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Exam Code: Math 251 L01 Bauer

Calculators and one page of notes are permitted.

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50 MINUTES

UNIVERSITY OF CALGARY

MIDTERM EXAM

SESSION 2004
FALL SEMESTER

DATE: 29 October 2004
TIME: 9:00-9:50

Please show all your work clearly.
Incorrect answer with work shown may receive partial credit, while correct answers with no work shown may receive no credit.
Please write your student number on every page, beginning with this one.
Good luck!

For Markers Only	
1	/15
2	/10
3	/15
4	/10
5	/10
6	/10
$\sum$	/70

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1 Evaluate the following limits. Justify your answers carefully.

[10 points] (a) $\lim_{x\to 1} \frac{\sqrt{x}-1}{1-x}$

Multiply on the top and bottom by the conjugate:

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{1-x}\frac{\sqrt{x}+1}{\sqrt{x}+1} = \lim_{x\to 1} \frac{x-1}{(1-x)(\sqrt{x}+1)} = \lim_{x\to 1} \frac{-1}{\sqrt{x}+1} = -1/2$$

The last step follows from the fact that the function $\sqrt{x}+1$ is continuous.

[10 points] (b) $\lim_{x\to\infty} \frac{\sin^2(x)}{x^2}$

Unfortunately, the limit we did in class related to finding the derivative of $\sin x$ is

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Since that limit occurs as $x \to 0$, and we are interested in a limit as $x \to \infty$, we can't use it.

Instead, we'll have to use the Squeeze Theorem. We start with the inequality $0 \leq \sin^2(x) \leq 1$. Multiply through by $1/x^2$ (note that $1/x^2 > 0$):

$$0 \leq \frac{\sin^2(x)}{x^2} \leq \frac{1}{x^2}$$

Now take limits:

$$0 = \lim_{x\to\infty} 0 \leq \lim_{x\to\infty} \frac{\sin^2(x)}{x^2} \leq \lim_{x\to\infty} \frac{1}{x^2} = 0$$

By the Squeeze Theorem, we can conclude that $\lim_{x\to\infty} \frac{\sin^2(x)}{x^2} = 0$.

[10 points]

2 For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx + 1 & \text{if } x \leq 3; \\ cx^2 - 1 & \text{if } x > 3. \end{cases}$$

For this function to be continuous, we need to know that $\lim_{x \to a} f(x) = f(a)$ for all a. Since $f(x)$ is a polynomial when $x \neq 3$, it is continuous for all $x \neq 3$ for any value of c. On the other hand, at $x = 3$, we need to know that

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x) = f(3)$$

That is, we need to know that

$$3c + 1 = 9c - 1$$

Solving this equation is not difficult, we obtain $c = 1/3$. So the function is continuous for $c = 1/3$.

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3 Find $\frac{dy}{dx}$ for the following:

[10 points] (a) $y = \sqrt{\frac{x^3+1}{x^3-1}}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\frac{x^3+1}{x^3-1}}}\frac{(x^3-1)(3x^2)-(x^3+1)(3x^2)}{(x^3-1)^2}$$

Any simplification of this is acceptable.

[10 points] (b) $xy = \cot(xy)$

First differentiate implicitly:

$$x\frac{dy}{dx} + y = -\csc^2(xy)(x\frac{dy}{dx} + y)$$

Now, remember to expand the right hand side:

$$x\frac{dy}{dx} + y = -x\csc^2(xy)\frac{dy}{dx} - y\csc^2(xy)$$

Finally, solve for $\frac{dy}{dx}$:

$$(x + x\csc^2(xy))\frac{dy}{dx} = -y - y\csc^2(xy)$$

$$\frac{dy}{dx} = \frac{-y}{x}\frac{1+\csc^2(xy)}{1+\csc^2(xy)} = -\frac{y}{x}$$

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[10 points]

4 If an arrow is shot straight upward on the moon with a velocity of 58 m/s, it's height in meters after t seconds is given by

$$h = 58t - \frac{4}{5}t^2$$

How long will it take for the arrow to reach the peak of its trajectory?

The arrow reaches the top of its trajectory when the velocity is 0. To compute the velocity, take the derivative:

$$v(t) = h'(t) = 58 - \frac{8}{5}t$$

We solve to find out when the velocity is equal to 0:

$$0 = 58 - \frac{8}{5}t$$

$$\frac{8}{5}t = 58$$

$$t = 58\frac{5}{8}$$

It takes $58\frac{5}{8}$ seconds for the arrow to reach the peak of its trajectory.

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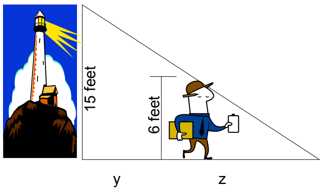
[15 points]

5 A street light is at the top of a 15 foot tall pole. A man 6 feet tall walks away from the pole with a speed of 5 ft/s along a straight sidewalk. How fast is the tip of his shadow moving when he is 40 ft from the pole?



In the diagram above, y is the distance from the streetlight to the man, and z is the length of the man's shadow. We know that $\frac{dy}{dt} = 5$. We are asked to find $\frac{dy}{dt} + \frac{dz}{dt}$ when $y = 40$. Note that $\frac{dz}{dt}$ is the rate at which the man's shadow is *lengthening*, but we are acutally asked to find how fast the tip of his shadow is *moving*. For this we need to add the speed of the man, since the shadow will move when he does. I didn't take off marks for anyone who just found $\frac{dz}{dt}$, since finding the rate at which the shadow is lengthening is a reasonable interpretation of the question.

Either way, we need to find $\frac{dz}{dt}$. For this we need an equation which relates y and z. Using similar triangles, we have

$$\frac{15}{y+z} = \frac{6}{z}$$

Since we'll have to differentiate, it's wise to simplify this expression to

$$15z = 6y + 6z$$

or

$$9z = 6y.$$

Differentiate both sides:

$$9\frac{dz}{dt} = 6\frac{dy}{dt}$$

and now plug in $\frac{dy}{dt} = 5$ to find $\frac{dz}{dt} = (2/3)(5) = 10/3$. That is, the shadow is lengthening at a rate of 10/3 feet per second. The tip of the shadow is moving at a rate of $5 + 10/3 = 25/3$ feet per second.

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[15 points]

6 Use a linear approximation to estimate the value of $\tan(51^o)$.

Let $f(x) = \tan(x)$. Then $f'(x) = \sec^2(x)$. The linear approximation to the function $f(x)$ near $x = a$ is

$$f(x) \approx f'(a)(x-a) + f(a)$$

We want to approximate the function $\tan(x)$ at $x = 51^o$. For this, we'll need to choose an a which we can evaluate and which is reasonably close to 51^o. The closest angle whose values we know is $a = 45^o$. One last thing to worry about... the derivative of $\tan(x)$ that we calculated above relies on the fact that x is an angle measured in radians. It's easy to solve this problem, we'll just convert everything into radians: $51^o = \frac{51\pi}{180} rad$ and $45^o = \pi/4 rad$. Now just plug into the formula above:

$$\tan(51) \approx \sec^2(\pi/4)(51\pi/180 - \pi/4) + \tan(\pi/4)$$

That is, $\tan(51) \approx 2(\pi/30) + 1 = \pi/15 + 1$.

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[10 points]

7 Ler $f(x)$ be a differentiable function. Use the *definition of the derivative* to prove the reciprocal rule:

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{[f(x)]^2}$$

The definition of the derivative says that $g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$. We need to evaluate this definition for $g(x) = \frac{1}{f(x)}$. Here goes:

$$\lim_{h\to 0}\frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h} = \lim_{h\to 0}\frac{1}{h}\frac{f(x)-f(x+h)}{f(x)f(x+h)} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\frac{-1}{f(x)f(x+h)}$$

Now, since $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x)$, we can simplify this to

$$= f'(x)\lim_{h\to 0}\frac{-1}{f(x)f(x+h)}$$

Finally, since f is differentiable it is continuous, so to evaluate the limit, simply plug in $h = 0$:

$$= f'(x)\frac{-1}{f(x)f(x)} = \frac{-f'(x)}{[f(x)]^2}$$

as desired.

END OF EXAM