Higher Order Hochschild Homology and Its Decompositions

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August 1, 2008

Abstract

In this paper, we provide a decomposition of a generalization of Hochschild homology, $HH_*^{S^1\wedge Y}(A)$ for commutative algebras A over a field k of characteristic 0. We show that $HH_*^{S^1\wedge Y}(A)$ is a Hopf algebra whose structure maps are induced by the pinch and fold maps on the circle, S^1. Following Loday and Gerstenhaber and Schack we use Eulerian idempotents to obtain a decomposition of $HH_*^{S^1\wedge Y}(A)$. Finally, we show that this decomposition recovers a decomposition of higher order Hochschild homology discovered by Pirashvili.

1 Introduction

Our main result provides a rational decomposition of higher order Hochschild homology and its generalizations. This result recovers two similar results. The first is a result of Loday [**?**] and Gerstenhaber and Schack [**?**]. They independently showed that the reduced Hochschild chain complex is a differential graded Hopf algebra. Using the Eulerian idempotents associated to this Hopf algebra, they obtained a rational decomposition of Hochschild homology. McCarthy geometrically interpreted this decomposition in [**?**] using the edgewise subdivision of a simplicial set and the r-fold covering map of the circle.

More recently, Pirashvili extended this decomposition to higher Hochschild homology [**?**]. Let k be a field of characteristic zero, and let Δ be the simplicial category and Γ be the category of finite pointed sets (see section **??**). One can think of Hochschild homology as the homology of the chain complex associated to the simplicial vector space

$$\begin{tikzcd} \Delta^{op} \arrow[r, "S^1"] & \Gamma \arrow[r, "L_A"] & Vect_{/k} \end{tikzcd}$$

where S^1 is the simplicial circle and L_A is the functor which takes a finite pointed set of cardinality n to $M \otimes A^{\otimes n}$ for any commutative k algebra A and A-module M (see §**??** for a precise definition). Pirashvili extended the definition to "n-th higher order Hochschild homology" by considering instead the simplicial vector space $L_A \circ S^n$. Furthermore, he obtained a decomposition of higher Hochshild homology which recovers the previously known decomposition of Hochschild homology in the case $n = 1$.

The goal of this paper is to study functors of the form $L_A \circ X$ for any simplicial finite pointed set X. We obtain a decomposition of this functor for any simplicial finite poined set X of the form $S^1 \wedge Y$ which recovers the two decompositions just mentioned. Following the methods employed by McCarthy [**?**] to give a geometric description of the decomposition of HH_*, we use the edgewise subdivision of a simplicial set to provide a simplicial version of the usual pinch map on a circle. We then show that the pinch map and the fold map on the S^1 coordinate of $S^1 \wedge Y$ induce a differential graded Hopf algebra structure on the chain complex associated to $L_A \circ (S^1 \wedge Y)$. Using this structure, we are able to provide decompositions analogous to those of Loday and Gerstenhaber and Schack. Finally, we are able to show that in the case $Y = S^{n-1}$, the image of the r-fold covering map on S^n (induced by the r-fold covering map on the first S^1-coordinate) is the same on the summands of our decomposition as well as the summands of the decomposition provided by Pirashvili. Hence this recovers Pirashvili's decomposition.

This paper is organized as follows. In section **??**, we present notation and conventions to be used in the rest of the papers. In section **??** we outline a model for the geometric realization of simplicial sets of the form $F \circ X$. where X is a simplicial set and F is a functor of finite sets. This model is homeomorphic to the standard geometric realization, but more suited to our purposes. The following section describes operators on the realization of $F \circ X.$. In particular we construct a simplicial version of the n-fold covering map using the pinch and fold maps on the simplicial spaces of the form $(S^1 \wedge Y).$, which will play an important role. A remark at the end of this section is the key to understanding this operation for the simplicial vector space $L_A \circ X.$ where $X.$ is a suspension. Section **??** explains how to perform these operations on the chain level. In section **??**, we show that the chain complex associated to $L_A \circ X.$ for a suspension $X.$ is actually a differential graded Hopf algebra. Finally, in section **??** we use the Eulerian idempotents to obtain the desired decomposition.

This work was completed during my graduate study at the University of Illinois. I am indebted to all who encouraged me there in this work, including Matthew Ando, Maria Basterra, Dan Grayson and Manos Lydakis. I am especially grateful to Randy McCarthy. Without his good wishes, I would never have begun. Finally, I would like to thank the referee for many helpful suggestions.

2 Conventions

The simplicial category Δ is the category of finite simplicial sets and order preserving monotonic maps. Denote an object of Δ by $[n] = \{0, 1, \cdots, n\}$. Let $Sets_*$ be the category whose objects are basepointed sets and whose morphisms are all basepoint preserving maps. The category $Sets_*$ has a full subcategory Γ of all finite basepointed sets. We also denote objects of $Sets_*$ and Γ by $[n]$ where 0 is the basepoint. We are interested in the category of simplicial finite pointed sets. These are contravariant functors $X : \Delta \to \Gamma$ satisfying certain axioms (see, e.g. [**?**]). We write $X.$ for a simplicial set, where $\cdot$ stands for the simplicial dimension.

Let k be a field of characteristic 0. Let V_k be the category of k-vector spaces. The free functor $k[-]$ is the functor from $Sets_*$ to V_k which produces for any set $S \in Sets_*$ a k-vector space $k[S]$ with basis S. If $X.$ is any simplicial finite pointed set, then $k[X.]$ is a simplicial finite dimensional k-vector space, defined by $k[X.]_n = k[X_n]$. The functor $k[-]$ is left adjoint to the forgetful functor U from V_k to $Sets_*$. When considered as a functor of simplicial sets, $k[-]$ is left adjoint to the forgetful functor from simplicial vector spaces to simplicial sets which is also defined degree-wise. Let $C.(X)$ be the associated chain complex with $C_n(X) = k[X_n]$ and the boundary map δ_n^X is given by the alternating sum of the face maps.

If $X.$ and $Y.$ are simplicial finite pointed sets, then $X. \vee Y.$ is the bisimplicial set whose (p, q)-th set is the one point union of the sets X_p and Y_q, whose common point is the 0 element of each set. Let $C..(X \vee Y)$ be the associated chain bicomplex with $C_{pq} = k[X_p \vee Y_q]$. The coproduct of $X.$ and $Y.$ in the category of simplicial finite pointed sets is the simplicial finite pointed set $(X \vee Y).$, which is the diagonal of $X. \vee Y.$. In other words, this is the simplicial complex with $(X \vee Y)_n = X_n \vee Y_n$. These are related by the Eilenberg-Zilbur theorem, which says that $\pi_*((X \vee Y).) \cong H_*(Tot\, C..(X \vee Y))$ (see, e.g. [**?**] for this statement of Eilenberg-Zilbur). The notation denoting a bisimplicial set versus its diagonal simplicial set can be cumbersome. To alleviate this, in the special case of the bisimplicial set $S^1_. \wedge Y.$, we denote its diagonal simplicial set by $SY.$.

3 Realizations of a functor F evaluated at $X.$

Let $X.$ be a simplicial finite pointed set and let $F : \Gamma \to \mathrm{Sets}_*$ be a covariant functor. The composition $F \circ X.$ is a pointed simplicial set, and one can form the usual geometric realization, $|F \circ X.|$. The main purpose of this section is to rewrite $|F \circ X.|$ in terms of powers of $|X.|$.

Applying the functor $\mathrm{Hom}_\Gamma([n], -)$ to the simplicial set $X.$ results in a simplicial set which in dimension n is $\mathrm{Hom}_\Gamma([n], X_n)$. Define the *$X.$-realization of F*, to be the set

$$|F|_{X.} = \coprod_{n \in \mathbb{N}} F([n]) \times |\mathrm{Hom}_\Gamma([n], X.)|/\!\sim.$$

Let $[\alpha]$ denote the equivalence class of $\alpha \in \mathrm{Hom}_\Gamma([m], X.)$ in $|\mathrm{Hom}_\Gamma([m], X.)|$. The relation $\sim$ is given as follows: if $f : [n] \to [m]$ is a morphism in Γ, and if $f_* u$ denotes $F(f)(u)$ for $u \in F([n])$ and $f^*([\alpha]) = [\alpha \circ f]$, then $(f_* u, [\alpha]) \sim (u, f^*[\alpha])$.

The geometric realization $|\mathrm{Hom}_\Gamma([n], X.)|$ is defined to be

$$\coprod_{l \in \mathbb{N}} \mathrm{Hom}_\Gamma([n], X_l) \times \Delta^l/\!\sim$$

where $\sim$ denotes the usual relation (see e.g. [**?**]). We may write an element of $|F|_{X.}$ as the equivalence class of a triple (u, α, t), where $u \in F([n])$, $\alpha \in \mathrm{Hom}_\Gamma([n], X_l)$, and $t \in \Delta^l$. On the other hand, an element of $|F \circ X.| = \coprod_{m \in \mathbb{N}} F(X_m) \times \Delta^m/\!\sim$ is the equivalence class of a pair (v, s). An explicit homeomorphism σ from $|F|_{X.}$ to $|F \circ X.|$ is given by

$$\sigma(u, \alpha, t) = (F(\alpha)(u), t).$$

It is not difficult to check that σ is well defined on equivalence classes. It has an inverse $\sigma^{-1} : |F \circ X| \to |F|_X$ defined on $(v, s) \in F(X_l) \times \Delta^l$ by

$$\sigma^{-1}(v, s) = (v, id, s) \in F([n]) \times \mathrm{Hom}_\Gamma([n], X_l) \times \Delta^l$$

The map σ^{-1} is again well defined on $|F \circ X.|$.

There is a more concise way of showing that $|F|_{X.} \cong |F \circ X.|$. This is accomplished by viewing $|F|_{X.}$ as a certain coend. We may write $|F|_{X.}$ as a tensor product of functors with equivalences following from the properties of tensors and Yoneda's Lemma:

$$\begin{aligned} F(-) \otimes_\Gamma |\mathrm{Hom}_\Gamma(-, X.)| &= F(-) \otimes_\Gamma \mathrm{Hom}_\Gamma(-, X.) \otimes_\Delta \Delta^{\cdot} \\ &= (F \circ X). \otimes_\Delta \Delta^{\cdot} \end{aligned}$$

and $(F \circ X). \otimes_\Delta \Delta^{\cdot}$ is $|F \circ X|$.

Either way, we have the following:

Lemma 3.1. $|F|_{X.} \cong |F \circ X.|$.

The construction of $|F|_{X.}$ is suitably natural in $X.$. Let $T : X. \to X'_{\cdot}$ be a natural transformation of pointed simplicial sets. Then T induces a continuous map $\overline{T}_F : |F|_{X.} \to |F|_{X'_{\cdot}}$ for each functor $F : \Gamma \to Sets_*$. The map $\overline{T}$ is given by:

$$\overline{T}_F : (u, \alpha) \to (u, T \circ \alpha)$$

for $u \in F([n])$ and $\alpha \in \mathrm{Hom}_\Gamma([n], X.)$. This is well defined on equivalence classes, since for $f \in \mathrm{Hom}_\Gamma([n], [m])$ and $\beta \in \mathrm{Hom}_\Gamma([m], X.)$ with $(f_* u, [\beta]) \sim (u, f^*[\beta])$, then

$$\begin{aligned} \overline{T}_F(f_* u, [\beta]) &= (f_u, [T \circ \beta]) \\ \overline{T}_F(u, f^*[\beta]) &= (u, T \circ f^*[\beta]) = (u, f^*[T \circ \beta]) \end{aligned}$$

This induces a commutative diagram:

$$\begin{tikzcd}
{|F|_{X.}} \arrow[r] \arrow[d] & {|F|_{X'_.}} \arrow[d] \\
{|F \circ X.|} \arrow[r] & {|F \circ X'_.|}.
\end{tikzcd}$$

The construction is also natural in F.

It is possible to express $|\mathrm{Hom}_\Gamma([n], X.)|$ in terms of $|X.|$. Since $[n] = \bigvee_n [1]$, we have

$$\mathrm{Hom}_\Gamma([n], X.) \cong \mathrm{Hom}_\Gamma([1], X.)^{\times n}.$$

If $\alpha \in \mathrm{Hom}_\Gamma([n], X.)$, there is an explicit identification

$$\alpha \mapsto \alpha_1 \times \cdots \times \alpha_n \tag{1}$$

where α_i is defined by the property that $\alpha_i(1) = \alpha(i)$. The isomorphism $\mathrm{Hom}_\Gamma([1], X.) \simeq X.$ is given by identifying a map in $\mathrm{Hom}_\Gamma([1], X.)$ with the image of 1, an element of $X.$.

Lemma 3.2. *There is an equivalence of topological spaces*

$$|F|_{X.} = \coprod_{n \in \mathbb{N}} F([n]) \times |X.|^{\times n}/\sim$$

Proof. The proof is now a series of equivalences:

$$\begin{aligned}
|F|_{X.} &= \coprod_{n \in \mathbb{N}} F([n]) \times |\mathrm{Hom}_\Gamma([n], X.)|/\sim \\
&\simeq \coprod_{n \in \mathbb{N}} F([n]) \times |\mathrm{Hom}_\Gamma([1], X.)^{\times n}|/\sim \\
&\simeq \coprod_{n \in \mathbb{N}} F([n]) \times |X.^{\times n}|/\sim \\
&\simeq \coprod_{n \in \mathbb{N}} F([n]) \times |X.|^{\times n}/\sim
\end{aligned}$$

One can describe the equivalence relations at each stage by identifying α with $(\alpha_1, \ldots, \alpha_n)$ as in Equation **??** and by then identifying the α_i with elements of $\alpha_i(1)$ of $X.$. The equivalence relations are preserved at each stage of the equivalence. □

4 Operators on $|F|_{X.}$

We want to define operators on $|F \circ X.|$. In particular, we will be interested in finding a way to operate on $|F \circ S^1.|$ by the r-fold covering map of the circle. One way to obtain the r- fold cover is to first pinch the circle into a bouquet of r-circles, and then fold these circles together again. The fold map exists simplicially because the wedge product is the coproduct in the category of simplicial sets. However, the pinch map is not a simplicial map. One can make a simplicial pinch map by approximating the usual model for the simplicial circle with one which has r 0-cells instead of 1 0-cell. To complete the construction, identify the 0-cells to one point. To do this, we use edgewise subdivisions of simplicial sets [**?**].

If $[p]$ and $[q]$ are objects of Δ, the *concatenation* of $[p]$ and $[q]$ is the new set $[p] * [q] = [(p+1)+(q+1)-1]$. This is the new set obtained by taking the union of the sets $[p]$ and $[q]$, and setting aside one basepoint. Let $[n]^{*r}$ be the concatenation of $[n]$ with itself r times. Define a functor $\mathrm{sd_r}$ from Δ to Δ by

$$\mathrm{sd_r}([n]) = [n]^{*r} = [r(n+1)-1].$$

The *r-th edgewise subdivision of $X.$*, $\mathrm{sd_r}X.$, is the composition of functors $X. \circ \mathrm{sd_r}$. Thus, $\mathrm{sd_r}X_{n-1} = X_{rn-1}$. Since $\Delta^{n-1} = \{(t_1, \cdots, t_n) | \sum_i t_i = 1\}$ we have a map $d_r : \Delta^{n-1} \to \Delta^{rn-1}$ given by

$$t = (t_1, \cdots, t_n) \to (s_1, \ldots, s_r)$$

where each $s_i = t/r$. We have the following lemma:

Lemma 4.1 ([**?**],[**?**])**.** *The map* $1 \times d_r : X_{rn-1} \times \Delta^{n-1} \to X_{rn-1} \times \Delta^{rn-1}$ *induces a homeomorphism* $D_r : |\mathrm{sd_r}X.| \cong |X.|$.

We will be concerned in particular with the simplicial circle

$$S^1_{\cdot} = \mathrm{Hom}_\Delta(\cdot, [1])/\partial$$

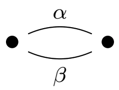
where ∂ identifies the map whose image is identically 0 with the map whose image is identically 1.

Recall that the non-degenerate simplices of $S^1_{\cdot}$ correspond to the maps $\alpha \in \mathrm{Hom}_\Delta(-, [1])$ which are injective. In particular any simplex of dimension two or higher must be degenerate.

Example 4.1. We want to compute the geometric realization of $\mathrm{sd}_2 S^1_{\cdot}$. To describe the geometric realization $|\mathrm{sd}_2 S^1_{\cdot}|$, we need only look at simplices of dimensions zero and one. The others will be degenerate.

The set $\mathrm{Hom}_{\Delta}([0] * [0], [1])$ has three elements. The map which is identically one is identified with the map which is identically zero by ∂. The other map is the identity map on [1]. Thus there are two 0-simplices in $\mathrm{sd}_2 S^1$.

The set $\mathrm{Hom}_{\Delta}([1] * [1], [1])$ has five elements, and again we identify two of these to the basepoint. The degeneracy map is given by precomposition by the unique map from [1] to [0]. Only two of the remaining elements are injective on one component. These represent the non-degenerate simplices, α and β. The face maps are given by precomposition with the two maps from [0] to [1]. These maps identify the end points of the 1-simplices to opposite 0-simplices. The geometric realization is a circle with two 0-simplices and two 1-simplices, like this:



Let $Y_{\cdot}$ be any other simplicial finite pointed set. The suspension of $Y_{\cdot}$ is the simplicial finite pointed set given by

$$\begin{array}{rrl} (S^1 \wedge Y)_{\cdot} : & \Delta \to & \mathrm{Fin}_* \\ & [n] \mapsto & S^1_n \wedge Y_n . \end{array}$$

For brevity, we denote the suspension by $SY_{\cdot}$.

Corollary 4.2. $|\mathrm{sd}_{\mathrm{r}}(SY)_{\cdot}| \cong |(SY)_{\cdot}|$.

Proof. This follows immediately from Lemma **??** and the fact that the smash product commutes with realizations. $\square$

Following McCarthy [**?**], we define a *system of natural operators on* $S^1_{\cdot}$ to be a family of simplicial maps $\{\phi^r : \mathrm{sd}_{\mathrm{r}} S^1_{\cdot} \to S^1_{\cdot}\}_{r \in \mathbb{Z}_+}$ making the following diagrams commute:

$$\begin{tikzcd} sd_{rs} S^1_{\cdot} \arrow[r, "\mathrm{sd}_{\mathrm{r}}(\phi^s)"] \arrow[dr, "\phi^{rs}"'] & \mathrm{sd}_{\mathrm{r}} S^1_{\cdot} \arrow[d, "\phi^r"] \\ & S^1_{\cdot} \end{tikzcd}$$

Let $Y_\cdot$ be a fixed simplicial set. Any such family of operators gives rise to a family $\{\Phi^r\}$ of maps by letting Φ^r be the composite

$$\begin{tikzcd}
{|SY|} \arrow[r, "\cong"] & {|S^1| \wedge |Y|} \arrow[d, "D_r^{-1}\wedge 1"'] & & \\
 & {|\mathrm{sd_r} S^1| \wedge |Y|} \arrow[r, "|\phi^r|\wedge 1"] & {|S^1| \wedge |Y|} \arrow[r, "\cong"] & {|S^1 \wedge Y|}
\end{tikzcd} \tag{2}$$

Example 4.2. In this example we construct a system of natural operators $\{\phi^r\}$ on $S^1_\cdot$ for which $\{\Phi^r\}$ will correspond to the r-fold covering map of the circle. For $1 \leq k \leq r$, let i_k be the basepointed (sends 0 to 0) inclusion of $[n]$ into the k-th copy of $[n]$ in $[n]^{*r}$. For any simplicial set $X_\cdot$, the map $i_k : [n] \mapsto [n]^{*r}$ induces a natural transformation $\mathrm{sd_r} X_\cdot \to X_\cdot$. Assembling the i_k's, we get a natural transformation $p^r : \mathrm{sd_r} X_\cdot \to X_\cdot^{\times r}$. When $X_\cdot = S^1_\cdot$, the image of this map lies in the simplicial set $(S^1 \vee \cdots \vee S^1)_\cdot$.

Define ϕ^r to be the composite of the fold map $+$ with p^r. Thus $\{\phi^r : \mathrm{sd_r} S^1_\cdot \to S^1_\cdot\}$. For $\alpha \in \mathrm{sd_r} S^1_n = \mathrm{Hom}_\Delta(\mathrm{sd_r}[n], [1])/\partial$, we have

$$\phi^r(\alpha) = \begin{cases} \alpha \circ i_k & \text{if } \alpha \circ i_k \text{ is surjective for some } k \\ * & \text{if no such } k \text{ exists.} \end{cases}$$

One can check that the maps $\{\phi^r\}$ form a natural system. Thus there are maps $\{\Phi^r : |(SY)_\cdot| \to |(SY)_\cdot|\}$. If $Y_\cdot$ is a point, i.e., the functor which sends each $[n]$ to $\{0\}$, then $|(SY)_\cdot| = |S^1_\cdot|$ and this map corresponds to the r-fold covering map of the circle.

If $Y = S^{d-1}_\cdot$ then Φ^r is a map from $S^d_\cdot$ to $S^d_\cdot$. In this case Φ^r induces multiplication by r on the top dimension of homology, since Φ^r is an r-fold covering map on the suspension coordinate.

We define operations on $\pi_*(|F \circ X_\cdot|)$ using the family of operators Φ^r from Example **??**. By Lemma **??**, we have

$$|F|_X = \coprod_{n \in \mathbb{N}} F([n]) \times |X_\cdot|^{\times n}/\sim$$

In the case where $X_\cdot = (SY)_\cdot$ is actually a suspension, we define $\Phi^r : |F|_{(SY)_\cdot} \to |F|_{(SY)_\cdot}$ by

$$\Phi^r = \coprod_n 1 \times (\Phi^r)^{\times n} : \coprod_{n \in \mathbb{N}} F([n]) \times |(SY)_\cdot|^{\times n}/\sim \;\to \coprod_{n \in \mathbb{N}} F([n]) \times |(SY)_\cdot|^{\times n}/\sim.$$

Theorem 4.3. *The map* $\Phi^r : |F|_{(SY).} \to |F|_{(SY).}$ *is well defined.*

Proof. We prove more generally that if $\Psi : X. \to Y.$ is any simplicial map of simplicial finite pointed sets, then there is a well-defined map $\tilde{\Psi} : |F|_{X.} \to |F|_{Y.}$ defined diagonally on the $|X.|^{\times n}$ part of $|F|_{X.}$.

We only need to show that the operations preserve the equivalence relation $\sim$. As in Lemma **??**, we have

$$|F|_{X.} \simeq \coprod_{n \in \mathbb{N}} F([n]) \times |X_{.}^{\times n}|/\sim.$$

Let $\alpha \in |X^{\times n}| \cong |Hom_\Gamma([1], X.)^{\times n}|$ be represented as the equivalence class of

$$((\alpha_1(1), \ldots, \alpha_n(1)), t)$$

where each $\alpha_i \in Hom_\Gamma([1], X_k)$ and $t \in \Delta^k$. Let u be an element of the set $F([m])$ and choose a map $f : [m] \to [n]$ in Γ. The equivalence $\sim$ is given by

$$(f^*u, (\alpha_1(1), \ldots, \alpha_n(1)), t) \sim (u, f_*(\alpha_1(1), \ldots, \alpha_n(1)), t)$$

where $f^*u = F(f)(u)$ is an element of $F([n])$, and $f_*(\alpha_1(1), \ldots, \alpha_n(1))$ is $(\alpha_{f(1)}, \ldots, \alpha_{f(m)})$.

Apply Ψ. We have

$$\begin{aligned}
&\tilde{\Psi}(f^*u, (\alpha_1(1), \cdots, \alpha_n(1)), t) \\
&= (f^*u, (\Psi \circ \alpha_1(1), \cdots, \Psi \circ \alpha_n(1)), t) \\
&\sim (u, (f_*((\Psi \circ \alpha_1)(1), \cdots, (\Psi \circ \alpha_n)(1)), t) \\
&= (u, ((\Psi \circ \alpha_{f(1)})(1), \cdots, (\Psi \circ \alpha_{f(m)})(1)), t) \\
&= \tilde{\Psi}(u, (\alpha_{f(1)}(1), \cdots, \alpha_{f(m)}(1)), t) \\
&= \tilde{\Psi}(u, f_*(\alpha_1(1), \ldots, \alpha_n(1)), t)
\end{aligned}$$

Thus, $\tilde{\Psi}$ preserves the equivalence $\sim$ and $\tilde{\Psi}$ is well defined. □

Remark 4.4. The results of §**??** and §**??** can be extended to functors to k-vector spaces. Let F be a functor from Γ to V_k. There is a version of the tensor product of functors in this situation analogous to the one we used for simplicial sets which uses direct sum instead of disjoint union, tensor product over k instead of products, and the quotient of the subspace generated by the equivalence relation $\sim$ rather than just the quotient by the relation $\sim$ (see [**?**] for details). Denote this tensor product by $\otimes_{k,\Gamma}$ to distinguish it from $\otimes_\Gamma$ of §**??**. We have

$$F([n]) \cong F([-]) \otimes_{k,\Gamma} k[\mathrm{Hom}_\Gamma([-],[n])].$$

This equivalence is obtained exactly as in Lemma **??**. Since $\mathrm{Hom}_\Gamma([n], X.)$ is $(X.)^{\times[n]}$ (see §**??**), we can express $F \circ X.$ as a tensor product more succinctly as

$$|F \circ X| = F([-]) \otimes_{k,\Gamma} k[(X.)^{\times[-]}].$$

5 The chain level

We now want to show that if the operations Φ^r of the last section are extended to simplicial vector spaces, they are actually well defined on the normalized chain complexes. First we define a new map Φ^r on the chain level which agrees with the map from §**??** on homology. Since Φ^r is built out of maps $\phi_r : \mathrm{sd_r} S^1_\cdot \to S^1_\cdot$, to show that these operations are well defined on the chain level we need an equivalence

$$D(r) : C.(F \circ X) \to C.(F \circ \mathrm{sd_r} X)$$

to serve as the inverse of $D_r : |\mathrm{sd_r} X.| \to |X.|$ in our old setting. This equivalence is provided by McCarthy [**?**]. The construction is included here for completeness.

Let $\mathrm{Hom}^*_\Delta([n],[m])$ denote the set of maps from $[n]$ to $[m]$ in Δ which send 0 to 0. Let Σ_n be the n-th symmetric group. The set $S(r,n)$ consists of pairs

$$(\sigma, f) \in \Sigma_n \times Hom^*_\Delta([n],[r-1])$$

subject to the condition that if $i < j$ and $\sigma(i) > \sigma(j)$, then $f(i) < f(j)$ for $1 \leq i,j \leq n$. For $\eta = (\sigma, f) \in S(r,n)$, define a map

$$\epsilon^{op}_\eta \in \mathrm{Hom}^*_\Delta([n],[rn+r-1])$$

by $\epsilon^{op}_\eta(i) = \sigma(i) + f(i)(n+1)$ for $1 \leq i \leq n$. Let $\epsilon^{op}_\eta(0) = 0$. This map bins the image of σ into the copies of $[n]$ in $sd_r n$ to obtain a map which is monotone. Define a corresponding map

$$\epsilon_\eta \in \mathrm{Hom}^*_\Delta([rn+r-1],[n])$$

by $\epsilon_\eta(k) = \max\{i | \epsilon_\eta^{op}(i) \leq k\}$. The map $D_n(r) : C_n(F \circ X) \to C_n(F \circ \mathrm{sd_r} X)$ is given on $x \in C_n(F \circ X)$ by

$$D_n(r)(x) = \sum_{\eta \in S(r,n)} \mathrm{sgn}(\eta) \epsilon_\eta^*(x)$$

where $\mathrm{sgn}(\eta) = \mathrm{sgn}(\sigma)$ and ϵ_η^* is the map on the chain level induced by $F \circ X(\epsilon_\eta)$.

Example 5.1. Let A be a commutative k-algebra and M an A-module. Define $L_{(M,A)}$ to be the functor from the category of finite pointed sets to the category of A-modules which takes $[n]$ to $M \otimes_k A^{\otimes_k n}$. If $f : [m] \to [n]$ then $L_{(M,A)}(f)(a_0 \otimes a_1 \otimes \cdots \otimes a_m) = b_0 \otimes b_1 \otimes \cdots \otimes b_n$ where

$$b_j = \prod_{\{i | f(i) = j\}} a_i,$$

and $b_j = 1$ if $f^{-1}(j) = \emptyset$. The chain complex associated to the functor $L_{(M,A)} \circ S^1_\cdot$ is the Hochschild homology chain complex, with coefficients in M. When $M = A$, we will simply write $= L_A \circ X_\cdot$. An element of the chain complex $C_\cdot(L_A \circ S^1)$ is written $(a_0, a_1, \cdots, a_n)$.

On the chain level, the term of $D_n(r)(a_0, \ldots, a_n)$ associated to $\eta = (\sigma, f)$ is given by the assignment of $(a_0, \ldots, a_n)$ to

$$(a_0, b_1, \ldots, b_n, 1, b_{n+1}, \ldots, b_{2n}, \ldots, 1, b_{(r-1)n+1}, \ldots, b_{rn}) \tag{3}$$

where

$$b_j = \begin{cases} a_i & j = \sigma(i) + f(i)(n+1); \\ 1 & \text{otherwise.} \end{cases}$$

This definition appears quite technical, so we provide an illustrative example. Think of $(b_1, \ldots, b_{rn})$ as an array of n columns and r rows. The jth column has entries 1 or $a_{\sigma(j)}$ such that there is only one entry in each column not equal to 1. Reading the array from the upper left entry across each row to the lower right entry, the order of $(a_1, \ldots, a_n)$ is preserved. For example, the element of $C_3(L_A \circ sd_2(S^1))$ given by

$$\begin{array}{ccccccc} \{ & a_0 & 1 & a_1 & 1 & & \\ & & 1 & a_2 & 1 & a_3 & \} \end{array}$$

corresponds to the element of $S(2,3)$ which is the permutation $(12)(3)$ and the map in $\mathrm{Hom}^*_\Delta([3],[1])$ which sends $1 \to 0$, $2 \to 1$ and $3 \to 1$. The permutation governs the columns while the map $f \in \mathrm{Hom}^*_\Delta([3],[1])$ governs the rows. The map $D_3(r)$ sums along all possible arrangements like this.

From [**?**] we have the following lemma:

Lemma 5.1. *([**?**]) The maps $D_*(r)$ assemble to give a chain map $D(r)$ which passes to the normalized chain complex. Furthermore, for all $r \in \mathbb{N}$, the chain map $D(r)$ is a quasi- isomorphism with $H_*(D(r)) = \pi_*(D_r)^{-1}$.*

From the bisimplicial set $X_{\cdot\cdot} = S^1_{\cdot} \wedge Y_{\cdot}$ form the simplicial set $X_{\cdot p} = S^1_{\cdot} \wedge Y_p$. There is a chain map

$$D(r) \wedge 1_p : C_{\cdot}F \circ (S^1 \wedge Y_p) \to C_{\cdot}F \circ (\mathrm{sd}_{\mathrm{r}} S^1 \wedge Y_p).$$

Since $D(r)$ is functorially constructed, we can then extend this to the map

$$\mathrm{Tot}(D(r) \wedge 1) : \mathrm{Tot}(C_{**}F \circ (S^1 \wedge Y)) \to \mathrm{Tot}(C_{**}F \circ (\mathrm{sd}_{\mathrm{r}} S^1 \wedge Y)).$$

In other words, the identity maps 1_p assemble to give the identity map on the chain bicomplex in the Y direction. Since $D(r) \wedge 1$ is a quasi-isomorphism, this map gives a chain equivalence. By the Eilenberg-Zilber Theorem (e.g. [**?**] p. 238), we may consider $D(r) \wedge 1$ as a map of chain complexes as opposed to chain bi-complexes. If $\mathcal{D}$ is the diagonal of a simplicial bi-complex, then we have

$$\begin{tikzcd}
\mathrm{Tot} C_{**}F \circ (S^1 \wedge Y) \arrow[r, "D(r)\wedge 1", "\simeq"'] & \mathrm{Tot} C_{**}F \circ (\mathrm{sd}_{\mathrm{r}} S^1 \wedge Y) \arrow[d, "\simeq"'] \\
C_*F \circ \mathcal{D}(S^1 \wedge Y) \arrow[u, "\simeq"'] \arrow[r, "D(r)\wedge 1"] & C_*F \circ \mathcal{D}(\mathrm{sd}_{\mathrm{r}} S^1 \wedge Y)
\end{tikzcd}$$

where the vertical maps are the Eilenberg-Zilber equivalences. Since this is a composition of chain equivalences, the map

$$D(r) \wedge 1 : C_*F \circ (S^1 \wedge Y) \to C_*F \circ (\mathrm{sd}_{\mathrm{r}} S^1 \wedge Y)$$

is also a chain equivalence. Thus we have proved:

Lemma 5.2. *There is a natural chain equivalence*

$$D(r) \wedge 1 : C_*F \circ (S^1 \wedge Y) \to C_*F \circ (\mathrm{sd}_{\mathrm{r}} S^1 \wedge Y)$$

Note that in our notation, the map $D(r)$ is a chain equivalence both for $\mathrm{Tot}C_{**}F \circ (S^1 \wedge Y))$ and $C_*F \circ \mathcal{D}(S^1 \wedge Y))$. The discussion before Lemma **??** indicates that this ambiguity is not misleading.

The previous two lemmas prove:

Theorem 5.3. *If Φ^r is given by*

$$\Phi^r : C.F \circ (S^1 \wedge Y) \xrightarrow{D(r)\wedge 1} C.F \circ (\mathrm{sd_r}S^1 \wedge Y) \xrightarrow{F(\phi_r \wedge 1)} C.F \circ (S^1 \wedge Y)$$

then Φ^r is a well defined map of chain complexes.

6 Hopf Algebra Structure

In this section, we use the geometric structure of $L_A \circ (SY).$ to obtain a Hopf algebra structure on the chain level. We continue to let A denote a commutative algebra over a field k of characteristic 0.

Definition 6.1. A graded Hopf algebra up to homotopy over A is a chain complex $B.$ (concentrated in non-negative degrees) of A-modules together with chain maps

$$\Delta : B. \to Tot(B. \otimes B.) \qquad \text{(comultiplication)}$$

$$\mu : Tot(B. \otimes B.) \to B. \qquad \text{(multiplication)}$$

satisfying:

- The multiplication map μ is a coalgebra map or, equivalently, the comultiplication map Δ is an algebra map;
- The (co)multiplication map is (co)associative up to homotopy.
- There is a unit map which is a coalgebra map, and a counit map which is an algebra map.

If in addition there is an A-linear map $d : B_n \to B_{n-1}$ for all $n > 0$ satisfying $d^2 = 0$ and $d(ab) = (da)(b) + (-1)^p a(db)$, then B_* is a differential graded Hopf algebra up to homotopy.

We want to show that $B. = C.L_A(SY)$ is a differential graded Hopf algebra. We simplify some computations first.

Since L_A is coproduct preserving when A is commutative, we have an equivalence of bicomplexes:

$$C..L_A((SY) \vee (SY)) = C..(L_A(SY) \otimes L_A(SY)).$$

In what follows, we use these two formulations interchangeably.

Recall that the total chain complex Tot $C..L_A(S^1 \vee S^1)$ is quasi-isomorphic to the diagonal complex $C.L_A(S^1 \vee S^1)$ via the Eilenberg-Zilber theorem. Let $\mathfrak{f} : C.L_A(S^1 \vee S^1) \to \text{Tot } C..L_A(S^1 \vee S^1)$ denote the Alexander-Whitney map (which gives one direction of the Eilenberg-Zilber quasi-isomorphism). Since the Alexander-Whitney map is defined simplicially, we can extend by the identity on the Y component so that $C.L_A((S^1 \vee S^1) \wedge Y) = C.L_A(SY \vee SY)$ is quasi-isomorphic to Tot $C..L_A(SY \vee SY)$.

We begin by defining candidates for the multiplication and comultiplication maps. Define a map $\mu : \text{Tot}C..L_A((SY) \vee (SY)) \to C.L_A(SY)$ by the commuting diagram

$$\begin{tikzcd}
\text{Tot}C..L_A((SY) \vee (SY)) \arrow[d, "\mathfrak{f}^{-1}"'] \arrow[dr, "\mu"] & \\
C.L_A((S^1 \vee S^1) \wedge Y) \arrow[r, "C.L_A(+\wedge 1_Y)"] & C.L_A(SY)
\end{tikzcd}$$

where $+$ is the simplicial fold map $+ : (S^1 \vee S^1). \to S^1_{\cdot}$. The inverse of the Alexander Whitney map is the shuffle map, denoted by $\mathfrak{f}^{-1}$. Define another map $\Delta : C.L_A(SY) \to TotC..L_A((SY) \vee (SY))$ by the commuting diagram

$$\begin{tikzcd}
C.L_A(SY) \arrow[rr, "\Delta"] \arrow[d, "="'] & & TotC..L_A((SY) \vee (SY)) \\
C.L_A(S^1 \wedge Y) \arrow[r, "D(2)\wedge 1"] & C.L_A(\mathrm{sd}_2 S^1 \wedge Y) \arrow[r, "p^2\wedge 1"] & C.L_A((S^1 \vee S^1) \wedge Y) \arrow[u, "\mathfrak{f}"']
\end{tikzcd}$$

where the bottom row is the simplicial version of the pinch map on the circle defined on the chain level in §**??**.

Consider the special case when $Y = *$. In this case, the chain complex $C.L_A(S^1)$ computes Hochschild homology. The Hopf algebra structure on this chain complex has been studied extensively.

Recall ([**?**]) that the comultiplication on the Hochschild chain complex is given by

$$\tilde{\Delta}(a_0, a_1, \ldots, a_n) = \sum_{i=0}^{n} (a_0, a_1, \ldots, a_i) \otimes (1, a_{i+1}, \ldots, a_n)$$

for $(a_0, \ldots, a_n) \in L_A(S^1)[n]$. The multiplication map is defined using the (n, m)-shuffle maps. An (n, m)- shuffle σ is an element of the $(n+m)$-th symmetric group satisfying the property that the restriction maps $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n+m\}$ and $\sigma : \{n+1, \ldots, n+m\} \to \{1, \ldots, n+m\}$ are monic, order preserving functions. Let $sgn(\sigma)$ be the sign of the permutation σ. Multiplication is defined on the Hochschild chain complex by

$$\tilde{\mu}((a_0, a_1, \ldots a_n) \otimes (a'_0, a_{n+1}, \ldots, a_{n+m})) = \sum sgn(\sigma)(a_0 a'_0, a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n+m)})$$

where the sum is taken over all (n, m)-shuffles, σ.

Proposition 6.2. *When $Y = *$, the map Δ is homotopy equivalent to the map $\tilde{\Delta}$ and the map μ is equal to the map $\tilde{\mu}$ of [**?**].*

Proof. The map μ is defined to be the shuffle map followed by the simplicial fold map. The map $\mathfrak{f}^{-1}$ is defined on an element

$$a \otimes b = (a_0, a_1, \ldots, a_n) \otimes (a'_0, a_{n+1}, \ldots, a_{n+m})$$

of $C_n(L_A(S^1)) \otimes C_m(L_A(S^1)) = C_{n,m} L_A(S^1 \vee S^1)$ by

$$\mathfrak{f}^{-1}(a \otimes b) = \sum sgn(\sigma) s_{\sigma(m+n)} \cdots s_{\sigma(n+1)} a \otimes s_{\sigma(n)} \cdots s_{\sigma(1)} b$$

where s_i are induced by the degeneracy maps on S^1, and the sum is taken over (n, m)-shuffles. The definition of the functor L_A from Example **??** implies that the image of the degeneracy map s_i on $a = (a_0, a_1, \ldots, a_n)$ is $(a_0, a_1, \ldots, a_{i-1}, 1, a_i, \ldots, a_n)$. That is, s_i inserts a “1” in the i-th slot. So, we may rewrite $\mathfrak{f}^{-1}(a \otimes b)$ as

$$\sum sgn(\sigma)(a_0, b_1, \ldots, b_{n+m}) \otimes (a'_0, b'_1, \ldots b'_{n+m})$$

where the sum is taken over (n, m)-shuffles and $b_i = 1$ if $i = \sigma(i)$ and similarly for b'_i.

Now we apply the fold map. The image of the fold map is the sum over (n, m)-shuffles σ:

$$+\big(\sum sgn(\sigma)(a_0, b_1, \dots, b_{n+m}) \otimes (a'_0, b'_1, \dots b'_{n+m})\big) = \\ \sum sng(\sigma)(a_0a'_0, b_0b'_0, \dots, b_{n+m}b'_{n+m}).$$

For each $1 \leq i \leq n+m$, exactly one of b_i or b'_i is equal to 1 since σ is a permutation of $\{1, \dots, n+m\}$. The other is equal to some a_k. Furthermore, the order of the a_k's has been preserved exactly so that this is equal to the sum over σ's

$$\sum sgn(\sigma)(a_0a'_0, a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(n+m)})$$

which is exactly the image of $\tilde{\mu}$.

The map Δ is defined to be the chain equivalence $D(2)$ followed by the pinch map p^2 followed by the Alexander Whitney map $\mathfrak{f}$. Let $(a_0, \dots, a_n) \in C_n(L_A(S^1))$. From Example **??**, if $S(2,n) \subset \Sigma_n \times Hom^*_\Delta([n],[1])$ then

$$D(2)(a_0, \dots, a_n) = \sum_{(\sigma,f)\in S(2,n)} \operatorname{sgn}(\sigma) \begin{pmatrix} a_0, & b_1, & \dots, & b_n \\ 1, & b_{n+1}, & \dots, & b_{2n} \end{pmatrix} \tag{4}$$

where for a given term, $b_i = a_j$ if $i = \sigma(j) + f(j)n$ for some $1 \leq i \leq n$ or else $b_i = 1$. Recall that only one of b_i and b_{n+i} is not 1, and that $\sigma(1) + f(1)n < \cdots < \sigma(n) + f(n)n$. The image of the pinch map is

$$\operatorname{sgn}(\sigma)(a_0, b_1, \dots, b_n) \otimes (1, b_{n+1}, \dots, b_{2n}). \tag{5}$$

For $a \otimes b \in C.(S^1 \vee S^1)$ the Alexander-Whitney map is given by

$$\mathfrak{f}(a \otimes b) = \sum_{i=1}^{n} \tilde{d}^{n-i} a \otimes d_0^i b$$

where $\tilde{d}$ is the iterated last face operator. Apply $\mathfrak{f}$ to Equation **??** to obtain for each (σ, f)

$$\operatorname{sgn}(\sigma)\sum_{i=1}^{n}(b_{i+1}\cdots b_n a_0, b_1, \dots b_i) \otimes (b_{n+1}\cdots b_{n+i}, b_{n+i+1}, \dots, b_{2n})$$

For each $0 \leq i \leq n$, there is exactly one pair $(\sigma, f) \in S(2,n)$ which produces a non-degenerate element. Specifically, that pair is $\sigma = id$ and $f(k) = 0$ for $k \leq i$ and $f(k) = 1$ for $k > i$. Any other choice will produce degenerate elements. Therefore, on normalized chain complexes, this agrees with $\tilde{\Delta}$.

$\square$

An immediate consequence of this proposition is that μ and Δ make the chain complex $C.L_A(S^1)$ into a graded commutative Hopf algebra.

Lemma 6.3. *The map Δ is an algebra map.*

Proof. We need to show that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Tot}C_{\cdot\cdot}L_A((SY) \vee (SY)) \arrow[r, "\mu"] \arrow[d, "D(2)\otimes D(2)"', "\simeq"] & C.L_A(SY) \arrow[d, "\simeq"', "D(2)"] \\
\mathrm{Tot}C_{\cdot\cdot}L_A((\mathrm{sd}_2 S^1 \wedge Y) \vee (\mathrm{sd}_2 S^1 \wedge Y)) \arrow[r, "\mu"] \arrow[d, "\Delta_p \vee \Delta_p"'] & C.L_A(\mathrm{sd}_2 S^1 \wedge Y) \arrow[d, "\Delta_p"] \\
\mathrm{Tot}C_{\cdot\cdot}(((S^1 \vee S^1) \wedge Y) \vee ((S^1 \vee S^1) \wedge Y)) \arrow[r, "\mu\vee\mu"] & C.L_A((S^1 \vee S^1) \wedge Y)
\end{tikzcd}$$

Note that $\mathrm{sd}_2(S^1 \vee S^1)_\cdot = (\mathrm{sd}_2 S^1 \vee \mathrm{sd}_2 S^1)_\cdot$ since

$$\mathrm{sd}_2(S^1 \vee S^1)_n = S^1_{rn-1} \vee S^1_{rn-1} = (\mathrm{sd}_2 S^1 \vee \mathrm{sd}_2 S^1)_n.$$

First we want to show that the top square of the rectangle commutes on homology. We use the identification [**?**]

$$H_*(C.L_A(S^1)) = \pi_*(|L_A(S^1)|)$$

where we regard $L_A(S^1)$ on the right side as a simplicial set. By Lemma 4.1, $H_*(D_*(r)) = \pi_*(D_r)^{-1}$. So it suffices to show that D_r commutes with the fold map. However, this follows immediately since the map $1 \times d_r$ (the map which defines D_r levelwise) commutes with the fold maps $S^1 \vee S^1 \to S^1$ and $\mathrm{sd_r}\, S^1 \vee \mathrm{sd_r}\, S^1 \to \mathrm{sd_r}\, S^1$.

Now, to show that the bottom square commutes on homology means that the diagram commutes after $\pi_*(|\ |)$ has been applied. But $|L_A(S^1)| = L_A \otimes_{k,\Gamma} k[(S^1_\cdot)^\cdot]$. So the Lemma follows by the commuting of the pinch and fold maps on $|S^1|$ up to homotopy. $\square$

As a consequence, we have the

Theorem 6.4. *The higher Hochschild homology $H_*(L_A(SY))$ is a commutative graded Hopf algebra.*

Proof. We have verified the first two conditions of Definition **??**. The unit and counit maps are easily verified to be the inclusion and projection which give isomorphisms with A and the degree 0 part of $C.L_A(SY)$.

It remains to show that the usual boundary map $d = \sum_{i=o}^{p}(-1)^i d_i$ is a derivation with respect to μ. The proof is straightforward, and so only the main idea is presented here. If $a \otimes b$ is an element of $C_{p,q}L_A((SY) \vee (SY))$ then we want to show that

$$d(\mu(a \otimes b)) = \mu(da \otimes b) + (-1)^p \mu(a \otimes db).$$

Since $\mu = + \circ \mathfrak{f}^{-1}$, it suffices to show that

- $(d \otimes d)(\mathfrak{f}^{-1}(a \otimes b)) = \mathfrak{f}^{-1}(da \otimes b) + (-1)^p \mathfrak{f}^{-1}(a \otimes db)$
- $d(+(a \otimes b)) = +(d(a \otimes b))$

Here, we are using the fact that the boundary map for the diagonal complex $C..(L_A(SY) \vee (SY))$ is $d \otimes d$.

The second equation follows immediately from the fact that the fold map $+$ is defined simplicially, and so commutes with all face maps d_i.

Let σ be a (p, q)-shuffle, so that

$$(-1)^{sgn(\sigma)} s_{\sigma_{p+q}} \cdots s_{\sigma_{p+1}} a \otimes s_{\sigma_p} \cdots s_{\sigma_1} b$$

is a typical summand of $\mathfrak{f}^{-1}(a \otimes b)$. Then $d \otimes d$ applied to this element is

$$\sum_{i=0}^{p+q} (-1)^i sgn(\sigma) d_i s_{\sigma_{p+q}} \cdots s_{\sigma_{p+1}} a \otimes d_i s_{\sigma_p} \cdots s_{\sigma_1} b.$$

To show that $d \otimes d \circ \mathfrak{f}^{-1} = \mathfrak{f}^{-1}(1 \otimes d + (-1)^p d \otimes 1)$, we make repeated use of the simplicial identity

$$d_i s_j = \begin{cases} s_{j-1} d_i & j > i; \\ 1 & i = j \text{ or } i = j+1; \\ s_j d_{i-1} & i > j+1. \end{cases}$$

Choose $0 < i \leq p+q$. Suppose that $i \in \{\sigma_{p+1}, \ldots, \sigma_{p+q}\}$. We have two cases. If $i-1$ is in $\{\sigma_1, \ldots, \sigma_p\}$, then let $i = \sigma_k$ for some $p+1 \leq k \leq p+q$ and let $i-1 = \sigma_j$ for some $1 \leq j \leq p$. Then

$$\begin{aligned} &d_i s_{\sigma_{p+q}} \cdots s_{\sigma_{p+1}} a \otimes d_i s_{\sigma_p} \cdots s_{\sigma_1} b \\ &\quad = s_{\sigma_{p+q}-1} \cdots s_{\sigma_{k+1}-1} s_{\sigma_{k-1}} \cdots s_{\sigma_{p+1}} a \otimes s_{\sigma_p -1} \cdots s_{\sigma_{j+1}-1} s_{\sigma_{j-1}} \cdots s_1 b \\ &\qquad = d_i s_{\omega_{p+q}} \cdots s_{\omega_{p+1}} a \otimes d_i s_{\omega_p} \cdots s_{\omega_1} b \end{aligned}$$

where ω is the (p,q)-shuffle with $\omega = \tau\sigma$ and τ is the transposition which exchanges σ_k and σ_j. Clearly, $sgn(\sigma) = -sgn(\omega)$, so that these elements cancel. In this case, d is trivially a derivation with respect to $\mathfrak{f}^{-1}$.

If i and $i-1$ are both in $\{\sigma_{p+1}, \ldots, \sigma_{p+q}\}$, let $\sigma_k = i$ and $\sigma_{k-1} = i-1$ for some $p+1 < k \leq p+q$. There exists $1 \leq j \leq p$ with $\sigma_j < i-1 < i < \sigma_{j+1}$. So

$$\begin{aligned} &d_i s_{\sigma_{p+q}} \cdots s_{\sigma_{p+1}} a \otimes d_i s_{\sigma_p} \cdots s_{\sigma_1} b \\ &\quad = s_{\sigma_{p+q}-1} \cdots s_{\sigma_{k+1}-1} s_{\sigma_{k-1}} \cdots s_{\sigma_1} a \otimes s_{\sigma_p -1} \cdots s_{\sigma_{j+1}-1} s_{\sigma_j} \cdots s_1 d_{i-j} b \\ &\qquad = s_{\sigma'_{p+q-1}} \cdots s_{\sigma'_{p+1}} a \otimes s_{\sigma'_p} \cdots s_{\sigma'_1} d_{i-j} b \end{aligned}$$

where σ' is a $(p, q-1)$-shuffle. To ensure that this is in the image of $(-1)^p \mathfrak{f}^{-1}(1 \otimes d)$, we must make sure that this has the correct sign. So far, we haven't changed the original sign $(-1)^i sgn(\sigma)$. As in [**?**], we have $\mathrm{sgn}(\sigma) = (-1)^{\epsilon(\sigma)}$ where

$$\epsilon(\sigma) = \sum_{i=1}^{p} \sigma_i - (i-1).$$

By comparison,

$$\epsilon(\sigma') = \sum_{i=1}^{p} \sigma'_i - (i-1) = \sum_{i=1}^{j} \sigma_i - (i-1) + \sum_{i=j+1}^{p} (\sigma_i - 1) + (i-1) = \epsilon(\sigma) + (j-p)$$

This implies that $(-1)^p(-1)^{i-j} sgn(\sigma') = (-1)^i sgn(\sigma)$. Together with our previous calculation, we can conclude that this summand is in the image of $(-1)^p \mathfrak{f}^{-1}(1 \otimes d)$.

The cases $i = 0$, and $i \in \{\sigma_1, \ldots, \sigma_q\}$ are treated similarly. Reversing the calculation, it is easy to show that the image of summands of $\mathfrak{f}^{-1} d \otimes 1 +$

$(-1)^p \mathfrak{f}^{-1}(1\otimes d)$ are in the image of $(d\otimes d)\mathfrak{f}^{-1}$ as well. This verifies that d is a derivation with respect to μ.

□

7 Eulerian idempotent decomposition

In this section it is essential that k is a field of characteristic 0 and A is a commutative k-algebra. In a commutative Hopf algebra $\mathcal{H}$, there is an operation on the self maps of $\mathcal{H}$ called the convolution. Let μ be the multiplication map on $\mathcal{H}$ and Δ be the comultiplication. The convolution of two maps f and g is

$$f * g = \mu \circ (f \otimes g) \circ \Delta.$$

The convolution operator is an associative operation, so that $f * g * h$ is well defined. When $\mathcal{H}$ is $C_*L_A \circ S^1$, the r-th λ-operation is defined to be $\lambda^r = id^{*r}$ (this differs from the original definition of λ^r of [**?**] and [**?**] but it is equivalent, see [**?**]). Let $\overline{\lambda}^r = C_*L_A(\phi^r) \circ D(r)$, where $\phi^r : \mathrm{sd_r} S^1_\cdot \to S^1_\cdot$ is the map of Example **??**. Notice that $\overline{\lambda}^r$ is a self map of $C_*L_A \circ S^1$. From Example 3.8 of [**?**] $\overline{\lambda}^r = \lambda^r$ (up to a sign). We can extend the λ-operations to $C_*L_A \circ (SY)$ by setting $\overline{\lambda}^r = C_*L_A(\phi^r \wedge 1) \circ D(r)$.

Because λ^r is defined on the chain level, it induces a self map of HH_*^{SY}. Since $HH_*^{SY}(A)$ is a commutative graded Hopf algebra, there exist Eulerian idempotents $e^{(i)}$ with the relationship for $n \geq 1$

$$\lambda^r|_{HH_n^{SY}(A)} \quad = \quad r e_n^{(1)} + \cdots + r^n e_n^{(n)} \tag{6}$$

where $e_n^{(i)} = e^{(i)}|_{HH_n^{SY}(A)}$ (see [**?**]).

Theorem 7.1. *Let Y be any simplicial finite pointed set. Then $HH_*^{SY}(A)$ decomposes as*

$$HH_n^{SY}(A) = e_n^{(1)} HH_n^{SY}(A) \oplus \cdots \oplus e_n^{(n)} HH_n^{SY}(A)$$

where $e_n^{(r)} HH_n^{SY}(A)$ is the n-th homology of the chain complex $e_n^{(r)} F_A \circ (S^1 \wedge Y)$.

Proof. First observe that by Loday ([**?**] Proposition 4.5.9), the idempotents commute with the boundary of the Hochschild complex, which is induced by the face maps of $S^1_\cdot$. Since $\wedge$ is a bifunctor, the idempotents will also commute with the boundary of the higher Hochschild chain complex. We have

$$e_n^{(1)} + \cdots + e_n^{(n)} = id,$$

so that the Eulerian idempotents split $L_A \circ (SY)_n$ into

$$e_n^{(1)} L_A \circ (SY)_n \oplus \cdots \oplus e_n^{(n)} L_A \circ (SY)_n.$$

Taking homology, this yields the desired decomposition. □

The following Corollary is the special case $Y = S^{d-1}$.

Corollary 7.2. *There is a decomposition of higher Hochschild homology given by*

$$HH_n^{S^d}(A, M) = e_n^{(1)} HH_n^{S^d}(A, M) \oplus \cdots \oplus e_n^{(n)} HH_n^{S^d}(A, M).$$

Observe that because of the relationship (1) given above, λ^r is multiplication by r^k on $e_n^{(k)} HH_n^{S^d}(A, M)$.

8 Agreement of the Decompositions

The goal of this section is to show that the operators Φ^r act like "eigenweights" on two decompositions of HH^{S^d}. The first decomposition is obtained by considering the Eulerian idempotents in §**??** and the second decomposition is given by Pirashvili in [**?**]. One important difference between the two decompositions is that by using the Eulerian idempotents, one finds decompositions of HH^{SY} for any simplicial set Y while the decomposition obtained by Pirashvili applies only to spheres, HH^{S^d}. We will show that when $Y_\cdot = S^{d-1}_\cdot$, the first decomposition agrees with the second decomposition. We will review Pirashvili's decomposition here.

Let B_* be the chain complex in $\mathrm{Func}(\Gamma, \mathrm{V_k})$ defined by $B_n(m) = k[(S^d_m)^{[n]}]$. Pirashvili discovered a hyperhomology spectral sequence obtained from $L_A \otimes_{k,\Gamma} B_*$ by taking a projective resolution $(L_A)_*$ of L_A in the category $\mathrm{Func}(\Gamma, \mathrm{V_k})$. This spectral sequence exists for any simplicial set, but in the case of spheres, Pirashvili shows that the spectral sequence collapses at the E^2 page:

Proposition 8.1 (Proposition 1.6 of [**?**])**.** *There is a first quadrant spectral sequence with*

$$E^2_{p,q} = \mathrm{Tor}^\Gamma_p(H_q(B_*), L_A)$$

converging to $H_{p+q}(B_* \otimes_{k,\Gamma} L_A)$. *This spectral sequence collapses at the* E^2 *page and one has the decomposition*

$$H_n(B_* \otimes_{k,\Gamma} L_A) \simeq \oplus_{p+dj=n} \mathrm{Tor}^\Gamma_p(H_{dj}(B_*), L_A).$$

The bi-functor Tor^Γ_p refers to the derived functors of the tensor product $\otimes_{k,\Gamma}$.

We want to consider the image of Φ^r on this decomposition. To do so, we must first understand $\Phi^r(H_*(B_*))$.

Lemma 8.2. *The map* Φ^r *acts on* $\big(\mathrm{Tor}^\Gamma_p(H_{dj}(B_*), L_A)\big)$ *by multiplication by* r^j.

Proof. For any n, $H_*(B_*)[n] = H_*((S^d)^{\times[n]})$. By the Künneth theorem, $H_*((S^d)^{\times[n]}) = H_*(S^d)^{\otimes n}$. Let $H_*(S^d) = k \oplus kx$ where x is a generator of dimension d. Then

$$H_*(S^d)^{\otimes n} = \bigoplus_{j=0}^{n} \big(\oplus_{(\binom{n}{j})} kx \otimes \cdots \otimes kx \big)$$

where $kx \otimes \cdots \otimes kx$ is the j-fold tensor product. Consequently, $H_*((S^d)^{\times[n]}) = 0$ if $* \neq dj$ for some j and $H_{dj}((S^d)^{\times[n]}) = \bigoplus_{(\binom{n}{j})} kx \otimes \cdots \otimes kx$.

The map Φ^r acts on $H_*(S^d)$ by multiplication by r on the top dimensional homology group since Φ^r is the r-fold covering map on the first coordinate of $(S^1 \wedge S^{d-1})$.. Then Φ^r acts as the identity on k and by multiplication by r on kx. We see that Φ^r applied to $H_{dj}((S^d)^{\times[n]})$ acts by multiplication by r^j. This action does not depend on n, so actually Φ^r acts by multiplication by r^j on $(H_{dj}(B_*))$.

Since Φ^r is defined on the chain level, this computation passes to the Tor groups so that we obtain the result. □

Theorem 8.3. *The two decompositions of higher order Hochschild homology agree. That is,*

$$e^{(j)}_n HH^{S^d}_n(A) = \mathrm{Tor}^\Gamma_p(H_{dj}(B_*), F_A).$$

Proof. The map $\lambda^r = \Phi^r$ plays the role of a linear operator on the k-vector space $HH_n^{S^d}(A)$. Since k is a field, the image of Φ^r determines its subspace decomposition. The result follows. $\square$

Notice that this means that $e_n^{(j)} HH_n^{S^d}$ vanishes whenever $j > (d/n)$. Joint work with R. McCarthy indicates that an explanation of this may be given by looking at the decomposition of higher order Hochschild homology in another way (using Goodwillie calculus). We explain this result in a subsequent paper.

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