Quiz 5
Math 349 Lecture 01

Tuesday, November 30 2004

NAME:

JUSTIFY YOUR ANSWERS. Answer each question in the space provided. A correct answer without work shown may be worth 0 points, while an incorrect answer with full justifiction may be worth partial credit. Each question is worth 5 points.

1. Let $f(x, y, z) = \sin(x \ln(y) + z)$ and $g(t)$ be the function defined parametrically by $g(t) = (2t, \sqrt{t}, t)$.

   (a) Find ∇f.

   (b) Use the Chain Rule to find $h'(1)$ where $h = f \circ g$. That is, $h(t) = f(x, y, z)$ where $x = 2t$, $y = \sqrt{t}$, $z = t$.

   (a) $\nabla f(x, y, z) = \langle \ln(y) \cos(x \ln(y) + z), (x/y) \cos(x \ln(y) + z), \cos(x \ln(y) + z) \rangle$.

   (b) In addition to the information stored in $\nabla f(x, y, z)$, we will also need to use the partial derivatives:

   $$\frac{dx}{dt} = 2 \qquad \frac{dy}{dt} = \frac{1}{2\sqrt{t}} \qquad \frac{dz}{dt} = 1$$

   The chain rule says that $h'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial x}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$. That is:

   $$h'(t) = \ln(y) \cos(x \ln(y) + z)(2) + (x/y) \cos(x \ln(y) + z)\frac{1}{2\sqrt{t}} + \cos(x \ln(y) + z)(1)$$

   When $t = 1$, $x(1) = 2$, $y(1) = 1$ and $z(1) = 1$. This becomes:

   $$h'(1) = 0 + (2) \cos(1)(1/2) + \cos(1) = 2 \cos(1).$$

   By the way, this isn't the only way to approach this problem. One could also approach this by using substitution. However, you were asked to use the chain rule, so I've only presented that solution.

2. Find an equation of the tangent plane to $f(x, y) = \arcsin(x^2 - y + 1)$ at $a = -1$, $b = 2$. Use the linearization to approximate the value of f at $x = -0.9$, $y = 1.8$.
   (Hint: $\frac{d}{dt}(\arcsin(t)) = \frac{1}{\sqrt{1-t^2}}$.)

   The tangent plane is $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$. Taking the appropriate derivatives, we get:

   $$f_x(x, y) = \frac{2x}{\sqrt{1 - (x^2 - y + 1)^2}} \qquad f_y(x, y) = \frac{-1}{\sqrt{1 - (x^2 - y + 1)^2}}$$

   and we have $f_x(-1, 2) = -2$ and $f_y(0, 2) = -1$. We also need $f(-1, 2) = \arcsin(0) = 0$. The equation of the tangent plane is then

   $$z = 0 - 2(x + 1)) - (y - 2) = -2x - y.$$

The tangent plane also gives the approximation of $f(x, y)$ near the point (a, b). Thus,

$$f(0.9, 1.8) \approx -2(-0.9) - (1.8) = 0.$$

3. Find the second derivatives $\frac{\partial^2 f}{\partial x^2} = f_{xx}$ and $\frac{\partial^2 f}{\partial y^2} = f_{yy}$ of

$$f(x, y) = \frac{xy^2}{x^2 + y}.$$

The first partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{y^2(x^2 + y) - 2x^2y^2}{(x^2 + y)^2} = \frac{-x^2y^2 + y^3}{(x^2 + y)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(2xy)(x^2 + y) - xy^2}{(x^2 + y)^2} = \frac{2x^3y + xy^2}{(x^2 + y)^2}$$

Now take the appropriate derivatives of these:

$$\frac{\partial^2 f}{\partial x^2} = \frac{(-2xy^2)(x^2 + y)^2 - 2(x^2 + y)(2x)(-x^2y^2 + y^3)}{(x^2 + y)^4} = \frac{2x^3y^2 - 6xy^3}{(x^2 + y)^3}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{(2x^3 + 2xy)(x^2 + y)^2 - 2(x^2 + y)(2x^3y + xy^2)}{(x^2 + y)^4} = \frac{2x^5}{(x^2 + y)^3}$$