Quiz 3
Math 349 Lecture 01

Tuesday, October 19 2004

NAME:

JUSTIFY YOUR ANSWERS. Answer each question in the space provided. A correct answer without work shown may be worth 0 points, while an incorrect answer with full justifiction may be worth partial credit. Each question is worth 5 points.

1. Is the series
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n\sqrt{n}}$$
absolutely convergent, conditionally convergent or divergent? Explain.

   First we test for absolute convergence. That is, we have to determine whether or not the series
$$\sum_{n=1}^{\infty} \frac{1}{2^n\sqrt{n}}$$
is convergent. When $n \geq 1$, $\sqrt{n} \geq 1$ so $2^n\sqrt{n} \geq 2^n$. Inverting this, we have
$$\frac{1}{2^n\sqrt{n}} \leq \frac{1}{2^n}$$
The series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $r = 1/2$, hence it is convergent. By the comparison test, so is our series $\sum_{n=1}^{\infty} \frac{1}{2^n\sqrt{n}}$. Since the series is absolutely convergent, it is convergent.

2. What is the center, radius and interval of convergence for
$$\sum_{n=2}^{\infty} \frac{(x-1)^n}{\ln n}.$$
The center is $c = 1$. To find the radius of convergence, calculate $R = 1/L$ where
$$L = \lim_{n\to\infty} \frac{\ln n}{\ln n + 1} = \lim_{n\to\infty} \frac{n+1}{n} = 1$$
(using L'Hopital's rule). Thus the radius of convergence is $R = 1/1 = 1$. So the interval of convergence is *at least* $(0, 2)$. To find the precise interval of convergence, we need to test the endpoints. That is, we need to check the series with $x = 0$ and $x = 2$. The first one is the series
$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$$

which converges by the alternating series test: $\frac{1}{\ln n} > 0$ when $n \geq 2$, $\ln n$ is increasing so $\frac{1}{\ln n}$ is decreasing, and $\lim_{n\to\infty} \frac{1}{\ln n} = 0$. The endpoint $x = 2$ corresponds to the series

$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$

which diverges. One can show this several different ways. One way is to realize that $\frac{1}{n \ln n} \leq \frac{1}{\ln n}$, and the first series diverges by the integral test. Thus, the interval of convergence is $[0, 2)$.

3. Expand $f(x) = \frac{1}{1+3x}$ in powers of $(x - 1)$. Find the interval where the representation is valid.

   First, rewrite $f(x)$ in terms of $(x - 1)$:

$$\frac{1}{1+3x} = \frac{1}{1+3(x-1)+3} = \frac{1}{4+3(x-1)} = \frac{1}{4(1+\frac{3(x-1)}{4})}$$

   Now use the power series representation

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n$$

   which is valid for $-1 < t < 1$. Substituting $t = \frac{3(x-1)}{4}$, we have

$$\frac{1}{1+3x} = \sum_{n=0}^{\infty} (-1)^n (\frac{3(x-1)}{4})^n$$

   This is valid for $-1 < \frac{3(x-1)}{4} < 1$, or $-1/3 < x < 7/3$.