# Take Home Quiz

Due Friday, April 1 (no joke!)

NAME:

This is an EXTRA quiz and is optional in the sense that I will only count 4 out of the total of 6 quizzes for the semester. In order to get credit for this quiz, your solution must be PERFECT. Of course, the definition of perfect is subjective - I am unlikely to fail you for small errors like a sign error, etc. However, you will receive 0/15 on the quiz if there are ANY misunderstandings about spherical coordinates or integration. Please turn your quizzes in to me in class on April 1 (or before that in my office). I will accept no late quizzes.

Choose ONE of the problems below.

1. Evaluate the iterated integral

$$\int_0^1 \int_{\sqrt{1-y^2}}^{\sqrt{8-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} x^2+y^2+z^2 \; dz\; dx\; dy \; + \int_1^{\sqrt{8}} \int_0^{\sqrt{8-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} x^2+y^2+z^2 \; dz\; dx\; dy$$

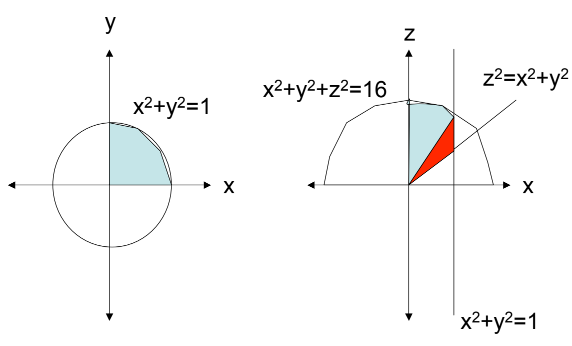
   by changing to spherical coordinates.

2. Evaluate the iterated integral

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} x^2+y^2+z^2 \; dz\; dx\; dy$$

   by changing to spherical coordinates.

   Most people attempted this problem. Here are the cross sections of this shape in the $xy$-plane and in the $xz$-plane (the section in the $yz$-plane is the same).



The first graph tells us that the domain of $\theta$ will be $0 \le \theta \le \pi/2$. The second graph tells us that we need TWO integrals - since the bounds for $\rho$ are not the same in the blue area and in the red area. In the blue area, we have $0 \le \rho \le 4$. In the red area we have $0 \le \rho \le 1/\sin(\phi)$ (this last is the spherical version of $x^2+y^2 = 1$ - use the change of coordinates $x = \rho\sin(\phi)\cos(\theta)$ etc.). Now we have to solve for important values of $\phi$. The blue area meets the red area where the sphere $x^2+y^2+z^2 = 16$ intersects the cylinder $x^2+y^2 = 1$. When this happens,

$$\phi = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \arctan(1/\sqrt{15}).$$

I'm using the arctangent formula that I showed you in class, and I'm using that $x^2+y^2 = 1$ and $z = \sqrt{16-x^2-y^2}$. However, if you prefer to use arcsine (since that is suggested by the diagram), it is also correct. You should get $\phi = \arcsin(1/4)$. Thus, in the blue area of the second graph we have $0 \le \phi \le \arctan(1/\sqrt{15})$. The next important value of $\phi$ occurs when the cylinder $x^2+y^2 = 1$ intersects the cone $z = \sqrt{x^2+y^2}$. When this happens, $\phi = \arctan(1) = \pi/4$. Thus, in the red area of the graph we have $\arctan(1/\sqrt{15}) \le \phi \le \pi/4$. Finally, the integrand $x^2+y^2+z^2 = \rho^2$. In spherical coordinates, the integral becomes

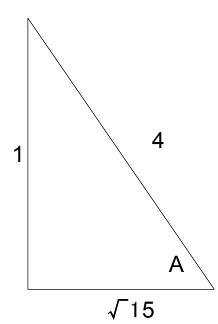
$$\int_0^{\pi/2}\int_0^{A}\int_0^{4} \rho^4\sin(\phi)\,d\rho\,d\phi\,d\theta + \int_0^{\pi/2}\int_Z^{\pi/4}\int_0^{1/\sin(\phi)} \rho^4\sin(\phi)\,d\rho\,d\phi\,d\theta$$

where $A = \arctan(1/\sqrt{15})$.

The first integral in this sum is pretty easy to compute. We have

$$\int_0^{\pi/2}\int_0^{A}(1/5)\rho^5|_0^4\sin(\phi)\,d\phi\,d\theta = \int_0^{\pi/2}(1024/5)(\cos(A)-cos(0))\,d\theta = (512/5)\pi(1-\cos(A)).$$

We can simplify $\cos(\arctan(1/\sqrt{15}))$ if we want to. I'll type this computation up ONCE so you can see it, but you weren't required to do this. Here is a right triangle with angle A satisfying the property that $\tan(A) = 1/\sqrt{15}$. That is, $A = \arctan(1/\sqrt{15})$.




Then $\cos(A) = \sqrt{15}/4$.

The second integral is a little bit harder, but it isn't so bad if you just use a table. We have

$$\int_0^{\pi/2}\int_A^{\pi/4}(1/5)\rho^5|_0^{1/\sin(\phi)}\sin(\phi)\,d\phi\,d\theta = 1/5\int_0^{\pi/2}\int_A^{\pi/4}\csc^4(\phi)\,d\phi\,d\theta$$

$$= 1/5 \int_0^{\pi/2} -(1/3)\csc^2(\phi)\cot(\phi) - (2/3)\cot(\phi)\big|_A^{\pi/4}\, d\theta$$

$$= (\pi/10)(-2/3 - 2/3 + 1/3\csc^2(A)\cot(A) + 2/3\cot(A))$$

$$= (\pi/10)(-4/3 + (1/3)(4^2)(\sqrt{15}) + (2/3)(\sqrt{15})) = -(2/15)\pi + (3/5)\sqrt{15}\pi$$

Adding the two of these, we get $\frac{1534\pi}{15} - 25\sqrt{15}\pi$.

I had expected you guys to learn a great deal about spherical coordinates with this problem. What I found while grading it was that you also learned a GREAT deal about "reasonable" numbers. It's true that most problems are doctored so that the answers are integers, or integer multiples of $\pi$, or 0, etc. HOWEVER, most problems that arise in nature DO NOT share this property. One word of advice ... it is SILLY to mix exact numbers like $\pi$ with DECIMAL APPROXIMATIONS. One of these is correct to infinitely many significant digits, while the other is correct to only a few. Personally, I prefer to leave the decimal approximations out of the picture for this reason. It's up to you, but don't MIX!