On vanishing Tate cohomology and decompositions in Goodwillie calculus

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Abstract

Our main result is that if F is a functor from a pointed category $\mathcal{C}$ to spectra, the Goodwillie tower of F evaluated at X splits rationally when evaluated on the co-H-objects of $\mathcal{C}$. In other words, rationally the degree n approximation to F is the product of the first n derivatives of F. We demonstrate that the derivatives of $F(X)$ in this case are easy to identify. When the Goodwillie tower converges, the splitting of the Goodwillie tower gives a decomposition of $F(X)$. We use this to recover the rational decompositions of Hochschild and higher Hochschild homology [P00], [L98], [GS87]. Finally, we extend the main theorem to include dual calculus to recover the Poincaré-Birkhoff-Witt theorem.

1 Introduction

Let F be a functor from any pointed category $\mathcal{C}$ to any abelian category. As an application of [G90], [G92] and [G02], Johnson-McCarthy provide a

Goodwillie calculus theory for such functors ([JM4]). There is a tower

$$\begin{tikzcd}
 & \vdots \arrow[d, "q_{n+1}"] \\
F(X) \arrow[ur] \arrow[r, "p_n"] \arrow[dr, "p_{n-1}"] & P_nF(X) \arrow[d, "q_n"] \\
 & P_{n-1}F(X) \arrow[d] \\
 & \vdots
\end{tikzcd}$$

of universal degree n approximations $P_nF(X)$ to $F(X)$. The derivatives $D_nF(X)$ of $F(X)$ are the fibers of the maps q_n. This model of calculus can be extended to functors F from $\mathcal{C}$ to spectra, as in [McC02]. In this paper, we provide a criterion on X for the Goodwille tower of F, in the sense of Johnson and McCarthy, to split rationally when evaluated at X. In particular, we show that if X is an object with a "comultiplication" map ∇ from X to the coproduct $X \vee X$ – in other words, if X is a co-H-object in $\mathcal{C}$ – then rationally the first map of the fiber sequence

$$D_nF(X) \longrightarrow P_nF(X) \xrightarrow{q_n} P_{n-1}F(X)$$

has a splitting map. This results in the main theorem of the paper:

Theorem 1.1. *If F is a homotopy functor and X is a co-H-object of $\mathcal{C}$ then*

$$P_nF(X) \simeq \prod_{i=1}^{n} D_iF(X)$$

whenever the Tate cohomology vanishes. Consequently $P_\infty F(X) \simeq \prod_{n\geq 1} D_nF(X)$.

The proof of the main theorem is constructive. We first reduce to the case where F is a degree n functor. In that case, $P_nF(X) \simeq F(X)$ and we construct maps

$$D_nF(X) \longrightarrow F(X) \xrightarrow{F(\nabla)} F(\bigvee_n X) \longrightarrow D_nF(X)\ . \tag{1}$$

Most of the proof goes into showing that this composite is induced by the norm map (§3), which in turn induces the Tate map. Thus, the splitting occurs whenever the Tate map is invertible, which happens when the Tate cohomology vanishes.

The constructed map (1) is also invertible when F takes values in rational spectra. We also state the hypotheses of the theorem for this situation.

This theorem is now due to [K]. However, our proof uses different methods and is independent of this work.

Since X is a co-H-object, it is equipped with "covering maps" Φ^r defined by

$$X \xrightarrow{\nabla^r} \bigvee_r X \xrightarrow{+} X$$

where $+$ is the fold map. An important consequence of the proof of the main theorem is that the maps Φ^r induce multiplication by r^n on $D_nF(X)$, and hence can be used to identify the layers of $F(X)$. We use this idea to show that the known rational decompositions of (higher) Hochschild homology ([GS87], [L98], [P00], [B]) are actually splittings of the Goodwillie tower of the forgetful functor from augmented commutative k-algebras to k-modules. We then compute the layers of the tower associated to higher Hochschild homology, and show that they are just suspensions of the layers of the tower associated to Hochschild homology.

We also prove a version of the main theorem for dual calculus, which can be thought of as formally dual to calculus. A brief description of this theory is in section 6. In this case, we examine the H-objects of $\mathcal{C}$. Using the dual version of our main theorem, we are able to show that rationally, the Poincaré-Birkhoff-Witt theorem is a decomposition of the universal enveloping algebra of a free Lie algebra into the dual layers of a dual Goodwillie tower associated to it.

This paper is organized as follows. In section 2 we give a very brief summary of the prerequisites in Goodwillie calculus. We pay particular attention to the cross effects of a functor, which are essential ingredients of the proof. A more detailed account of the calculus tower, and in particular the degree n approximation of a functor, can be had from [JM4] and [M] (for the case where the target category is spectra) or [McC02]. In section 3 we describe the norm map and its relationship to the Tate map, which will be essential in constructing the splitting map to $D_nF(X) \to P_nF(X)$. In section 4 we make explicit our definition of co-H-objects. We also check that the properties we require of F are preserved by cr_n, D_n and P_n. Section 5 is the statement and proof of the main theorem. We conclude by giving an extensions of the main theorem; to dual calculus (section 6).

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2 Preliminaries in Goodwillie Calculus

In a now famous series of three papers [G90], [G92] [G02], Goodwillie built a theory of calculus for homotopy functors of spaces or spectra. This theory asserts that any homotopy functor can be approximated by a kind of Taylor series of polynomial functors, where a functor is polynomial of degree n if it behaves nicely on $\mathbf{n}$-cubical diagrams. Let $\mathbf{n} = \{1, \ldots, n\}$ be a finite set of n elements. Let$P(\mathbf{n})$ be the category whose objects are the subsets of $\mathbf{n}$ and whose morphisms are ordered inclusions. An $\mathbf{n}$-cube is a (possibly contravariant) functor from $P(\mathbf{n})$ to $\mathcal{C}$. If the terminal entry is homotopy equivalent to the homotopy colimit of the functor evaluated on $P(\mathbf{n}) - \mathbf{n}$, then the cubical diagram is co-Cartesian. A cubical diagram is strongly co-Cartesian if every face of the diagram is co-Carteisan. Similarly, if the initial entry is equivalent to the homotopy limit of the functor evaluated on $P(\mathbf{n}) - \emptyset$, then the diagram is Cartesian. An n-excisive functor is one which takes strongly co-Cartesian $\mathbf{n}$-cubical diagrams to Cartesian diagrams. When $n = 2$, this means that the functor takes push-out diagrams to pull-back diagrams.

Following Goodwillie's work, Johnson-McCarthy [JM4] outlined a related theory of calculus for functors from a basepointed cateogory C with all finite coproducts to the category of chain complexex. In this case, a functor was polynomial of degree n if it satisfied a kind of homotopy additivity. The additivity property is measured by the cross effect, which is the total fiber of a particular $\mathbf{n}$-cube.[1] Let $\vee$ denote the coproduct in $\mathcal{C}$. Let $g : \mathbf{n} \to \mathrm{obj}(\mathcal{C})$ be a function. The function g generates a list of objects $X_1, \ldots, X_n$ of $\mathcal{C}$. Define an $\mathbf{n}$-cube χ_g to be the contravariant functor which is given on a subset S of

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[1] The cross effects were also used by Goodwillie to classify the n-homogeneous functors.

n by

$$\chi_g(S) = \bigvee_{c \notin S} g(c)$$

and which takes an inclusion $S \hookrightarrow S'$ to the projection

$$\bigvee_{d \notin S} g(d) \to \bigvee_{c \notin S'} g(c).$$

By convention, $\chi_g(\mathbf{n}) = *$ (the base point of $\mathcal{C}$). Let F be a functor from $\mathcal{C}$ to chain complexes. Let $P_0(\mathbf{n})$ be the subcategory of $P(\mathbf{n})$ consisting of non-empty subsets and let $X_i = g(i)$.

Definition 2.1. The n-th cross effect, $\mathrm{cr}_n F(X_1, \ldots, X_n)$, is the fiber of the map

$$\mathrm{holim}_{S \in P(\mathbf{n})} F(\chi_g(S)) \to \mathrm{holim}_{S \in P_0(\mathbf{n})} F(\chi_g(S)).$$

Note that $\mathrm{holim}_{S \in P(\mathbf{n})} F(\chi_g(S)) = F(\vee_{i=1}^n X_i)$.

When $n = 2$, the cross effect is the total fiber of the square diagram

$$\begin{tikzcd}
F(X_1 \vee X_2) \arrow[r] \arrow[d] & F(X_2) \arrow[d] \\
F(X_1) \arrow[r] & F(*)
\end{tikzcd}$$

making it clear that a functor is degree 2 if and only if $F(X_1 \vee X_2) \simeq F(X_1) \times F(X_2)$. That is, the degree 2 functors are the additive functors. In the case where $X_1 = \cdots = X_n = X$, we denote $cr_n F(X, \ldots, X)$ more concisely as just $\perp_n F(X)$.

In the Johnson-McCarthy theory of calculus, a functor is then degree n if $cr_n F \simeq 0$. Indeed, they show that $\perp_n F$ forms a cotriple, and define the polynomial degree n approximation to F by

$$P_n F(X) := hocolim(\ \perp_n^* F(X) \xrightarrow{+} F(X)\ )$$

where the map $+$ is induced by the fold map.

In this paper we will use the Johnson-McCarthy model of calculus for homotopy functors adjusted for functors F from a pointed simplicial model category C to a category $\mathcal{S}$ of rational spectra. We prefer the category of functors with smash products (FSP's) (see e.g. [McC] for details) as a model for spectra. Rational spectra are modules over the Eilenberg-MacLane spectrum $H\mathbb{Q}$. Viewed as an FSP, this is the spectrum given by

$$H\mathbb{Q}(S) = \mathbb{Q}[S]/\mathbb{Q}[*]$$

where S is a pointed set, $\mathbb{Q}[S]$ is the free rational algebra on S and $*$ is the basepoint. One can check that this recovers the usual Eilenberg-MacLane spectrum. However, any model of rational spectra will do. The Johnson-McCarthy theory for functors in this setting has been used in e.g. [McC]. If $\mathcal{C}$ is the category of either topological spaces or spectra, then [M] shows that for functors satisfying mild conditions (e.g. commuting with realizations) then the Johnson-McCarthy model and Goodwillie's model of calculus are in fact equivalent.

We finish this section with a few more details about our fundamental building blocks, the cross effects. Where proofs are not included, the reader should refer to [JM4] for detailed statements with proof.

The cross effect $\mathrm{cr}_n F(X_1, \ldots, X_n)$ is an n multi-variable functor from $C^{\times n}$ to $\mathcal{S}$. It is often more helpful to think of $\mathrm{cr}_n F(X_1, \ldots, X_n)$ as the "total fiber" of the cube $F(\chi_g)$. That is, the object obtained by repeatedly taking parallel fibers of $F(\chi_g)$. For example, using this viewpoint it is easier to see that the n-th cross effect is related to the $(n+1)$-st cross effect by (e.g.)

$$\begin{aligned} &\mathrm{cr}_{n+1} F(X_1, X_2, \ldots, X_{n+1}) \\ &\qquad\qquad\qquad \times \mathrm{cr}_n F(X_1, X_3, \ldots, X_{n+1}) \\ &\qquad\qquad\qquad \times \mathrm{cr}_n F(X_2, X_3, \ldots, X_{n+1}) \\ &\qquad\qquad\qquad\qquad\qquad \simeq \mathrm{cr}_n F(X_1 \vee X_2, X_3, \ldots, X_{n+1}). \quad (2) \end{aligned}$$

One of the most important consequences of being degree n is that $\mathrm{cr}_n F$ is an n multivariable functor which is linear in each variable. That is, e.g.

$$\mathrm{cr}_n F(X \vee Y, X_2, \ldots, X_n)$$

is equivalent to

$$\mathrm{cr}_n F(X, X_2, \ldots, X_n) \times \mathrm{cr}_n F(Y, X_2, \ldots, X_n).$$

Lemma 2.2. *If F is a degree n functor, then there exists a natural equivalence*

$$\perp_n F(\bigvee_{i=1}^{m} X_i) \;\simeq\; \prod_{\alpha \in \mathrm{Hom}(n,m)} \mathrm{cr}_n F(X_{\alpha(1)}, \ldots, X_{\alpha(n)}) \qquad (3)$$

Proof. This is a consequence of the fact that for degree n functors, $\mathrm{cr}_n F$ is linear in each variable. $\square$

Remark 2.3. Let $u_k : \vee_{i=1}^m X_i \to X_k$ $(1 \leq k \leq m)$ be the map which is the identity on the X_k component of $\vee_{i=1}^m X_i$ and which sends X_j to the basepoint for $j \neq k$. The weak equivalence of lemma 2.2 can be represented by the map $\omega : \perp_n F(\vee_{i=1}^m X_i) \to \prod_{\alpha \in \mathrm{Hom}(n,m)} \mathrm{cr}_n F(X_{\alpha(1)}, \ldots, X_{\alpha(n)})$ defined by

$$\omega = \Big(\prod_{\alpha \in \mathrm{Hom}(n,m)} \mathrm{cr}_n F(u_{\alpha(1)}, \ldots, u_{\alpha(n)}) \Big) \circ \Delta$$

where Δ is the diagonal map $\perp_n F(\vee_{i=1}^m X_i) \to \prod_{\mathrm{Hom}(n,m)} \perp_n F(\vee_{i=1}^m X_i)$. Let $c_k : X_k \to \vee_{i=1}^m X_i$ be the dual map to u_k which includes the k-th summand. Keeping in mind that coproducts and products in $\mathcal{S}$ agree, we have a homotopy inverse to ω given by the composition of

$$\coprod_{f \in \mathrm{Hom}(n,m)} \perp_n F(X) \xrightarrow{\coprod_{f \in \mathrm{Hom}(n,m)} \perp_n F(c_{f(1)}, \ldots, c_{f(n)})} \coprod_{f \in \mathrm{Hom}(n,m)} \perp_n F(\underset{m}{\vee} X) \xrightarrow{+} \mathrm{cr}_n F(\underset{m}{\vee} X)$$

where $+$ is the fold (or sum) map which is always defined on the coproduct.

Lemma 2.4. *If F is a degree n functor, then there exists a natural equivalence*

$$\mathrm{cr}_n(\perp_n F)(X_1, \ldots, X_n) \ \simeq \ \prod_{\sigma \in \Sigma_n} \mathrm{cr}_n F(X_{\sigma(1)}, \ldots, X_{\sigma(n)}). \qquad (4)$$

Proof. We will provide here a computational proof relying on 2.1. However, to get a better feel for this lemma apply lemma 2.2 to the cube corresponding to $\mathrm{cr}_n \perp_n F$ for $n = 2$ or $n = 3$.

Let $g(i) = X_i$ and let cS denote the complement in $\mathbf{n}$ of a subset S of $\mathbf{n}$. Using the definition of the cross effect and lemma 2.2, we compute that

$$\mathrm{cr}_n \perp_n F(X_1, \ldots, X_n) = \mathrm{fiber}\{\underset{S \in P(\mathbf{n})}{\mathrm{holim}} \perp_n F(\underset{c \in cS}{\vee} X_c) \to \underset{S \in P_0(\mathbf{n})}{\mathrm{holim}} \perp_n F(\underset{c \in cS}{\vee} X_c)\}$$

is equivalent to the fiber of

$$\underset{S \in P(\mathbf{n})}{\mathrm{holim}} \prod_{\sigma \in \mathrm{Hom}(\mathbf{n}, cS)} \mathrm{cr}_n F(X_{\sigma(1)}, \ldots, X_{\sigma(n)}) \to \underset{S \in P_0(\mathbf{n})}{\mathrm{holim}} \prod_{\sigma \in \mathrm{Hom}(\mathbf{n}, cS)} \mathrm{cr}_n F(X_{\sigma(1)}, \ldots, X_{\sigma(n)}).$$

The first homotopy limit is just $\prod_{\sigma \in \mathrm{Hom}(\mathbf{n},\mathbf{n})} \mathrm{cr}_n F(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$ (since this limit diagram has an initial object) and the second homotopy limit is $\prod_{\sigma \in \mathrm{Hom}_0(\mathbf{n},\mathbf{n})} \mathrm{cr}_n F(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$ where $\mathrm{Hom}_0(\mathbf{n},\mathbf{n})$ are the non-bijective set maps from $\mathbf{n}$ to itself (one can easily check that this satisfies the universal property). So this is equivalent to

$$\begin{aligned} \mathrm{fiber}\{ \prod_{\sigma \in \mathrm{Hom}(\mathbf{n},\mathbf{n})} \mathrm{cr}_n F(X_{\sigma(1)}, \ldots, X_{\sigma(n)}) \to & \prod_{\sigma \in \mathrm{Hom}_0(\mathbf{n},\mathbf{n})} \mathrm{cr}_n F(X_{\sigma(1)}, \ldots, X_{\sigma(n)})\} \\ = & \prod_{\sigma \in \mathrm{Bij}(\mathbf{n},\mathbf{n})} \mathrm{cr}_n F(X_{\sigma(1)}, \ldots, X_{\sigma(n)}) \end{aligned}$$

where $\mathrm{Bij}(\mathbf{n},\mathbf{n}) := \Sigma_n$ is the group of bijective maps. □

Remark 2.5. Using remark 2.3, one can represent the weak equivalence of 2.4 by a map $\omega' : \mathrm{cr}_n \bot_n F(X) \to \prod_{\mathrm{Bij}(n,n)} \bot_n F(X)$ defined by

$$\omega' = \big(\prod_{\alpha \in \mathrm{Bij}(n,n)} \mathrm{cr}_n F(u_{\alpha(1)}, \ldots, u_{\alpha(n)}) \big) \circ \Delta \circ i$$

where $i : \mathrm{cr}_n \bot_n F(X) \to \bot_n F(\vee_n X)$ is the natural map. This is accomplished by applying the map ω to $\bot_n F(\vee_{c \in cS} X_c)$ for each $S \in P(\mathbf{n})$ (or $P_0(\mathbf{n})$).

Theorem 2.6. *(Lemma 3.9 [JM4]) If F is a degree n functor from $\mathcal{C}$ to $\mathcal{S}$ then $D_n F \simeq (\mathrm{cr}_n F)_{h\Sigma_n}$.*

Theorem 2.7. *(Remark 2.8 [JM4]) If F is a degree n functor from $\mathcal{C}$ to $\mathcal{S}$ then $F \simeq P_n F$.*

By using cofibers instead of fibers in the definition of the cross effect, we obtain another cross effect, called the co-cross effect, $\widetilde{\mathrm{cr}}_n$. Let $P_1(\mathbf{n})$ denote the full subcategory of $P(\mathbf{n})$ consisting of proper subsets of $\mathbf{n}$. Let $\widetilde{\chi}$ be the covariant functor from $P(\mathbf{n})$ to $\mathcal{C}$ with $\widetilde{\chi}(S) = \vee_{c \in S} g(c)$ and which takes inclusions in $P(\mathbf{n})$ to inclusions in $\mathcal{C}$.

Definition 2.8. We define the co-cross effect $\widetilde{\mathrm{cr}}_n F(X_1, \ldots, X_n)$, to be the cofiber of the map

$$\mathrm{holim}_{S \in P_1(\mathbf{n})} F(\widetilde{\chi}_g(S)) \to \mathrm{holim}_{S \in P(\mathbf{n})} F(\widetilde{\chi}_g(S)).$$

Again, we have $\widetilde{\perp}_n F(X) := \widetilde{cr}_n F(X, \ldots, X)$ for the diagonal functor. In the category of spectra, this should cause no confusion. Since cofibration and fibration sequences are equivalent in $\mathcal{S}$, we have that $\mathrm{cr}_n F \simeq \widetilde{\mathrm{cr}}_n F$. There is a convenient map

$$F(\vee_n X) \to \widetilde{\perp}_n F(X)$$

(dual to the map ρ) which we will need.

Lemma 2.9. *If F is a degree n functor, then there exist natural equivalences*

$$\begin{aligned} \widetilde{\perp}_n F(\bigvee_n X) &\simeq \coprod_{\mathrm{Hom}(n,n)} \widetilde{\perp}_n F(X) && (5) \\ \widetilde{\perp}_n \widetilde{\perp}_n F(X) &\simeq \coprod_{\Sigma_n} \widetilde{\perp}_n F(X). && (6) \end{aligned}$$

Proof. This is the analogue of lemmas 2.2 and 2.4, and the proof is straightforward. ☐

3 The Tate Map

If a finite group G acts on a k-module M, there is a natural map $t : M_G \to M^G$ induced by the norm map. The norm map is constructed using the diagonal Δ, the action of g for each $g \in G$, and the addition map $+$ as follows:

$$\begin{tikzcd} M \arrow[r, "\Delta"] \arrow[rrr, bend left, "t'"] & \oplus_{|G|} M \arrow[r, "\oplus_{g \in G} g"] & \oplus_{|G|} M \arrow[r, "+"] & M \end{tikzcd}$$

We can extend this to a map $t : M_G \to M^G$ by making two observations. First, t' extends in the following diagram (where p is the projection map onto the orbits)

$$\begin{tikzcd} M \arrow[r, "t'"] \arrow[d, "p"'] & M \\ M_G \arrow[ur, dashed] & \end{tikzcd}$$

since for each $g \in G$, $t'(m) = t'(gm)$. Thus t' extends to M_G since it is well defined on each orbit. Second, note that the image of t' actually lands in M^G

since $t'(m) = \sum_{g\in G} gm$ so that for any $h \in G$,

$$ht'(m) = h\sum_{g\in G} gm = \sum_{g\in G} hgm = \sum_{g\in G} gm = t'(m)$$

since G is finite. Thus we have a map t which factors t' as:

$$\begin{tikzcd}
M \arrow[r, "t'"] \arrow[d, "p"'] & M \\
M_G \arrow[r, "t"] & M^G \arrow[u, hook, "i"']
\end{tikzcd}$$

where i is the inclusion of the fixed points. The map t is an equivalence whenever the order of G is invertible. In fact, the inverse is given by $p \circ i$ and we have $p \circ i \circ t[m] = |G|[m]$.

We wish to extend this map to the category $\mathcal{S}$ of $H\mathbb{Q}$-modules. Such constructions can be found in e.g. [WW], [GM], [McC]. Here we provide a brief overview. Suppose that E is an $H\mathbb{Q}$-module with an action of a finite group G. This time, we must use the weak equivalence between the product and the coproduct. The following diagram defines T':

$$\begin{tikzcd}
E \arrow[r, "\Delta"] & \prod_{g\in G} E \arrow[r, "\prod_{g\in G} g\times(-)"] & \prod_{g\in G} E & \coprod_{g\in G} E \arrow[l, hook', "\simeq"'] \arrow[r, "+"] & E
\end{tikzcd}$$

Motivated by this diagram, we make the following definition

Definition 3.1. Let I be an indexing set and $\{X_i\}_{i\in I}$ be a collection of object of $\mathcal{S}$. Let $f_{i,j} : X_i \to X_j$ be maps in $\mathcal{S}$ for each $i, j \in I$ and $g_{i,j} : X_i \xleftarrow{\simeq} X_j$ be weak equivalences in $\mathcal{S}$ for each $i, j \in I$. A *weak map* in $\mathcal{S}$ from $X_{i_m} \to X_{i_n}$ is a collection of objects $X_{i_m}, X_{i_{m+1}}, \ldots, X_{i_{m+k}} = X_{i_n}$ and arrows $f_{i,j}$ and $g_{i,j}$ such that for any adjacent pair $X_{i_{m+j}}$, $X_{i_{m+j+1}}$ of objects in the collection, there is either a map $f_{i_{m+j},i_{m+j+1}}$ or a weak equivalence $g_{i_{m+j},i_{m+j+1}}$ between them.

A typical weak map might look like a zig-zag

$$X_i \xrightarrow{f_{i,j}} X_j \xleftarrow[\simeq]{g_{j,k}} X_k.$$

This weak map is denoted $X_i \overset{g_{j,k}^{-1}\circ f_{i,j}}{\dashrightarrow} X_j$, where the dashed line denotes that it's a weak map.

Note that each weak map has associated to it a map in the homotopy category. We say that a diagram of weak maps commutes if the corresponding diagram of maps in the homotopy category commutes. There is weak map T which extends the weak map T' to homotopy orbits and homotopy fixed points just as t extended t' in the case of modules. This weak map is called the Tate (weak) map. A construction of t can be found in [McC].

We will need the following properties of the Tate map.

Lemma 3.2. *The Tate map is the restriction of the Norm map to homotopy orbits. That is, the following diagram of weak maps commutes:*

$$\begin{tikzcd}
E \arrow[r, "\Delta"] \arrow[d] & \prod_G E \arrow[r, "\prod_{g\in G} g"] & \prod_G E \arrow[r, dashed, "\simeq"] & \coprod_G E \arrow[r, "+"] & E \\
E_{hG} \arrow[rrrr, dashed, "T"] & & & & E^{hG} \arrow[u, hook]
\end{tikzcd}$$

where T refers to the composite that defines the Tate map and the composite of the top (weak) maps is the norm map.

Proposition 3.3. *Let Σ_n be the n-th symmetric group. The composition*

$$T_{h\Sigma_n} \xrightarrow{t} T^{h\Sigma_n} \longrightarrow T \longrightarrow T_{h\Sigma_n}$$

is a rational equivalence.

4 The category of homotopy comonoids

Let $\mathcal{C}$ be a basepointed simplicial model category.

Definition 4.1. Let X be a cofibrant object of $\mathcal{C}$. We say that X is a *co-H-object* of $\mathcal{C}$ if there exists a map $\nabla : X \to X \vee X$ which is coassociative up to homotopy, and counital up to homotopy with counit $c : X \to *$.

The co-H-objects are the comonoids of $\mathcal{C}$ (up to homotopy). They form a subcategory of $\mathcal{C}$ whose morphisms from X to Y are the maps of $f \in \mathcal{C}$

which make the following diagrams commute:

$$\begin{tikzcd}
X \arrow[r, "\nabla_X"] \arrow[d, "f"'] & X \vee X \arrow[d, "f \vee f"] \\
Y \arrow[r, "\nabla_Y"'] & Y \vee Y
\end{tikzcd} .$$

For any object X of $\mathcal{C}$, recall that the fold map, $+$, is the unique map which exists by the universal property defining the coproduct:

$$\begin{tikzcd}
& X \vee X \arrow[dd, "\exists!"', "+"] & \\
X \arrow[ur, "i_1"] \arrow[dr, "="'] & & X \arrow[ul, "i_2"'] \arrow[dl, "="] \\
& X &
\end{tikzcd}$$

where i_j is the inclusion into the j-th component of $X \vee X$. Since the fold map is unital with respect to the unit $u : * \to X$, we can think of the fold map as a unital multiplication with which $\mathcal{C}$ is already equipped.

The co-H-objects of $\mathcal{C}$ are exactly those whose comultiplication maps ∇ act like algebra maps with respect to the multiplication map defined by the fold map. In other words, for a co-H-object X the following diagram commutes up to homotopy:

$$\begin{tikzcd}
\vee_n X \arrow[r, "\vee_n \nabla^n"] \arrow[d, "+^n"'] & \vee_n(\vee_n X) \arrow[r, "\tau"] & \vee_n(\vee_n X) \arrow[d, "\vee_n +^n"] \\
X \arrow[rr, "\nabla^n"] & & \vee_n X
\end{tikzcd} . \tag{7}$$

If one arranges $\vee_{\mathrm{n}} \vee_{\mathrm{n}} X$ in an $n \times n$-array, one can think of the map τ as the transpose map.

The first examples of co-H-objects are co-H-spaces in the category of pointed topological spaces. In particular, the basepointed circle S^1 is a co-H-space. The map ∇ in this case is the pinch map which identifies the basepoint with its antipodal point. Since the circle is a co-H-space, so are all suspensions $\Sigma X = S^1 \wedge X$ using the map $\nabla \wedge 1$. For the main theorem, we will be considering a co-H-object X and a functor $F : \mathcal{C} \to \mathcal{S}$ which preserves weak equivalences. If F preserves weak equivalences, then when F is applied to diagram 7 the resulting diagram still commutes up to homotopy. The following lemma will allow us to use F, $\mathrm{cr}_n F$, P_n and $D_n F$ interchangeably with respect to diagrams which commute up to homotopy.

Lemma 4.2. *Let $F : \mathcal{C} \to \mathcal{S}$ be a functor which preserves weak equivalences. Then $\mathrm{cr}_{n+1}F$, $P_n F$ and $D_n F$ preserve weak equivalences between cofibrant objects.*

Proof. Recall (Definition 2.1) that $\mathrm{cr}_{n+1}F(X)$ is the fiber of

$$\mathrm{holim}_{P(\mathbf{n+1})} F(\chi_g) \to \mathrm{holim}_{P_0(\mathbf{n+1})} F(\chi_g).$$

We claim that if $\alpha : A \to B$ is a weak equivalence then $\mathrm{cr}_{n+1}F(\alpha) : \mathrm{cr}_{n+1}F(A) \to \mathrm{cr}_{n+1}F(B)$ is also a weak equivalence, as long as A and B are both cofibrant. This is true because if A and B are both cofibrant, then

$$\vee_k \alpha : \vee_k A \to \vee_k B$$

is again a weak equivalence for all $k \leq n+1$. Thus α induces a weak equivalence on each term $F(\chi_g)$ of the cube defining $\mathrm{cr}_{n+1}F$, and hence induces weak equivalences on $\mathrm{holim}_{P(\mathbf{n+1})} F(\chi_g)$ and $\mathrm{holim}_{P_0(\mathbf{n+1})} F(\chi_g)$. Then the map induced on the fiber is also a weak equivalence.

Each of the remaining constructions defining $P_n F$ and $D_n F$ is a homotopy construction involving only $\mathrm{cr}_{n+1}F$ and F.

□

Remark 4.3. We only require $\mathcal{C}$ to be a model category in order to provide us with the correct notion of "weak equivalences" in $\mathcal{C}$, and co-H-objects are only cofibrant to insure that diagrams involving co-H-objects which commute up to homotopy will still commute up to homotopy after $P_n F$ or other related functors have been applied. We can also define co-H-objects in categories $\mathcal{C}$ which are not model categories by requiring that all of the maps involved in the definition commute up to isomorphism. Since all functors preserve isomorphisms, lemma 4.2 is then unnecessary. For example, in the category of commutative algebras over k, the co-H-objects are almost the Hopf algebras. Notice that the co-H-objects only differ from Hopf algebras because they lack the antipodal map with which Hopf algebras are equipped.

5 The Splitting Theorem

The goal of this section is to prove the following:

Theorem 5.1. *If F is a functor from $\mathcal{C}$ to $\mathcal{S}$ and X is a co-H-object of $\mathcal{C}$ with a cocommutative comultiplication, then the fiber sequence*

$$D_nF(X) \xrightarrow{j} P_nF(X) \xrightarrow{q_n} P_{n-1}F(X)$$

splits on the homotopy category whenever the Tate map is an equivalence. Consequently,

$$P_\infty F(X) \simeq \prod_{n\geq 0} D_nF(X)$$

is a rational equivalence for each co-H-object of $\mathcal{C}$.

There are two situations in which the Tate map is an equivalence. First, if $\mathcal{S}$ is the category of *rational* spectra, then the Tate map is an equivalence. On the other hand, one can examine the Tate cohomology, which measures the failure of the Tate map being an equivalence.

Definition 5.2. [McC02] Let F be a homotopy functor from $\mathcal{C} \to \mathcal{S}$. Define the *$n$-th Tate cohomology of F at X* to be

$$Tate^n(F;X) := cofiber(\mathrm{cr}_nF(X)_{h\Sigma_n} \to \mathrm{cr}_nF(X)^{h\Sigma_n})$$

where the map from the homotopy orbits to the homotopy fixed points is the Tate map.

If the Tate cohomology is 0 for all n, then the Tate map is an equivalence.

Without loss of generality, we may assume F is a degree n functor by replacing F with P_nF, which is degree n by definition. This can be done since the following diagram from [JM4]:

$$\begin{tikzcd}
D_nP_nF \arrow[r] \arrow[d, "\simeq"', "p_nF"] & P_nP_nF \arrow[r] \arrow[d, "\simeq"', "p_nF"] & P_{n-1}P_nF \arrow[d, "\simeq"', "p_nF"] \\
D_nF \arrow[r] & P_nF \arrow[r] & P_{n-1}F
\end{tikzcd}$$

commutes and has weak equivalences as indicated (this replacement is why Lemma 4.2 is required).

The map j exists because D_nF is defined to be the fiber of the map q_n. We can reformulate j ([JM4]) in terms of cross effects. Since F is degree n, there are equivalences $D_nF(X) \simeq (\perp_n F(X))_{h\Sigma_n}$ and $P_nF(X) \simeq F(X)$. Thus, the above fiber sequence becomes

$$(\perp_n F(X))_{h\Sigma_n} \xrightarrow{j} F(X) \longrightarrow P_{n-1}F(X)$$

for each n. The map j is induced by the map $\rho : \perp_n F(X) \to F(\underset{n}{\vee} X)$ and the fold map in the following sense:

$$\begin{tikzcd}
\perp_n F(X) \arrow[rr, hook, "\rho"] \arrow[dd, two heads] & & F(\underset{n}{\vee} X) \arrow[dl, two heads] \arrow[dd, "F(+)"] \\
& F(\underset{n}{\vee} X)_{h\Sigma_n} \arrow[dr] & \\
(\perp_n F(X))_{h\Sigma_n} \arrow[ur, "\rho_{h\Sigma_n}"] \arrow[rr, hook, "j"] & & F(X)
\end{tikzcd} \tag{8}$$

There is a Σ_n action on $F(\vee_n X)$ by permuting the X's. However, the fold map $F(+) : \; F(\vee_n X) \longrightarrow F(X)$ has $F(+)(F(\sigma \cdot \vee_n X)) = F(+)(F(\vee_n X))$. Hence $F(+)$ factors through homotopy orbits. Since the inclusion ρ is Σ_n-equivariant, ρ extends to a map $\rho_{h\Sigma_n}$ on orbits. The map j is the map which we seek to provide a splitting for.

When X is a co-H-object, we may extend this diagram:

$$\begin{tikzcd}
\perp_n F(X) \arrow[r, hook, "\rho"] \arrow[d] & F(\underset{n}{\vee} X) \arrow[r, "\tau \circ F(\underset{n}{\vee} \nabla^n)"] \arrow[d, "F(+^n)"] & F(\underset{n}{\vee}(\underset{n}{\vee} X)) \arrow[d, "F(\underset{n}{\vee} +^n)"] \\
\perp_n F(X)_{h\Sigma_n} \arrow[r, hook, "j"] & F(X) \arrow[r, "F(\nabla^n)"] & F(\underset{n}{\vee} X) \arrow[d, "\delta"] \\
& & \widetilde{\perp}_n F(X)
\end{tikzcd} \tag{9}$$

where δ is the map dual to ρ (see definition 2.8).

If ∇ is cocommutative (hence ∇^n is cocommutative), then it is a Σ_n-fixed map and since δ is Σ_n-equivariant we have a factorization

$$\begin{tikzcd}
F(X) \arrow[rr, "F(\nabla^n)"] \arrow[dr] \arrow[dd, "\sigma"'] & & F(\underset{n}{\vee} X) \arrow[dd, "\delta"] \\
& F(\underset{n}{\vee} X)^{h\Sigma_n} \arrow[ur] \arrow[dl, "\delta^{h\Sigma_n}"] & \\
\widetilde{\perp}_n F(X)^{h\Sigma_n} \arrow[rr] & & \widetilde{\perp}_n F(X)
\end{tikzcd}$$

For functors whose target category is $\mathcal{S}$, $\perp_n F(X) \simeq \widetilde{\perp}_n F(X)$. We wish to show that the composite $\delta \circ F(\nabla^n) \circ F(+^n) \circ \rho$ is the norm map. If this composite gives the norm map on $ho(\mathcal{S})$, then the following diagram of weak maps commutes

$$\begin{tikzcd}
\perp_n F(X) \arrow[r, "\rho"] \arrow[d] & F(\underset{n}{\vee} X) \arrow[d, "F(+)"] & \\
\perp_n F(X)_{h\Sigma_n} \arrow[r, "j"] \arrow[dr, "T"] & F(X) \arrow[r, "F(\nabla^n)"] \arrow[d, "\sigma"'] & F(\underset{n}{\vee} X) \arrow[d, "\delta"'] \\
& \widetilde{\perp}_n F(X)^{h\Sigma_n} \arrow[r] & \widetilde{\perp}_n F(X)
\end{tikzcd} \tag{10}$$

where T is the Tate map. The key point is that the Tate map is rationally invertible. This will provide the splitting.

Using the diagram (7) from Section 4 as a central square, we can further extend the diagram to

$$\begin{tikzcd}
 & & \perp_n\perp_n F(X) \arrow[d, "\perp_n(p)"] \arrow[ddrr, "\eta_\tau"] & & \\
 & \perp_n F(X) \arrow[ur, "\eta_\Delta"] \arrow[r] \arrow[d, "\rho"] \arrow[dr, phantom, "A"] & \perp_n F(\underset{n}{\vee} X) \arrow[d, "\rho"] \arrow[dr, phantom, "C"] & & \\
\perp_n F(X) \arrow[ur, "="] \arrow[r, "\rho"] \arrow[d] & F(\underset{n}{\vee} X) \arrow[r, "\tau\circ F(\underset{n}{\vee}\nabla^n)"'] \arrow[d, "F(+^n)"'] & F(\underset{n}{\vee}\underset{n}{\vee} X) \arrow[r, "\delta"] \arrow[d, "F(\underset{n}{\vee}+^n)"'] \arrow[dr, phantom, "B"] & \widetilde{\perp}_n F(\underset{n}{\vee} X) \arrow[r, "\perp_n(\delta)"] \arrow[d] & \widetilde{\perp}_n\widetilde{\perp}_n F(X) \arrow[dl, "\eta_+"] \\
\perp_n F(X)_{h\Sigma_n} \arrow[r, "j"] \arrow[dr, "T"] & F(X) \arrow[r, "F(\nabla^n)"] \arrow[d, "\sigma"'] & F(\underset{n}{\vee} X) \arrow[r, "\delta"] \arrow[d, "\delta"'] & \widetilde{\perp}_n F(X) \arrow[dl, "="] & \\
 & \widetilde{\perp}_n F(X)^{h\Sigma_n} \arrow[r] & \widetilde{\perp}_n F(X) & &
\end{tikzcd}$$

(11)

The squares A and B commute by the functoriality of $\perp_n$ (resp. $\widetilde{\perp}_n$). Let η_τ be the composition of the indicated maps (so that triangle C commutes). We will show that the lifts η_Δ and η_+ exist. Then, by the commutativity of the diagram, it will suffice to show that $\eta = \eta_+ \circ \eta_\tau \circ \eta_\Delta$ is the norm map.

Lemma 5.3. *There exists a lift η_Δ such that the diagram*

$$\begin{tikzcd}
 & \perp_n\perp_n F(X) \arrow[d, hook] \\
\perp_n F(X) \arrow[ur, "\eta_\Delta"] \arrow[r, "\perp_n F(\nabla^n)"] \arrow[d, hook] & \perp_n F(\underset{n}{\vee} X) \arrow[d, hook] \\
F(\underset{n}{\vee} X) \arrow[r, "F(\underset{n}{\vee}\nabla^n)"] & F(\underset{n}{\vee}\underset{n}{\vee} X)
\end{tikzcd}$$

commutes. Moreover, η_Δ is homotopic to a diagonal map.

Proof. First, notice that by Remark 2.5 the following diagram commutes up to homotopy:

$$\begin{tikzcd}
\perp_n\perp_n F(X) \arrow[r, "\omega'", "\simeq"'] \arrow[d] & \prod\limits_{\mathrm{Bij}(n,n)} \perp_n F(X) \\
\perp_n F(\bigvee\limits_n X) \arrow[r, "\Delta"] & \prod\limits_{\mathrm{Bij}(n,n)} \perp_n F(\bigvee\limits_n X) \arrow[u, "u=\prod_{\mathrm{Bij}(n,n)} \mathrm{cr}_n F(u_{\alpha(1),\ldots,\alpha(n)})"']
\end{tikzcd}$$

where Δ is the appropriate diagonal map and $\alpha_i : \vee_n X \to X$ is the map which is the identity on the i-th copy of X and $*$ everywhere else.

If we rearrange this diagram, the lift becomes apparent:

$$\begin{tikzcd}
 & \prod\limits_{\mathrm{Bij}(n,n)} \perp_n F(X) & \perp_n\perp_n F(X) \arrow[l, "\omega'"', "\simeq"] \arrow[dl] \\
\perp_n F(X) \arrow[ur, "\eta_\Delta"] \arrow[r, "\perp_n F(\nabla)"] & \perp_n F(\bigvee\limits_n X) \arrow[u, "u\circ\Delta"'] &
\end{tikzcd}$$

In fact, since X is a co-H-object, the map ∇ is counital. Therefor $u_i \circ \nabla$ is homotopic to the identity on X for each $1 \leq i \leq n$. It follows that η_Δ is homotopic to the diagonal map. We will not distinguish between the lift η_Δ and the associated map $(\omega')^{-1} \circ \eta_\Delta$.

$\square$

Lemma 5.4. *There exists a lift η_+ such that the diagram*

$$\begin{tikzcd}
F(\bigvee\limits_n \bigvee\limits_n X) \arrow[r] \arrow[d, "F(+)"] & \widetilde{\perp}_n F(\bigvee\limits_n X) \arrow[r] \arrow[d, "\tilde{\perp}_n F(+)"'] & \widetilde{\perp}_n\widetilde{\perp}_n F(X) \arrow[dl, "\eta_+"] \\
F(\bigvee\limits_n X) \arrow[r] & \widetilde{\perp}_n F(X) &
\end{tikzcd}$$

commutes.

Proof. This is formally dual to Lemma 5.3. $\square$

Let τ be given by acting by the group action of Σ_n. In other words,

$$\tau : \prod_{f\in \mathrm{Bij}(n,n)} \perp_n F(X) \to \prod_{f\in \mathrm{Bij}(n,n)} \perp_n F(X)$$

is given on the factor indexed by $f \in \mathrm{Bij}(n,n)$ by shuffling each of the n variables of $\perp_n F(X)$ by f. Call τ the twist map.

Lemma 5.5. *The map η_τ, which is the composite*

$$\begin{tikzcd}
\perp_n\perp_n F(X) \arrow[r, "\omega'", "\simeq"'] \arrow[d] & \prod\limits_{\mathrm{Bij}(n,n)} \perp_n F(X) \arrow[d, hook] \arrow[drr, "\eta_\tau"] & & \\
\perp_n F(\underset{n}{\vee} X) \arrow[r, "\omega", "\simeq"'] & \prod\limits_{\mathrm{Hom}(n,n)} \perp_n F(X) \arrow[d, "\bar{\rho}"'] \arrow[dr, "\eta'_\tau"] & & \\
 & F(\underset{n}{\vee}\underset{n}{\vee} X) \arrow[r, "\bar{d}"'] & \prod\limits_{\mathrm{Hom}(n,n)} \widetilde{\perp}_n F(X) \arrow[r, two heads] \arrow[d, "\simeq"'] & \prod\limits_{\mathrm{Bij}(n,n)} \widetilde{\perp}_n F(X) \arrow[d, "\simeq"] \\
 & & \widetilde{\perp}_n F(\underset{n}{\vee} X) \arrow[r] & \widetilde{\perp}_n \widetilde{\perp}_n F(X)
\end{tikzcd}$$

is the twist map.

Proof. We will begin by examining the map η'_τ. Consider the factor $\perp_n F(X)$ of $\prod_{\mathrm{Hom}(n,n)} \perp_n F(X)$ indexed by $f \in \mathrm{Hom}(n,n)$. On the factor of

$$\prod_{\mathrm{Hom}(n,n)} \mathrm{cr}_n F(X)$$

indexed by f, notice that $\omega^{-1} : \perp_n F(X) \to \perp_n F(\vee_n X)$ is given by

$$\mathrm{cr}_n F(c_{f(1)}, \dots, c_{f(n)})$$

where $c_i : X \to \vee_n X$ is the inclusion into the i-th summand. Let $c_f := c_{f(1)} \vee \cdots \vee c_{f(n)}$. Then the map $\bar{\rho}$ is given on the f-th factor by either of the composites

$$\begin{tikzcd}
\perp_n F(X) \arrow[r, "\omega^{-1}", "\mathrm{cr}_n F(c_{f(1)},\dots,c_{f(n)})"'] \arrow[d, "\rho"'] & \perp_n F(\underset{n}{\vee} X) \arrow[d, "\rho"] \\
F(\underset{n}{\vee} X) \arrow[r, "F(c_f)"] & F(\underset{n}{\vee}\underset{n}{\vee} X)
\end{tikzcd}$$

Similar computations provide a factorization of $\bar{d}$, projected onto the g-th factor, using the map $u_i : \vee_n X \to X$ which is the identity on the i-th summand and 0 elsewhere (this is dual to c_i). Let $u_g := u_{g(1)} \vee \cdots \vee u_{g(n)}$. We obtain a commuting diagram

$$\begin{tikzcd}
\bot_n F(X) \arrow[r, "\omega^{-1}", "\bot_n F(c_{f(1)},\dots,c_{f(n)})"'] \arrow[d, "\rho"'] & \bot_n F(\bigvee_n X) \arrow[d, "\rho"] & \\
F(\bigvee_n X) \arrow[r, "F(c_f)"] \arrow[dr, dashed] & F(\bigvee_n \bigvee_n X) \arrow[r, "d"] \arrow[d, "F(u_g)"] & \widetilde{\bot}_n F(\bigvee_n X) \arrow[d, "\widetilde{\bot}_n F(u_{g(1)},\dots,u_{g(n)})"] \\
& F(\bigvee_n X) \arrow[r, "d"] & \widetilde{\bot}_n F(X)
\end{tikzcd} \tag{12}$$

The key point now is to understand the composition represented by the dashed arrow. By functoriality, we need only understand the composition $(u_g) \circ (c_{f(1)} \vee \cdots \vee c_f)$. To understand this composition, it is easiest to introduce the labels

$$\begin{tikzcd}
\begin{pmatrix} X_{1,*} \\ \vee \\ \vdots \\ \vee \\ X_{n,*} \end{pmatrix} \arrow[r, "(c_f)"] & \begin{pmatrix} X_{1,1} & \vee & \dots & \vee & X_{1,n} \\ \vee & & & & \vee \\ \vdots & & & & \vdots \\ \vee & & & & \vee \\ X_{n,1} & \vee & \dots & \vee & X_{n,n} \end{pmatrix} \arrow[d, "(u_g)"] \\
& \begin{pmatrix} X_{*,1} & \vee & \dots & \vee & X_{*,n} \end{pmatrix}
\end{tikzcd}$$

keeping in mind that $X_{*,*} = X$ for all choices of $*$'s. On any summand, $X_{i,*}$, we have

$$X_{i,*} \xrightarrow{c_{f(i)}} X_{i,1} \vee \cdots \vee X_{i,n} \xrightarrow{u_{g(f(i))}} X_{g(f(i)),f(i)}$$

which is non-zero if and only if $g(f(i)) = i$. Since this is true for all $1 \le i \le n$, this implies that f has an inverse and that $g = f^{-1}$. That is, $f \in \mathrm{Bij}(n, n)$.

Furthermore, the composition of $u_{f^{-1}}$ with c_f yields

$$\begin{pmatrix} X_{1,*} \\ \vee \\ \vdots \\ \vee \\ X_{n,*} \end{pmatrix} \longrightarrow \left(X_{1,f(1)} \quad \vee \quad \ldots \quad \vee \quad X_{n,f(n)} \right)$$

which is exactly the twist by the action of f.

We have shown the composition is exactly non-zero when $f \in \mathrm{Bij}$ so that diagram 12 actually defines

$$\eta_\tau : \prod_{\mathrm{Bij}(n,n)} \perp_n F(X) \to \prod_{\mathrm{Bij}(n,n)} \widetilde{\perp}_n F(X).$$

By the equivariance of ρ and d, we see that this extends to show that η_τ is the twist map. □

Now, looking at the composites of the maps of Lemmas 5.3, 5.4 and 5.5 we find by Lemma 3.2 that we have constructed the norm map. Thus the induced map T is indeed the Tate map which gives us the rational equivalence we sought. This concludes the proof of the Theorem 5.1.

Remark 5.6. A slight extension allows us to drop the hypothesis that ∇ is cocommutative. In the case where $\mathcal{S}$ is the category of rational spectra, it is possible to show that the map $Q \circ T$ in the diagram

$$\begin{tikzcd}
\perp_n F(X)_{h\Sigma_n} \arrow[r, "j"] \arrow[dr] & F(X) \arrow[r, "\nabla^n"] & F(\vee_n X) \arrow[d] \\
& \widetilde{\perp}_n F(X)^{h\Sigma_n} \arrow[r] \arrow[dr, "Q"] & \widetilde{\perp}_n F(X) \arrow[d] \\
& & \widetilde{\perp}_n F(X)_{h\Sigma_n}
\end{tikzcd}$$

is rationally invertible. In this case the factorization σ from the previous proof is not needed.

Example 5.1. The following example is due to Tom Goodwillie.

Let $F : Top \to S$ be the functor $F(X) = C_*(X \wedge X/\Delta)$ where Δ is the diagonal map $\Delta : X \to X \wedge X$. We show that rationally the Goodwillie tower of F splits when X is a co-H-space, but not in general.

First note that the cofiber sequence

$$\begin{tikzcd} X \arrow[r, "\Delta"] & X \wedge X \arrow[r] & \frac{X \wedge X}{\Delta} \end{tikzcd}$$

induces a short exact sequence

$$\begin{tikzcd} C_*(X) \arrow[r] & C_*(X \wedge X) \arrow[r] & C_*(\frac{X \wedge X}{\Delta}) \end{tikzcd} .$$

The functor $X \mapsto C_*(X)$ is a homogeneous functor of degree one and the functor $X \mapsto C_*(X \wedge X)$ is homogeneous of degree two. Since D_n (as well as P_n) preserve short exact sequences, we have that $F(X)$ must also be a functor of degree at most two.

We want to examine the fiber sequence

$$D_2F(X) \to P_2F(X) \to P_1F(X)$$

The following 3 by 3 diagram with exact rows and columns captures all of the essential information for the Goodwillie tower of $F(X)$:

$$\begin{tikzcd}
D_2C_*(X) \arrow[r] \arrow[d] & D_2C_*(X \wedge X) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & D_2F(X) \arrow[d] \\
P_2C_*(X) \arrow[r] \arrow[d] & P_2C_*(X \wedge X) \arrow[r] \arrow[d] & P_2F(X) \arrow[d] \\
D_1C_*(X) \arrow[r] & D_1C_*(X \wedge X) \arrow[r] & D_1F(X)
\end{tikzcd}$$

The columns are exact since we have a fiber sequence $D_2 \to P_2 \to P_1$ and, since all of the functors involved are reduced, $P_1 = D_1$. Since $C_*(X)$ is a homogeneous functor of degree 1, $D_2C_*(X) = 0$ and similarly, since $C_*(X \wedge X)$ is a homogeneous functor of degree 2, $D_1C_*(X \wedge X) = 0$. That means we have equivalences $D_2F(X) \simeq D_2C_*(X \wedge X) = C_*(X \wedge X)$ and $D_1F(X) \simeq \Sigma D_1C_*(X) = C_*(\Sigma X)$. Since F is of degree two, we also have that $P_2F(X) \simeq C_*(X \wedge X/\Delta)$. That is, we have an exact sequence

$$\begin{tikzcd} C_*(X \wedge X) \arrow[r] & C_*(X \wedge X/\Delta) \arrow[r] & C_*(\Sigma X). \end{tikzcd}$$

inducing the long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H_{*+1}(\Sigma X) \arrow[d, "\partial"] & & \\
& H_*(X \wedge X) \arrow[r] & H_*(X \wedge X/\Delta) \arrow[r] & H_*(\Sigma X) \arrow[d, "\partial"] & \\
& & & H_{*-1}(X \wedge X) \arrow[r] & \cdots
\end{tikzcd}$$

on homology. If this map splits, then the map ∂ is zero. It is easy to find spaces for which this can not be the case. One simple example is given by S^0. Also, notice that up to the suspension isomorphism the map ∂ can be expressed as

$$H_*(X) \xrightarrow{\Delta_*} H_*(X \wedge X) \ .$$

On (rational) cohomology, the map $\Delta^* : H^*(X \wedge X) \to H^*(X)$ is the cup product. Therefore, if this map is zero then X has trivial cup products. Since most spaces don't have trivial cup products, mostly this doesn't split. However, since when X is a co-H-space the Hopf algebra $H^*(X)$ is exterior on the indecomposables, so the cup products are trivial.

We end this section by showing that the layers $D_nF(X)$ can be identified by the image of a certain map. Let $\Phi^r : F(X) \to F(X)$ be the composite $\Phi^r = F(+) \circ F(\nabla^r)$. Note that in the case $X = S^1$, this induces an r-fold covering map of S^1.

Remark 5.7. The map $D_n(\Phi^r) := P_\infty(\Phi^r)|_{D_nF(X)}$ from $D_nF(X) \to D_nF(X)$ induces multiplication by r^n.

To see this, assume F is degree n as we did in the proof of Theorem 5.1. We have $D_nF(X) = D_1^{(n)} \perp_n F(X)$ by [JM4]. For degree n functors F we have $\perp_n F(\vee_k X) \cong \prod_{\mathrm{Hom}(n,r)=r^n} \perp_n F(X)$, so the following diagram describes

Φ^r:

$$\begin{tikzcd}
D_1^{(n)}\perp_n F(X) \arrow[d, "F_*(\nabla^r)"] \arrow[dr, "\Delta"] & \\
D_1^{(n)}\perp_n F(\bigvee\limits_r X) \arrow[r, "\simeq"] \arrow[d, "F_*(+)"] & \prod\limits_{\mathrm{Hom}(n,r)=r^n} D_1^{(n)}\perp_n F(X) \arrow[dl, "+"] \\
D_1^{(n)}\perp_n F(X) &
\end{tikzcd}$$

Now, by arguments analogous to Lemmas 5.3 and 5.4, the maps Δ and $+$ of the diagram are the diagonal and fold map (we make use of the fact that coproducts and products are weakly equivalent). Hence, Φ^r is multiplications by r^n.

5.1 Higher Hochschild Homology Revisited

Let A be a commutative algebra over a field k of characteristic zero. Let X be any finite pointed simplicial set. We can form a simplicial k-algebra by composing the functor $X.$ with the functor $-\otimes A$ from the category of finite pointed sets to A-algebras which takes the set $[[n]] = \{0, 1, \dots n\}$ to $[[n]] \otimes A = A^{\otimes_k n}$. The chain complex associated to the resulting simplicial algebra computes the Hochschild homology of A when $X \simeq S^1$ and computes the n-th higher Hochschild homology when $X \simeq S^n$. Denote the homology of the chain complex associated to $X.\otimes A$ by $HH_*^X(A)$. Rational decompositions of Hochschild homology have been studied extensively [L98], [GS87], [R93]. A rational decompositions of higher Hochschild homology which recovers the decomposition of Hochschild homology has been discovered by Pirashvili [P00] and a rational decomposition for $HH_*^{S^1\wedge X}(A)$ was found by the first author, recovering the decompositions of the other two cases [B].

The goal is to compute the layers of the decomposition of higher Hochschild homology in terms of the layers of Hochschild homology. The chain complex $(S^1 \wedge X) \otimes A$ is a commutative differential graded Hopf algebra [B]. Using the comultiplication map, consider $(S^1 \wedge X) \otimes A$ as a co-H-object in the category of commutative augmented A-algebras (the augmentation is given in degree 0; $(S^1 \wedge X)[0] \otimes A = A$). One should note that the comultiplication and multiplication structure maps making $(S^1 \wedge X) \otimes A$ into a Hopf algebra are induced by the same maps which make $S^1 \wedge X$ a co-H-space.

Let U be the forgetful functor from the category ${}_ACommm_A$ of commutative augmented A-algebras to the category $A - mod$ of A-modules. By Theorem 5.1, $HH_*^{S^1 \wedge X} = \underset{n}{\oplus} D_n U((S^1 \wedge X) \otimes A)$. The Goodwillie tower of the functor U has been computed [K-M]. Let M be an object of ${}_ACommm_A$. Then the linear part, $D_1U(M)$, is $I/I^2(M)$ where $I(M)$ is the augmentation ideal of M over A. The higher layers are given by $D_nU(M) = (I/I^2(M))^{\otimes n}_{\Sigma_n} = S^n(I/I^2(M))$ where S^n denotes the n-th part of the symmetric algebra. By Remark 5.7 the "r-fold cover map" Φ^r induces multiplication by r^n on each layer $D_nU((S^1 \wedge X) \otimes A)$ and one can use this to show that the decomposition using Theorem 5.1 agrees with the decomposition of [B] and hence also those of [P00], [L98] and [GS87].

Since D_1U is a linear functor, it commutes with suspensions. (Note that the suspension in ${}_ACommm_A$ is given by $S^1 \otimes -$ and the suspension in $A - mod$ is given by $\widetilde{\mathbb{Z}}[S^1] \otimes_A -$ where $\widetilde{\mathbb{Z}}[S^1]$ is the free module generated by S^1. If you compute this, use the more precise def'n that linear functors take cartesian to cocartesian.) Therefore $D_1U(S^d \otimes A) = \widetilde{\mathbb{Z}}[S^d] \otimes_A D_1U(S^0 \otimes A)$. Since $D_nU(S^d \otimes A) = (D_1U(S^d \otimes A))^{\otimes n}_{\Sigma_n}$ we have

$$\begin{aligned} D_nU(S^d \otimes A) &= (\widetilde{\mathbb{Z}}[S^d] \otimes D_1U(S^0 \otimes A))^{\otimes n}_{\Sigma_n} && (13) \\ &= [\widetilde{\mathbb{Z}}[S^{dn}] \otimes (D_1U(S^0 \otimes A))^{\otimes n}]_{\Sigma_n} && (14) \\ & && (15) \end{aligned}$$

where the action of Σ_n on the last line is given diagonally. On the first factor, Σ_n acts by permuting the copies of S^d. Each flip of factors S^d induces multiplication by $(-1)^d$ on homology. On the second factor, Σ_n acts by permuting the factors $D_1U(S^0 \otimes A)$. Taking the orbits of this action produces the homogeneous degree n part of the symmetric algebra. Taking both of these actions together, we have

$$D_nU(S^d \otimes A) = \begin{cases} \Sigma^{dn} S^n D_1U(S^0 \otimes A) & d \text{ is even;} \\ \Sigma^{dn} \Lambda^n D_1U(S^0 \otimes A) & d \text{ is odd,} \end{cases}$$

where Λ^n is the homogeneous degree n part of the exterior algebra. Finally, if one then computes that $D_nU(S^1 \otimes A) = \Sigma^n \Lambda^n D_1U(S^0 \otimes A)$ then we can express the layers of higher Hochschild homology in terms of Hochschild homology by realizing that

$$D_nU(S^d \otimes A) = \begin{cases} \Sigma^{n(d-1(}(-1)^d D_nU(S^1 \otimes A) & d \text{ is even;} \\ \Sigma^{n(d-1)} D_nU(S^1 \otimes A) & d \text{ is odd.} \end{cases}$$

This shows that the layers of higher Hochschild homology depend only on the layers of Hochschild homology.

6 Dual Calculus

There is a version of calculus which is strictly dual to the Goodwillie calculus tower which we have been using so far. To obtain this theory, one simply replaces homotopy limits with homotopy colimits, coproducts with products, fibers with cofibers, etc. For details, consult [McC02]. The following definitions (using the terminology of section 2) and results summarize what we need to know about dual calculus. Let $P_1(\mathbf{n})$ be the subcategory of $P(\mathbf{n})$ consisting of all *proper* subsets of $\mathbf{n}$.

Definition 6.1. Let $g : \mathbf{n} \to \mathcal{C}$ be a function defined by $g(i) = X_i$ for objects $X_i \in \mathcal{C}$. Let $\chi^g(S) = \prod_{c \in S} g(c)$ be a (covariant) functor from $P(\mathbf{n})$ (or $P_1(\mathbf{n})$) to $\mathcal{C}$ which takes inclusion of sets to inclusions of products. The n-th dual cross effect, $\mathrm{cr}^n F(X_1, \ldots, X_n)$ is the cofiber of the map

$$\mathrm{holim}_{S \in P_1(\mathbf{n})} F(\chi^g(S)) \to \mathrm{holim}_{S \in P(\mathbf{n})} F(\chi^g(S)).$$

Note that $\mathrm{holim}_{S \in P(\mathbf{n})} F(\chi^g(S)) = F(\prod_{i=1}^n X_i)$.

The dual cross effects can be thought of as the total cofibers of cubical diagrams. There is a natural map $\rho : F(\prod_n X) \to \mathrm{cr}^n F(X, \ldots, X) =: \perp^n F(X)$.

One can use the dual cross effects to define a codegree n approximation to F, and these assemble into a tower

$$\begin{tikzcd}
\vdots & \\
P^nF(X) \arrow[u] \arrow[r, "p^n"] & F(X) \\
P^{n-1}F(X) \arrow[u, "q^n"] \arrow[ur, "p^{n-1}"'] & \\
\vdots \arrow[u] &
\end{tikzcd}$$

which is universal with respect to maps to F from codegree n functors. The n-th coderivative of F, D^nF, is the cofiber of the map $q^n : P^{n-1}F \to P^nF$.

There is also a notion of dual co-cross effects. Let $\widetilde{\chi}$ be the contravariant functor with $\widetilde{\chi}(S) = \prod_{c \notin S} g(c)$.

Definition 6.2. We define the dual co-cross effect $\widetilde{\mathrm{cr}}^n F(X_1, \dots, X_n)$, to be the fiber of the map

$$\mathrm{hocolim}_{P(\mathbf{n})} F(\widetilde{\chi}_g) \to \mathrm{hocolim}_{P_0(\mathbf{n})} F(\widetilde{\chi}_g).$$

There is also a natural map $\tilde{\rho} : \widetilde{\perp}^n F(X) := \widetilde{\mathrm{cr}}^n F(X, \dots, X) \to F(\prod_n X)$.

To state the dual version of the theorem, we must describe the appropriate counterpart for co-H-objects. The following is the expected definition:

Definition 6.3. A fibrant object X of $\mathcal{C}$ is an H-object if X is equipped with a map $\mu : X \times X \to X$ which is unital and associative up to homotopy.

The unit map for μ is given by the inclusion of the basepoint. Let Δ be the diagonal map $\Delta : X \to X \times X$. The following diagram commutes up to homotopy:

$$\begin{tikzcd}
\prod_n X \arrow[r, "\tau\circ\prod\Delta^n"] \arrow[d, "\mu^n"'] & \prod_n\prod_n X \arrow[d, "\prod\mu^n"'] \\
X \arrow[r, "\Delta^n"] & \prod_n X
\end{tikzcd}$$

where τ is the map which transposes the entries of $\prod_n \prod_n X$. For an example of H-objects, consider the category $\mathcal{C}$ of pointed topological spaces. In this case, the H-objects are precisely the H-spaces.

The requirement that X be fibrant is necessary for the theorem so that we may replace the functor F by P^nF, in anology with the proof of Theorem 5.1. In particular, we want to know that if F preserved weak equivalences and if P^nF is applied to the diagram above, then the diagram will still commute up to homotopy. If the map μ is unital and associative up to isomorphism, X is not required to be fibrant and C need not be a model category as all diagrams will commute up to isomorphism.

Theorem 6.4. *If F is a functor from $\mathcal{C}$ to $\mathcal{S}$ which preserves weak equivalences and if X is an H-object of $\mathcal{C}$, then rationally the cofiber sequence*

$$P^{n-1}F(X) \xrightarrow{q^n} P^nF(X) \xrightarrow{j} D^nF(X)$$

splits in the homotopy category. Consequently,

$$P^\infty F(X) \simeq \prod_{n\geq 0} D^nF(X)$$

is a rational equivalence.

The proof of this dual version of our main theorem proceeds in essentially the same manner as the proof of the main theorem. Here is a sketch:

We reduce to the case where F is a codegree n functor. When F is codegree n, we have equivalences $P^nF(X) \simeq F(X)$ and $D^nF(X) \simeq \perp^n F(X)^{h\Sigma_n}$. We can then express the map j as in the following diagram:

$$\begin{tikzcd}
F(X) \arrow[rr, "j"] \arrow[dd, "F(\Delta)"'] \arrow[dr] & & \perp^n F(X)^{h\Sigma_n} \arrow[dd, "i"] \\
& F(\prod\limits_n X)^{h\Sigma_n} \arrow[ur, "\rho^*"] \arrow[dl] & \\
F(\prod\limits_n X) \arrow[rr, "\rho"] & & \perp^n F(X)
\end{tikzcd}$$

where $F(\Delta)$ factors through the fixed points since it is Σ_n-fixed, and ρ^* extends because ρ is Σ_n-equivariant.

Now, using the fact that X is an H-object and using the map $\widetilde{\rho} : \widetilde{\perp^n} F(X) \to F(\prod_n X)$, we can expand this diagram to

$$\begin{tikzcd}
\widetilde{\perp}^n F(X) \arrow[d, "\tilde{\rho}"] & & \\
F(\prod\limits_n X) \arrow[r, "F(\mu)"] \arrow[d, "F(\Delta)"] & F(X) \arrow[r, "j"] \arrow[d, "F(\Delta)"] & \perp^n F(X)^{h\Sigma_n} \arrow[d, "i"] \\
F(\prod\limits_n \prod\limits_n X) \arrow[r, "F(\mu\circ\tau)"] & F(\prod\limits_n X) \arrow[r, "\rho"] & \perp^n F(X)
\end{tikzcd}$$

which commutes. We claim that the "outside" map - that is, the composition $\rho \circ F(\mu \circ \tau) \circ F(\Delta) \circ \tilde{\rho}$ - is the norm map. Just as before, one can show this by using the equivalences $\perp^n F(\prod_n X) \simeq \prod_{Hom(n,n)} \perp^n F(X)$ and $\perp^n \perp^n F(X) \simeq \prod_{\Sigma_n} \perp^n F(X)$ and by showing that the relevant maps are the diagonal, twist and fold maps. Once this is done, we can further expand our diagram to

$$\begin{tikzcd}
\widetilde{\perp}^n F(X) \arrow[r, "q"] \arrow[d, "\tilde{\rho}"] & \perp^n F(X)_{h\Sigma_n} \arrow[dr, "T"] & \\
F(\prod\limits_n X) \arrow[r, "F(\mu)"] \arrow[d, "F(\Delta)"] & F(X) \arrow[r, "j"] \arrow[d, "F(\Delta)"] & \perp^n F(X)^{h\Sigma_n} \arrow[d, "i"] \arrow[ull, "i"'] \\
F(\prod\limits_n \prod\limits_n X) \arrow[r, "F(\mu\circ\tau)"] & F(\prod\limits_n X) \arrow[r, "\rho"] & \perp^n F(X)
\end{tikzcd}$$

where q and i are the relevant quotient and inclusion maps, respectively. The map T is the Tate map, and this commutes with the rest of the diagram as before because the outside composition is the norm map. In fact, the entire diagram commutes except possibly for the map i. We wish to show that i provides a splitting map for j. In other words, we'll show that $j \circ F(\mu) \circ \widetilde{\rho} \circ i$ is rationally homotopic to the identity map on $\mathrm{cr}^n F(X)^{h\Sigma_n}$. This is the same as showing that $T \circ q \circ i$ is rationally homotopic to the identity map. However, recalling the definition of the map T, we have that $T \circ q \circ i \sim N \circ i$ where N

is the norm map since N actually lands in the fixed points. This is rationally equivalent to the identity map on $\mathrm{cr}^n F(X)^{h\Sigma_n}$. Thus i provides the splitting.

Remark 6.5. Define a map $\Phi^r := F(\mu) \circ F(\Delta)$. Then $D^n(\Phi^r)$ induced multiplication by r^n. The argument for this is exactly dual to Remark 5.7.

6.1 The Poincaré-Birkhoff-Witt Theorem.

We seek to recover the rational version of the Poincaré-Birkhoff-Witt Theorem as an application of Theorem 6.4. First, we recall the definitions required to state this theorem. This background can be found in e.g. [W94].

Let Lie_k be the category of Lie algebras over a field k of characteristic 0. If A is any algebra over k, A can be thought of as a Lie algebra by giving it the bracket $[x, y] = xy - yx$, where $x, y \in A$. Denote A as a Lie algebra by $\mathcal{A}$. Let $\mathcal{G}$ be a Lie algebra over k with bracket $[\ ,\ ]$.

Definition 6.6. The universal enveloping algebra of $\mathcal{G}$ is an algebra over k, $U(\mathcal{G})$ with associated Lie algebra $\mathcal{U}(\mathcal{G})$ together with a morphism $i : \mathcal{G} \to \mathcal{U}(\mathcal{G})$ which is universal with respect to algebras over k. That is, if $f : \mathcal{G} \to \mathcal{A}$ is a map of Lie algebras, then there is a map $g : U(\mathcal{G}) \to A$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "i"] \arrow[dr, "f"'] & \mathcal{U}(\mathcal{G}) \arrow[d, "g_*"] \\
 & \mathcal{A}
\end{tikzcd}$$

where g_* is the map induced by g.

One can construct $U(\mathcal{G})$. Let $i : \mathcal{G} \to \mathcal{T}(\mathcal{G})$ be the inclusion of $\mathcal{G}$ into its tensor algebra induced by including $\mathcal{G}$ into the first graded piece of $T(\mathcal{G})$. Let I be the ideal of $T(\mathcal{G})$ generated by the relations

$$i([x, y]) = i(x)i(y) - i(y)i(x)$$

where $x, y \in \mathcal{G}$. Then $U(\mathcal{G}) = T(\mathcal{G})/I$.

Let $\pi : T(\mathcal{G}) \to U(\mathcal{G})$ be the quotient map and let $T_m(\mathcal{G}) = \oplus_{j=0}^{m} \mathcal{G}^{\otimes_k j}$. One can see that $U(\mathcal{G})$ inherits a grading from $T(\mathcal{G})$ by setting $U_m(\mathcal{G}) =$

$\pi(T_m(\mathcal{G}))$. If $\mathcal{G}$ is free as a module over k, then the Poincaré-Birkhoff-Witt theorem shows that $U_m(\mathcal{G})/U_{m-1}(\mathcal{G}) \cong S_m$ as k-modules, where $S_m := \mathcal{G}^{\otimes m}/\Sigma_m$ is the m-th homogeneous graded piece of the symmetric algebra, and that $U(\mathcal{G}) \cong S$ as k-modules, where S is the whole symmetric algebra. In fact, if $\{e_i\}_{i\in\mathcal{I}}$ is a basis for $\mathcal{G}$ as a k-module, then $\{e_{i_1} \otimes \cdots \otimes e_{i_m} | i_j \in \mathcal{I}; i_1 \leq \cdots \leq i_m; m \geq 1\}$ is a basis for $U(\mathcal{G})$ (where $\mathcal{I}$ is some indexing set).

From [MM65] we know that $U(\mathcal{G} \oplus \mathcal{G}) \cong U(\mathcal{G}) \otimes U(\mathcal{G})$ and that $U(\mathcal{G})$ is a cocommutative Hopf algebra. The multiplication structure map for the Hopf algebra is induced by concatenation on $T(\mathcal{G})$, and the comultiplication is induced by the diagonal map $\Delta : \mathcal{G} \to \mathcal{G} \oplus \mathcal{G}$. (unit and counit?) Let $\mathcal{C}$ be the category of cocommutative, coaugmented coalgebras over k. In $\mathcal{C}$, the product is given by $\otimes$ and the diagonal map for any object is given by the comultipication. The Hopf algebra structure makes $U(\mathcal{G})$ an H-object in the category $\mathcal{C}$.

Let F be the forgetful functor from $\mathcal{C}$ to the category of k-modules. We can now apply theorem 6.4 to $F(U(\mathcal{G}))$ to obtain a splitting of $U(\mathcal{G})$ as a k-module. We want to show that this splitting recovers the Poincaré-Birkhoff-Witt theorem. In other words, we want to show that $D^nF(U(\mathcal{G})) \cong S_n$.

Note that for $g \in U_1(\mathcal{G})/U_0(\mathcal{G})$, the image of the comultiplication map is $\Delta(g) = g \otimes 1 + 1 \otimes g$. Denote an element in $U_n(\mathcal{G})/U_{n-1}(\mathcal{G})$ by $(g_1, \ldots, g_n)$. By induction, we have

$$\Delta(g_1, \ldots, g_n) = \sum_{\substack{p+q=n \\ \sigma\in(p,q)-\text{shuffles}}} (g_{\sigma(1)}, \ldots, g_{\sigma(p)}) \otimes (g_{\sigma(p+1)}, \ldots, g_{\sigma(p+q)})$$

where σ, a (p,q)-shuffle means that σ is a permutation of the n g_i's with $\sigma(1) \leqslant \cdots \leqslant \sigma(p)$ and $\sigma(p+1) \leqslant \cdots \leqslant \sigma(p+q)$ (in some sense, this is the *opposite* of what is usually meant by a (p,q)-shuffle - in our case, we mean the analogy with a deck of cards to take a deck of cards and "unshuffle" - or sort - it into two piles without disturbing the order in each pile, rather than to shuffle two stacks of cards into a single deck). Since multiplication is induced by concatenation, the map $\Phi^2 = \mu \circ \Delta$ is

$$\mu \circ \Delta(g_1, \ldots, g_n) = \sum_{\substack{p+q=n \\ \sigma\in(p,q)-\text{shuffles}}} (g_{\sigma(1)}, \ldots, g_{\sigma(p)}, g_{\sigma(p+1)}, \ldots, g_{\sigma(p+q)}).$$

However, in $U(\mathcal{G}) = T(\mathcal{G})/I$, for any permutation $\sigma \in \Sigma_n$ we have

$$(g_{\sigma(1)}, \ldots, g_{\sigma(n)}) = (g_1, \ldots, g_n).$$

Therefore $\mu \circ \Delta$ is simply multiplication by the number of ways of shuffling $(g_1, \ldots, g_n)$ into two factors.

A quick induction shows that the number of (p, q)-shuffles with $p + q = n$ is 2^n. It is easy to see that there are only two shuffles if $n = 1$; they are $g \otimes 1$ and $1 \otimes g$. Any (p, q)-shuffle of $(g_1, \ldots, g_{n-1})$ determines exactly two shuffles of $(g_1, \ldots, g_n)$: one by adjoining g_n to the first factor of the shuffle to obtain a $(p+1, q)$ shuffle and the other by adjoining g_n to the second factor, resulting in a $(p, q+1)$ shuffle. It is not difficult to see that all shuffles of $(g_1, \ldots, g_n)$ can be obtained in this way. Therefore, there are twice as many shuffles of $(g_1, \ldots, g_n)$ as there were of $(g_1, \ldots, g_{n-1})$. This completes the induction, and so there are 2^n ways of shuffling $(g_1, \ldots, g_n)$ into two piles.

We now know that the map $\Phi^2 = \mu \circ \Delta$ induces multiplication by 2^n on $U_n(\mathcal{G})/U_{n-1}(\mathcal{G}) \cong S_n$. However we also know that Φ^2 induces multiplication by 2^n on $D_n F(U(\mathcal{G}))$. This shows that the two decompositions are the same - the map Φ^2 plays the role of a linear operator on the k-vector space $U(\mathcal{G})$ whose image determines the decomposition associated to it. Hence theorem 6.4 recovers the Poincaré-Birkhoff-Witt theorem.

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