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Exam Code: MATH 353 Winter '05

ONE HOUR

UNIVERSITY OF CALGARY

MATH 353 MIDTERM EXAM

SESSION 2005
WINTER SEMESTER

DATE: 4th March 2005
TIME: 2:00-3:00

NO GRAPHING CALCULATORS, BOOKS OR NOTES
scientific calculators are permitted

For Markers Only	
1	/5
2	/10
3	/10
4	/10
5	/10
6	/5
Σ	/50

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[5 points]

1 Let $S = \{(r,\theta) | 0 < r \leq \cos(\theta),\ 0 \leq \theta \leq \pi\}$ be a set in the plane described in polar coordinates.

(a) Is the set open, closed or neither?

(b) Find the boundary ∂S.

The boundary of the set is $\partial S = \{(r,\theta) | r = \cos(\theta)\}$. This includes the point at the origin. However, this point is not included in the set S. Therefore the set S can not be closed. On the other hand, the set S contains ALL of the other points in ∂S. Therefore the set S can not be open. The set is neither open nor closed since it contains some, but not all, of its boundary points.

[10 points]

2 A rectangular box with no top has a fixed volume of 100 cubic meters. The box is to be constructed of two materials. The material used for the bottom and front of the box is five times as costly per square meter as the material used for the back and the sides. What are the dimensions of the box which minimize the cost of the materials?

I will label the box as follows: the front of the box has length x, the side of the box has length y and the height of the box is z. I'll let you guys draw the picture that goes with this.

The volume of the box is fixed: $V = 100 = xyz$. This is my constraint equation. Let k be the cost of the materials used to build the sides and the back of the box per square unit. Then the cost to build the box is:

$$C(x,y,z) = 5k(xz + xy) + k(2yz + xz) = 6kxz + 5kxy + 2kyz$$

The important thing to remember here is that k is a constant. Using the method of Lagrange multipliers, we have

$$\nabla C(x,y,z) = \langle 6kz + 5ky, 5kx + 2kz, 6kx + 2ky \rangle$$

$$\nabla V(x,y,z) = \langle yz, xz, xy \rangle$$

This gives us the four equations:

$$6kz + 5ky = \lambda yz$$

$$5kx + 2kz = \lambda xz$$

$$6kx + 2ky = \lambda xy$$

$$xyz = 100$$

Multiply through by the missing variable in each of the first three equations to get

$$6kxz + 5kxy = 5kxy + 2kyz = 6kxz + 2kyz$$

We solve these pairwise. The first pair of equations leads to $6kxz = 2kyz$ or $y = 3x$ (as long as $z \neq 0$, which is consistent with $xyz = 100$). The second pair of equations leads to $5kxy = 6kxz$ or $z = (5/6)y = (5/2)x$ (as long as $x \neq 0$ which is consistent with $xyz = 100$). Now we plug these into the constraint equation to get $x(3x)(5/2)x = 100$ or $x^3 = 200/15 = 40/3$. Thus the minimum cost is attained when

$$x = \sqrt[3]{40/3}; \qquad y = 3\sqrt[3]{40/3}; \qquad z = (5/2)\sqrt[3]{40/3}$$

[10 points]

3 Evaluate $\int\int_T \frac{\sin(y)}{1-y}\, dA$ where T is the triangular region bounded by the x-axis, the y-axis and the line $y = 1 - x$.

The trick here is to realize that there is a singularity in the domain, since $\frac{\sin(y)}{1-y}$ is not defined when $y = 1$. The domain is:

$$T = \{(x, y) \mid 0 \leq x \leq 1;\ 0 \leq y \leq 1 - x\}$$

or, equivalently,

$$T = \{(x, y) \mid 0 \leq x \leq 1 - y;\ 0 \leq y \leq 1\}$$

We want to work with the second description since that one will allow us to integrate first with respect to x. We truncate the domain:

$$T' = \{(x, y) | 0 \leq x \leq 1 - y;\ 0 \leq y \leq b\}$$

where $0 \leq b \leq 1$. The integral is

$$\lim_{b \to 1^-} \int_0^b \int_0^{1-y} \frac{\sin(y)}{1-y}\, dx\, dy = \lim_{b \to 1^-} \int_0^b \sin(y)\, dy$$

This last is $\lim_{b \to 1^-} - \cos(b) + \cos(0) = 1 - \cos(1)$.

[10 points]

4 Evaluate $\int\int_S \frac{1}{x^2+y^2}\, dA$ where S is the region in the first quadrant bounded by the x-axis, the y-axis and the circle $x^2 + y^2 = 1$, if it is convergent.

The singularity in the domain occurs at the point $(0, 0)$. If you do not finish the problem by using polar coordinates, you'll want to use the truncated domain:

$$S' = \{(x, y) \mid b \leq x \leq 1;\ 0 \leq y \leq \sqrt{1 - x^2}\}$$

and take the limit as $b \to 0^+$.

However, the problem is greatly simplified by converting to polar coordinates. In that case, the domain becomes $R = \{(r, \theta) \mid 0 \le r \le 1;\ 0 \le \theta \le \pi/2\}$ and the truncated domain is

$$R' = \{(r, \theta) \mid b \le r \le 1;\ 0 \le \theta \le \pi/2\}$$

The integral is then

$$\lim_{b \to 0^+} \int_b^1 \int_0^{\pi/2} (1/r^2) r \ d\theta \ dr$$

$$= \lim_{b \to 0^+} (\pi/2)(\ln(1) - \ln(b))$$

However, the last limit diverges to infinity. Thus the integral does not converge.

[10 points]

5 Evaluate the integral

$$\int \int \int_R z \ dV$$

over the region R between the paraboloids $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$ using cylindrical or spherical coordinates.

It is easier to complete this problem using cylindrical coordinates. The circle of intersection of the two surfaces is

$$x^2 + y^2 = 4 - x^2 - y^2$$

or $x^2 + y^2 = 2$. The surfaces are $z = r^2$ and $z = 4 - r^2$. Thus the region in cylindrical coordinates is described by

$$R = \{(r, \theta, z) \mid 0 \le r \le \sqrt{2};\ 0 \le \theta \le 2\pi;\ r^2 \le z \le 4 - r^2\}$$

and the integral is

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{4-r^2} zr \ dz \ dr \ d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} (1/2) z^2 r |_{r^2}^{4-r^2} \ dr \ d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} (1/2)(16 - 8r^2 + r^4 - r^4) r \ dr \ d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} (8r - 4r^3) \ dr \ d\theta$$

$$= \int_0^{2\pi} 4r^2 - r^4 |_0^{\sqrt{2}} \ d\theta = 2\pi(8 - 4) = 8\pi$$

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[5 points] 6 The change of coordinates

$$x = r\sec(\theta)$$

$$y = r\tan(\theta)$$

simplifies a certain integral $\int\int_R f(x,y)\ dA$. What is the area element dA in terms of r and θ?

$$dA = \begin{vmatrix} \sec(\theta) & r\sec(\theta)\tan(\theta) \\ \tan(\theta) & r\sec^2(\theta) \end{vmatrix}$$

$$= r\sec(\theta)(\sec^2(\theta) + \tan^2(\theta)) = r\sec(\theta)$$

END OF EXAM