Quiz 4: Friday, December 3
Math 251 Lecture 01

NAME and/or ID: ____

Calculators, books, notes and the help of friends is NOT PERMITTED!
Part I: Multiple choice. Circle the correct answer. There will be no partial marks.

[3 pts]

1. Which of the following calculates the area below $y = 1 + x^2$, above the x-axis, between $x = 0$ and $x = 2$?

   (a) $\sum_{i=0}^{n-1} \left(1 + (\frac{2}{n}i)^2\right) (\frac{2}{n})$;
   (b) $\lim_{n\to\infty} \sum_{i=0}^{n-1} \left(1 + (\frac{2}{n}i)^2\right) (\frac{2}{n})$;
   (c) $\sum_{i=1}^{n} (\frac{2}{n}i)^2 (\frac{2}{n})$;
   (d) $\lim_{n\to\infty} \sum_{i=1}^{n} (\frac{2}{n}i)^2 (\frac{2}{n})$;

   It's easy to get this question down to two choices. The finite sums can not calculate the area. Area is calculated by a limit of a Riemann sum of the form:

   $$\lim_{n\to\infty} \sum_{i=0 \text{ or } 1}^{n-1 \text{ or } n} f(x_i)\Delta x$$

   where the "or" in the sum refers to the difference between the Left Hand method or the Right Hand method. To figure out whether the right answer is (b) or (d), we need to understand what the x_i and Δx are. Since the area is from 0 to 2, we have:

   $$\Delta x = \frac{(2-0)}{n} = 2/n$$

   If we divide the interval $[0, 2]$ into n equal pieces, this is the width of each piece. Then:

   $$x_i = 0 + (2/n)i$$

   Starting from the left most endpoint ($x = 0$), this indicates that we have to go accross i pieces of the interval, each of width $2/n$ to get to x_i. So, the Riemann sum is:

   $$\lim_{n\to\infty} \sum_{i=0 \text{ or } 1}^{n-1 \text{ or } n} f((2/n)i)(2/n)$$

   Since $f(x) = 1 + x^2$, this is

   $$\lim_{n\to\infty} \sum_{i=0 \text{ or } 1}^{n-1 \text{ or } n} \left(1 + (\frac{2}{n}i)^2\right) \frac{2}{n}$$

   The answer must be (b).

[3 pts]

2. Which of the following limits is the same as $\lim_{x\to 0}\frac{\cos^3(x)-1}{x^2}$?

   (a) $\lim_{x\to 0}\frac{\sin^3(x)}{2}$;

   (b) $\lim_{x\to 0}\frac{-3x^2\cos^2(x)\sin(x)-2\cos^3(x)}{x^4}$;

   (c) $\lim_{x\to 0}\frac{-3\cos^2(x)\sin(x)}{2}$;

   (d) $\lim_{x\to 0}\frac{-3\cos^2(x)\sin(x)}{2x}$.

   The first limit is an indeterminant form of the form $\frac{0}{0}$. So this limit could actually be equal to anything, including possibly any of the entries below. The best way to proceed is to use L'Hospital's rule:

$$\lim_{x\to 0}\frac{\cos^3(x)-1}{x^2}=\lim_{x\to 0}\frac{-3\cos^2(x)\sin(x)}{2x}$$

   Hey. lucky for us this is entry (d). If we use L'Hospital's rule once more we would find that the answer is actually $-3/2$, which precludes any of the other choices.

Part II. SHOW ALL YOUR WORK. Answer each question in the space provided. A correct answer without work shown may be worth 0 points, while an incorrect answer with work shown may be worth partial credit.

[7 pts]

3. Find $f(x)$ if $f''(x)=-\sin(3x)$, $f'(0)=1/3$, $f(0)=10$.

   An antiderivative of $f''(x)=-\sin(3x)$ is $f'(x)=(1/3)\cos(3x)+C$. Since $f'(0)=1/3+C=1/3$, we find that $C=0$. An antiderivative of $f'(x)=(1/3)\cos(3x)$ is $f(x)=(1/9)\sin(3x)+D$. Since $f(0)=0+D=10$, we find that $D=10$. Then

$$f(x)=(1/9)\sin(3x)+10.$$

[7 pts]

4. Find the dimensions of the rectangle of smallest diagonal whose perimeter is 10.

   My rectangles has sides of length x and y. We have to minimize the diagonal, which I named d. Using the Pythagorean theorem, we have

$$d^2=x^2+y^2 \qquad x\geq 0, y\geq 0.$$

   That means that we have to minimize the function $d=\sqrt{x^2+y^2}$. Unfortunately, this function has too many variables. So we have to use a constraint equation. In this case, the extra information is that the perimeter is equal to 10. That is,

$$2x+2y=10 \qquad 0\leq x,y\leq 5.$$

   If we solve this equation for y, we find $y=5-x$. Substitute this into the original equation to get a new equation to be minimized:

$$d=\sqrt{x^2+(5-x)^2} \qquad 0\leq x\leq 5.$$

   Take the derivative of the function:

$$d'=\frac{2x+2(5-x)(-1)}{2\sqrt{x^2+(5-x)^2}}.$$

This is equal to 0 when $2x - 5 = 0$ (that's the numerator), or in other words when $x = 5/2$. Now we have to check if this is the absolute minimum. Since we have endpoints, the easiest way to do that is by checking the value of the diagonal for the critical point and both endpoints:

$$d(5/2) = 5/\sqrt{2} \qquad d(0) = 5 \qquad d(5) = 5.$$

Now, $5/\sqrt{2} < 5$ so indeed this is a minimum. The dimensions of the rectangle with minimal diagonal and perimeter 10 is $5/2 \times 5/2$.