Quiz 4: Thursday, November 18
Math 251 Lecture 01

NAME and/or ID: ____

Calculators, books, notes and the help of friends is NOT PERMITTED!
Part I: Multiple choice. Circle the correct answer. There will be no partial marks.

[3 pts]

1. Only one of the functions below has an inverse. Find it.

   (a) $f(x) = e^{x^2}$;

   (b) $f(x) = \sin(\pi x)$ where $0 \le x \le 1$;

   (c) $f(x) = \cos(x)$ where $0 \le x \le \pi/2$;

   (d) $f(x) = |x|$.

   A function has an inverse if it is one-one. The functions (a), (b) and (d) are not one-one, here are some examples of why they aren't one-one:

$$e^{1^2} = e^{(-1)^2}$$

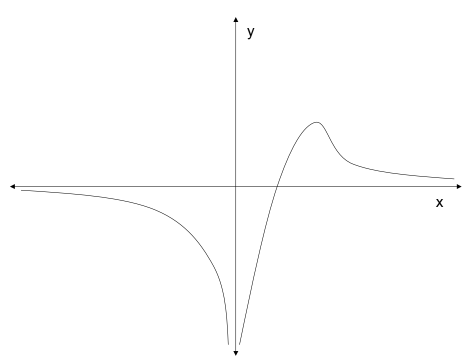
$$\sin(0) = \sin(\pi)$$

$$|-1| = |1|$$

   However, for (c), we have restricted the domain so that the function **is** one-one on its domain. In particular, on the domain $0 \le x \le \pi/2$ we have $f'(x) = -\sin(x) \le 0$. Thus the function is strictly decreasing on $0 \le x \le \pi/2$. Since the function is decreasing, it must be one-one.

[3 pts]

2. The graph of a function $y = f(x)$ is pictured below.



   Which one of the functions below could be $f(x)$?

(a) $f(x) = \frac{x-1}{x^2}$;

(b) $f(x) = \frac{x^2-1}{x^2}$;

(c) $f(x) = \frac{x^2-1}{x^4}$;

(d) $f(x) = \frac{\sqrt{1-x^2}}{x}$.

The function (b) has a horizontal asymptote at $y = 1$. The graph has a horizontal asymptote at $y = 0$, so this can't be the one. The function (c) only has two x-intercepts, at $x = \pm 1$. The graph has only one x-intercepts, so this can't be the one. The function (d) is an odd function. Our graph is not symmetric about the origin - so this can't be the one. By elimination, it must be (a). We can check further - (c) is neither even nor odd, has an intercept at $x = 1$, a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0$. So it looks good. If you are in a hurry, it's a good idea to tentatively choose (a) and move on to do the rest of the problems. If you really want to check that (a) is the right one, you might consider doing the following kind of analysis:

$$f'(x) = \frac{x^2 - 2x(x-1)}{x^4} = \frac{-x+2}{x^3}$$

$$f''(x) = \frac{-x^3 - 3x^2(-x+2)}{x^6} = \frac{2x-6}{x^4}$$

The first derivative tell us that the function is increasing on $(0, 2)$ and decreasing everywhere else. The second derivative tells us that the function is concave up on $(3, \infty)$ concave down everywhere else. We've already located the x-intercept and there are no y-intercepts because of the asymptote. We can check that $\lim_{x \to 0^+} f(x) = -\infty$ and similarly $\lim_{x \to 0^-} f(x) = -\infty$. This is enough information to determine that the graph is the one pictured.

Part II. SHOW ALL YOUR WORK. Answer each question in the space provided. A correct answer without work shown may be worth 0 points, while an incorrect answer with work shown may be worth partial credit.

[7 pts]

3. The half-life of a radioactive twinkie is 25,000 years (that is, after 25,000 years, one-half of the original sample of twinkie remains). How much of the twinkie remains after 25 years?

Radioactive decay is modelled by exponential decay. So, a function which represents the amount of twinkie remaining after t years is

$$P(t) = P_0 e^{kt}$$

where P_0 is the initial amount of twinkie and k is the decay constant. We can solve for the decay constant by using the given information about the half-life:

$$(1/2)P_0 = P_0 e^{25000k}$$

Simplify by dividing each side by P_0 and taking the logarithm:

$$1/2 = e^{25000k};$$

$$\ln(1/2) = 25000k.$$

Now we solve $k = \ln(1/2)/25000$ (you don't have to simplify this).

Now we can compute that when $t = 25$, $P(t) = P_0 e^{\ln(1/2)/1000}$. This is expressed as a fraction of the original P_0, which is enough.

[7 pts]

4. Let $f(x) = 2\sqrt{x} - x$. Find any critical points of $f(x)$. Classify the critical points as local maxima or minima or absolute maxima or minima.

To find the critical points, we have to take the derivative: $f'(x) = \frac{1}{\sqrt{x}} - 1 = \frac{1-\sqrt{x}}{\sqrt{x}}$. Now we need to solve for the points where $f'(x) = 0$. That is, we need to solve $1 - \sqrt{x} = 0$ (the function is 0 when the numerator is 0). This happens only once, when $x = 1$. Now we need to check to see if this is a local maximum or minimum. We can do this using the second derivative test:

$$f''(x) = -1/2(x)^{-3/2}$$

Note that this is ALWAYS less than 0 on the domain of $f(x)$. That means the function is always concave down. This ensures that the critical point is both a local max and an absolute max. This method is particularly nice, because we don' t have to worry about checking the endpoint $x = 0$ of the domain using this.