Name:________________________

Student Number: ________________________

Lab Section:________________________

Exam Code: PMAT 315

Books, notes and collaboration are *not permitted.*

50 MINUTES

UNIVERSITY OF CALGARY

MIDTERM EXAM

SESSION 2004
WINTER SEMESTER

DATE: 1 March 2004
TIME: 11:00 - 11:50

Please write your student number on every page, beginning with this one.
Good luck!

For Markers Only	
1	/5
2	/5
3	/5
4	/15
5	/15
6	/15
7	/15
$\sum$	/60

Student number:________________________

SECTION A

Answer the questions in the space provided. You do not need to show your work.

[5]

1 State the Finite Subgroup Test

A finite subset H of a group G is a subgroup if and only if H is closed.

[5]

2 Which one of these maps is a well defined map of sets?

- $\alpha : \mathbf{Z}/n \to \mathbf{Z}$ defined by $\alpha([x]) = 2x$.
- $\beta : \mathbf{Z}/n \to \mathbf{Z}/n$ defined by $\beta([x]) = [x - 1]$.

The map β is well defined because addition is well defined on $\mathbf{Z}/n$. The map α is not well defined. We have $[x] = [x + n]$ for any x, but $\alpha([x]) = 2x \neq 2x + 2n = \alpha[x + n]$.

[5]

3 Factor the following permutation into disjoint cycles:

$$(1\ 5)(1\ 6\ 7\ 8\ 9)(4\ 5)(1\ 2\ 3)$$

Multiply: $(1\ 2\ 3\ 6\ 7\ 8\ 9\ 5\ 4)$.

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SECTION B

Show all of your work. Each problem is worth 15 points. The best 3 of 4 solutions to the problems below will count towards your mark.

[15]

4 Let G be a group. Let $x \in G$ be a fixed element of G and let $H \leq G$ be a subgroup. Define

$$xHx^{-1} = \{xhx^{-1} | h \in H\}.$$

Show that xHx^{-1} is a subgroup of G.

Use the subgroup test. Let $e \in G$ be the identity element.

- $e \in xHx^{-1}$. Since H is a subgroup, $e \in H$. So $e = xex^{-1} \in xHx^{-1}$.
- xHx^{-1} contains inverses. Let $a = xhx^{-1} \in xHx^{-1}$ for some $h \in H$. Then compute $(xhx^{-1})^{-1} = xh^{-1}x^{-1}$. Since H is a subgroup, $h^{-1} \in H$. So $xh^{-1}x^{-1} \in xHx^{-1}$.
- H is closed. Let $a = xh_1x^{-1}$ and $b = xh_2x^{-1}$ be elements of xHx^{-1}, so that h_1 and h_2 are elements of H. Then $xh_1x^{-1}xh_2x^{-1} = xh_1h_2x^{-1} \in xHx^{-1}$ since $h_1h_2 \in H$.

[15]

5 Let G be a group and let $a, b \in G$. Define a relation on G by $a \sim b$ if $b = gag^{-1}$ for some element $g \in G$. Show that $\sim$ is an equivalence relation on G.

- Reflexive: Since $a = aaa^{-1}$ and $a \in G$, we have $a \sim a$.
- Symmetric: Suppose $a \sim b$ so that $b = hah^{-1}$ for some $h \in G$. Then $a = h^{-1}bh = h^{-1}b(h^{-1})^{-1}$, which is $a = gag^{-1}$ with $g = h^{-1}$. Since $h^{-1} \in G$, $b \sim a$.
- Transitive: Suppose $a \sim b$ and $b \sim c$. That is, suppose there are elements h_1 and h_2 in G with

$$b = h_1 a h_1^{-1}$$

$$c = h_2 b h_2^{-1}$$

Substituting the first equation into the second one, we have $c = h_2 h_1 a h_1^{-1} h_2^{-1} = (h_1 h_2) a (h_1 h_2)^{-1}$. Since $h_1 h_2 \in G$, $a \sim c$.

[15]

6 Recall that $A_n \leq S_n$ is the subgroup of even permutations of $\{1, \ldots, n\}$. Show that A_8 contains an element of order 15.

Recall that the order of any permutation $\sigma = \gamma_1 \cdots \gamma_n$ which has been written as a product of disjoint cycles is $lcm(|\gamma_1|, \ldots, |\gamma_n|)$. Also, $|\gamma_i|$ is the length of the cycle γ_i. So, it would suffice to find a 3-cycle and a disjoint 5-cycle in A_8, since their product will be a permutation of order 15.

Consider (1 2 3)(4 5 6 7 8). This permutation has oder 15. It can be written as 6 transpositions:

$$(1\ 3)(1\ 2)(4\ 8)(4\ 7)(4\ 6)(4\ 5).$$

Therefore, it is an element of order 15 in A_8.

[15]

7 Let G be a group with identity element e and let $a, b \in G$ with $|a| = m$ and $|b| = n$. If m and n are relatively prime, show that

$$\langle a \rangle \cap \langle b \rangle = \{e\}.$$

There are many very elegant solutions to this problem. Let $g \in \langle a \rangle \cap \langle b \rangle$. In each case we strive to show that $g = e$.

(a) Since $g \in \langle a \rangle$, $\langle g \rangle$ is a subgroup of $\langle a \rangle$. Similarly, since $g \in \langle b \rangle$, $\langle g \rangle$ is a subgroup of $\langle b \rangle$. By the Fundamental Theorem for Cyclic Groups,

$$|\langle g \rangle| \mid |\langle a \rangle| = m$$

and

$$|\langle g \rangle| \mid |\langle b \rangle| = n.$$

Since m and n are relatively prime, $|\langle g \rangle| = 1$. We must have $g = e$.

(b) Since $g \in \langle a \rangle$, $g = a^k$ for some integer k. So $g^m = (a^k)^m = (a^m)^k = e$. Similarly, since $g \in \langle b \rangle$, $g = b^j$ and we find $g^n = e$. The integers m and n are relatively prime, so there exist integers x and y such that

$$1 = xm + yn.$$

So, $g = g^1 = g^{xm+yn} = (g^m)^x (g^n)^y = e^x e^y = e$.

(c) Since $g \in \langle a \rangle$, we have $g = a^k$. Since $g \in \langle b \rangle$, we have $g = b^j$. Let $r = |g|$. Since $g^m = e$ (as in solution 2), we have $r \mid m$. Similarly, $r \mid n$. That is, $r = 1$. The only element of order 1 is $g = e$.

END OF EXAM