $B(D)$  exists if and only if a vertex of  $G$  is incident to an end segment in a heavy block. From Lemma 7, we have an upper bound on the number of edges in  $B(D)$ , which gives  $\alpha S < 2H + 5n$ . Using  $S > (1 - \sqrt{3\beta/2})m$  from Lemma 9,

the upper bound on  $H$  from (6), and the lower bound on  $\alpha$  from (7), we obtain

$$\begin{aligned}
\alpha S < 2H + 5n &\Rightarrow \alpha S < \frac{2S}{\beta c} \cdot \frac{n^2}{m^2} + 5n \\
&\Rightarrow \left( \alpha - \frac{2}{\beta c} \cdot \frac{n^2}{m^2} \right) \left( 1 - \sqrt{\frac{3\beta}{2}} \right) m < 5n \\
&\Rightarrow 0 < \frac{2 - 2\sqrt{3\beta/2}}{\beta c} \cdot \left( \frac{n}{m} \right)^2 + 5 \cdot \left( \frac{n}{m} \right) - \alpha \left( 1 - \sqrt{\frac{3\beta}{2}} \right) \\
&\Rightarrow 0 < \frac{2 - \sqrt{6\beta}}{\beta c} \cdot \left( \frac{n}{m} \right)^2 + 5 \cdot \left( \frac{n}{m} \right) - \left( \frac{1}{2} - \sqrt{\frac{3\beta}{2}} \right).
\end{aligned}$$

This is a quadratic inequality in  $n/m$ . Since  $\sqrt{3\beta/2} < 1/2$ , the constant term is negative, and the two roots have opposite signs. Therefore, we have

$$\frac{n}{m} > \frac{\beta c}{2(2 - \sqrt{6\beta})} \left( -5 + \sqrt{25 + \frac{4}{\beta c} (2 - \sqrt{6\beta}) \left( \frac{1}{2} - \sqrt{\frac{3\beta}{2}} \right)} \right).$$

This is maximized for  $\beta = 0.062$ , and gives  $m < 74.2n$ . □

## 4 Lower Bound Constructions

We complement the upper bounds in Theorems 1 and 2 with lower bound constructions. We construct an infinite family of graphs which admit  $\text{RAC}_1$  drawings and  $4.5n - O(\sqrt{n})$  edges. This shows that  $R_0 \neq R_1$  since every graph in  $R_0$  has at most  $4n - 10$  edges [7]. Let the vertices of  $G$  be points of the hexagonal lattice clipped in a square (Fig. 3). The edges of  $G$  are the hexagon edges and 6 diagonals with a bend in each hexagon. The diagonals connect every other vertex in the hexagon, and make a  $75^\circ$  angle with the side of the hexagon, and so they cross in right angles. The vertex degree is  $3 + 3 \cdot 2 = 9$  for all but at most  $O(\sqrt{n})$  lattice points around the bounding box. Hence the number of edges is  $4.5n - O(\sqrt{n})$ .

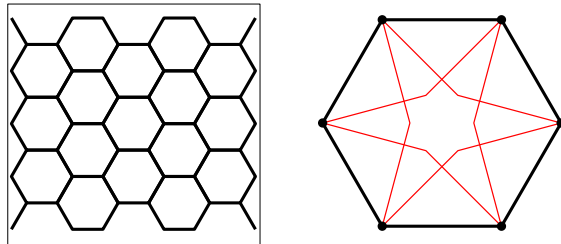


Figure 3: Lower bound construction for a  $\text{RAC}_1$  drawing in a hexagonal lattice

We also construct an infinite family of graphs which admit  $\text{RAC}_2$  drawings and  $7.8\dot{3}n - O(\sqrt{n})$  edges. This shows that  $R_1 \neq R_2$  since every graph in  $R_1$  has at most  $6.5n - 13$  edges by Theorem 1. Let the vertices of  $G$  be the vertices of an Archimedean tiling (12,12,3) clipped in a square. Refer to Fig. 4. In the tiling (12,12,3), we can assign two triangles to each 12-gon. The edges of  $G$  are

the edges of the tiling, a 6-regular graph of diagonals in each 12-gon, and two edges per 12-gon that go to vertices of the two adjacent triangles. The tiling and the diagonals of the 12-gons generate a vertex degree of  $3 + 2 \cdot 6 = 15$  at all but at most  $O(\sqrt{n})$  vertices (due to the boundary effect). The additional two edges between adjacent 12-gons and triangles increase the average degree to  $15 + \frac{2}{3} - O(1/\sqrt{n})$ . Hence the number of edges is  $\frac{47}{6}n - O(\sqrt{n}) = 7.8\dot{3}n - O(\sqrt{n})$ .

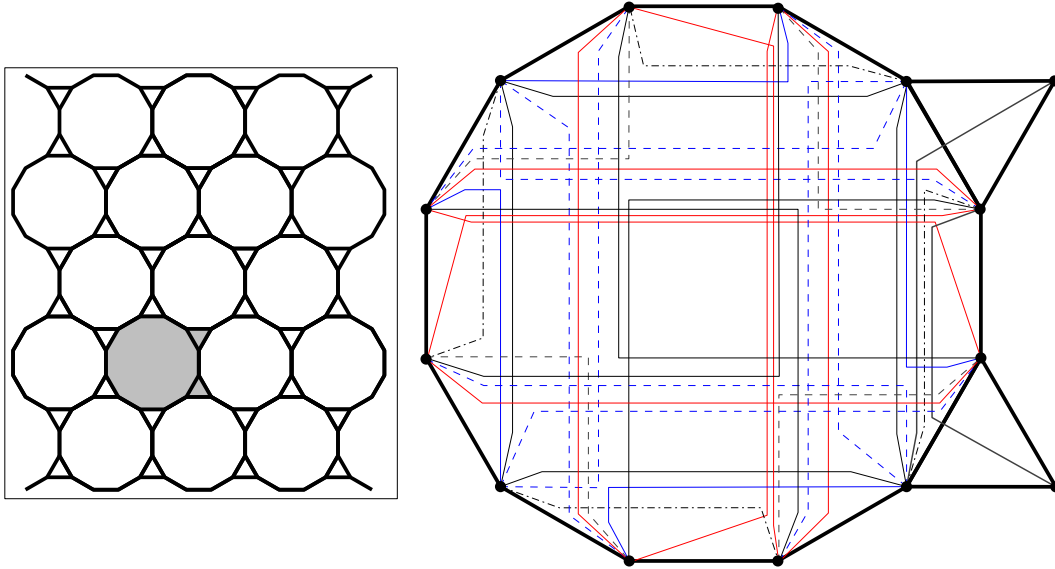


Figure 4: Lower bound construction for a  $RAC_2$  drawing in an Archimedean tiling (12,12,3)

## 5 Concluding Remarks

It remains an open problem to determine the maximum number of edges of a graph with  $n$  vertices in the classes  $R_1$  and  $R_2$ . Our upper bound in Theorem 1 may be slightly improved by refining the bound in Lemma 4. If we could strengthen the upper bound in Lemma 4 for small values of  $n$ , then (4) would improve. However, we did not pursue this direction as it would not lead to significant improvement without an extensive case analysis.

Let an  $\alpha AC_b^-$  drawing be a polyline drawing of a graph with  $b$  bends per edge where all crossings occur at angle *exactly*  $\alpha$ . It is easy to show that a graph with  $n$  vertices and an  $\alpha AC_0^-$  drawing has at most  $9n - 18$  edges. The edges in each “block” can be partitioned into 3 sets of noncrossing edges, and so the graph decomposes into 3 planar graphs. Every graph admits an  $\alpha AC_{\frac{2}{3}}^-$  drawing, since every affine transformation deforms *all* crossing angles uniformly in the construction by Didimo *et al.* [7]. Very recently, Ackerman *et al.* [1] proved that every graph on  $n$  vertices that admit  $\alpha AC_1^-$  or  $\alpha AC_2^-$  drawings have  $O(n)$  vertices.

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