Name:____________________

Lab Section:____________________

Exam Code: Math 251 L04 and L06/13

Calculators, books, notes and collaboration are *not permitted.*

120 MINUTES

UNIVERSITY OF CALGARY

FINAL EXAM

SESSION 2003
FALL SEMESTER

DATE: 22 December 2003
TIME: 3:30 - 5:30

Please write your student number on every page, beginning with this one.
Good luck!

For Markers Only	
1	/10
2	/6
3	/20
4	/10
5	/10
B	/39
Σ	/69

Student number:________________________

SECTION A

Show your work. An incorrect answer with work shown may receive partial credit, while correct answers with no work may receive no credit.

[10]

1 Find $\frac{dy}{dx}$ for $y = (2 + \cos(x))^{\ln(2+\cos(x))}$.

You must use logarithmic differentiation to solve this problem. Take the natural log of both sides to get:

$$\ln|y| = \ln|2 + \cos(x)|^{\ln(2+\cos(x))}$$

Bring down the exponent:

$$\ln|y| = \ln(2 + \cos(x))\ln|2 + \cos(x)|$$

Differentiate implicitly with respect to x - don't forget the product rule!

$$(1/y)y' = \frac{-\sin(x)}{2+\cos(x)}\ln|2+\cos(x)| + \frac{-\sin(x)}{2+\cos(x)}\ln(2+\cos(x))$$

Ok, simplify. Remember to plug in for y. Also, notice that the domain of the original function must have $2 + \cos(x) > 0$ (otherwise the natural logarithm in the exponent is not defined). So we can *drop* the absolute vaule signs here. The final answer is:

$$y' = (2+\cos(x))^{\ln(2+\cos(x))}\frac{-2\sin(x)}{2+\cos(x)}\ln(2+\cos(x))$$

[6]

2 Let $f(x)$ be differentiable on $[a, b]$. If $f'(x) > 0$ for every $x \in [a, b]$, explain why

$$\frac{f(b) - f(a)}{b - a} > 0.$$

Since $f(x)$ is differentiable, we can apply the Mean Value Theorem to f on $[a, b]$ to find that there is a value of c with $a < c < b$ and

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Since $f'(c) > 0$, we have $\frac{f(b)-f(a)}{b-a} > 0$ as well.

$f'(x)$	$x < \sqrt[3]{-2}$	$\sqrt[3]{-2} < x < 0$	$x > 0$
x^3+2	-	+	+
x^3	-	-	+
$f'(x)$	+	-	+

3 Let $f(x) = \frac{x^3-1}{x^2}$. The first two derivatives of $f(x)$ are:

$$f'(x) = \frac{x^3+2}{x^3}, \qquad f''(x) = \frac{-6}{x^4}.$$

[5]

(a) Find all asymptotes of $f(x)$.

The vertical asymptote is at $x = 0$. We have $\lim_{x\to 0^+} f(x) = -\infty$ and $\lim_{x\to 0^-} f(x) = \infty$.

There is a slant asymptote at $y = x$. To find this, write $f(x) = x - (1/x^2)$.

There is no horizontal asymptote.

[3]

(b) Find all x and y coordinates of the intercepts, critical points, and inflection points of $f(x)$, if they exist.

The intercepts are at $(1, 0)$. There is no y intercept (because of the asymptote).

The critical points are at $x = 0$, where there is a vertical asymptote, and at $(\sqrt[3]{-2}, -3(-2)^{-2/3})$.

There are no inflection points (except possibly $x = 0$).

[5]

(c) Find all of the intervals where $f(x)$ is increasing and all intervals where $f(x)$ is decreasing.

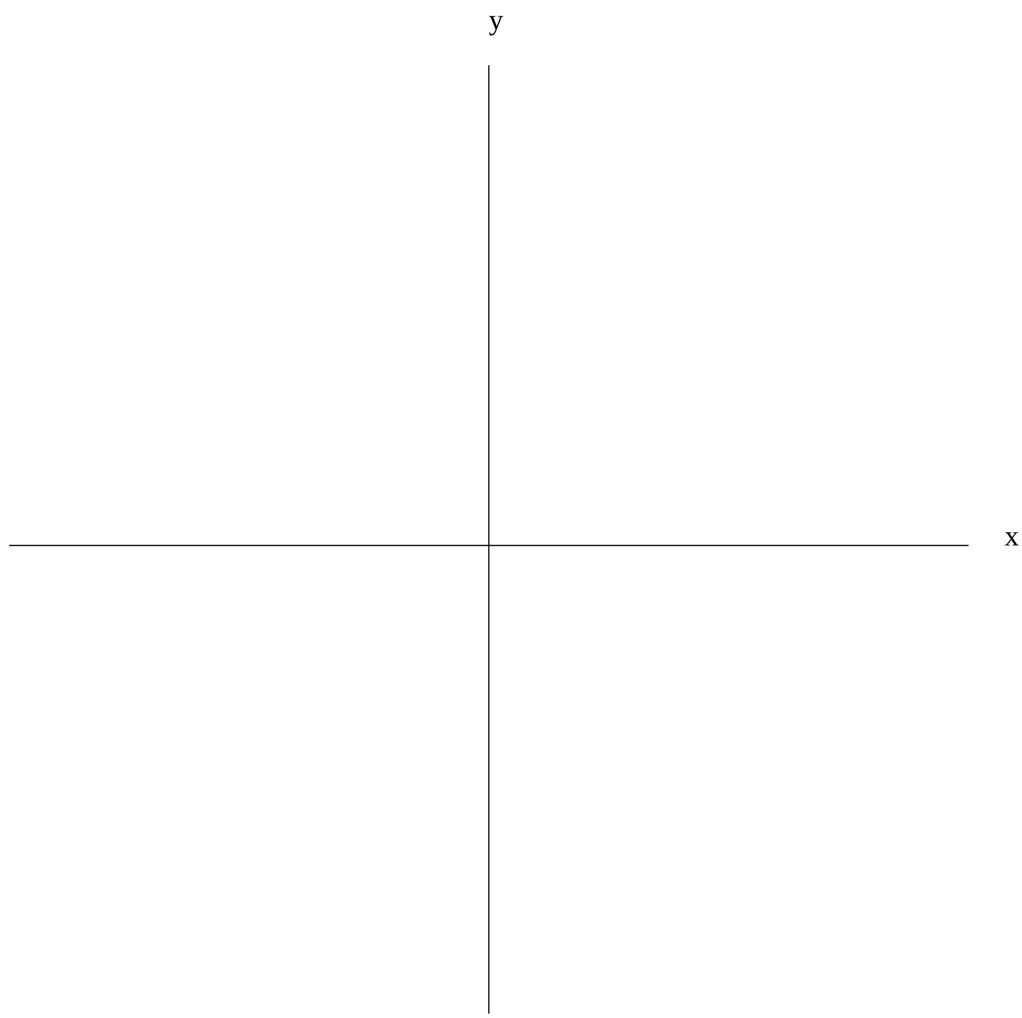
Here, $f'(x)$ is increasing on the intervals with $+$ and decreasing on the interval with $-$.

[5]

(d) Find all of the intervals where $f(x)$ is concave up and all intervals where $f(x)$ is concave down.

$f(x)$ is always concave down.

[2] (e) Sketch the graph of $f(x)$ on the axes below.



(to appear - it looks like an upside down u between the slant and the vertical symptote on the left, and it's just an increasing function approaching the slant on the right)

[10]

4 Special containers are manufactured, each having a square base, four identical sides, and no top. Each container holds a volume of 4000 cm^3. Find the container size having minimal cost. That is, find the length b of one side of the base and the height h of the sides such that the total surface area of the base and sides of the container is a minimum.

We need to minimize the surface area:

$$S = 4bh + b^2$$

The domain of this function is $b > 0$, $h > 0$. We need to eliminate b or h. We'll use the volume formula. The volume of a box with square base is $V = b^2h$. We have $V = 4000$. So,

$$h = \frac{4000}{b^2}.$$

When $b > 0$, we still just find $h > 0$. We can rewrite the surface area as a function of b:

$$S(b) = 4b\frac{4000}{b^2} + b^2$$

and simplify to get $S(b) = (16000/b) + b^2$. The domain is $b > 0$. Take the derivative:

$$S'(b) = (-16000/b^2) + 2b$$

The critical points of this function are at $b = 0$ and at $b = 2000$. Since $S(b)$ tends towards infinity as b tends to 0, $b = 0$ can not be a minimum. So our only candidate is $b = 2000$. Check by the 2nd derivative test:

$$S''(b) = 32000/b^3 + 2 > 0$$

when $b = 2000$. Since the function is concave up at $b = 2000$, it must be a minimum.

The dimensions of the container are $b = 2000$ cm and $h = 1$ cm.

[5]

5 Evaluate the integrals:

(a) $\int \frac{2x+1}{(x+1)^{3/2}} dx =$

First, use the substitution $u = x+1$. Then $du = dx$ and $2u - 1 = 2x + 1$. So we get the integral

$$\int \frac{2u-1}{u^{3/2}} du$$

Now, break up the fraction:

$$\int \frac{2u}{u^{3/2}} - \frac{1}{u^{3/2}} du$$

This simplifies to

$$\int 2u^{-1/2} - u^{-3/2} du = 4u^{1/2} + 2u^{-1/2} + C = 4(x+1)^{1/2} + 2(x+1)^{-1/2} + C.$$

[5]

(b) $\int \tan^2(x) \sec^2(x) dx =$

Let $u = \tan(x)$. Then $du = \sec^2(x) dx$. The integral is

$$\int u^2 du = (1/3)u^3 + C = (1/3)\tan^3(x) + C.$$

SECTION B

Circle the correct answer. Each one is worth 3 points.

6 For which value of k is the following function continuous

$$g(x) = \begin{cases} 2kx^2 - 1 & x \leq 1 \\ x^3 + kx & x > 1 \end{cases}$$

Ⓒ $k = 2$.

7 If $f(x)$ is continuous at $x = 2$, then

Ⓓ for every $n > 1$, you can find a value of x near $x = 2$ with $|f(x) - f(2)| < 1/n$.

In this problem, $\epsilon = 1/n$ and "x near 2" is the same as $|x - 2| < \delta$ for some δ.

8 If the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists and is equal to a positive number, then

Ⓓ f is increasing at x.

This is the definition of the derivative - when the derivative is positive, it must be increasing.

9 Compute

$$\lim_{x \to infty} \frac{\sqrt{4x^2 - x + 3}}{3x + 1}$$

Ⓒ 2/3.

10 The limit $\lim_{x \to 0} \frac{\ln(x+1)}{\sin(x)}$ is:

Ⓑ 1.

Use L'Hospital's rule.

11 The equation $(x + 1)(e^{(x+2)} + 1) = 0$ has

Ⓑ only one solution.

This equation can only be equal to 0 if either $x + 1 = 0$ or $e^{x+2} + 1 = 0$. Since $x + 1 = 0$ only when $x = -1$, and $e^{x+2} + 1$ is never 0, there must be only one solution.

You could also try to do this problem by using IVT, Rolle's theorem or by graphing the function. However, the 2nd derivative is quite difficult to manipulate in this case - I don't recommend doing it this way.

12 If the function $f(x)$ is decreasing on $(-\infty, \infty)$, then

ⓒ $f(x)$ can not have a local extremum.

The function $f(x) = -x^3$ is a function which is always decreasing. However, it has a point where $f'(x) = 0$ (so $f'(x) > 0$ is false), its graph lies above the x-axis for all $x < 0$ (so $f(x) < 0$ is false), and it has an inflection point.

13 For $y(x - y) = x^2 + y$ find y'.

ⓐ $y' = (2x - y)/(x - 2y - 1)$.

14 Classify the critical points of $f(x) = x^3 + x^{-1}$.

ⓐ f has one local min, and one local max.

15 A ten foot ladder is leaning against a vertical wall. If the foot of the ladder is being pulled away at a speed of 2 ft/s, how fast is the top of the ladder moving when it is still 6 feet off the ground?

ⓐ 8/3 ft/s.

You might have -8/3 ft/s. This is the velocity (indicates a direction), you only needed the speed.

16 Let f be the function defined by $f(x) = x^2$. A Riemann sum of f based on a partition of the closed interval $[3, 5]$ will have exactly four terms if

ⓑ the subintervals of the partition are defined by the set of points $x_0 = 3$, $x_1 = 7/2$, $x_2 = 4$, $x_3 = 9/2$, $x_4 = 5$.

The first sum has only 2 terms, and the other two represent the actual area under the curve - a Riemann sum with 4 terms will only give an APPROXIMATION of the area under the curve, usually it does not actually give the area.

17 If the function f is continuous everywhere then $\frac{d}{dx} \int_a^x f(t)\, dt$ is always

ⓒ $f(x)$.

This is the fundamental theorem of calculus.

18 $\int_1^2 \ln(x)\ dx$ is always

ⓐ positive.

This is the signed area under the curve $y = \ln x$ between $x = 1$ and $x = 2$. Since the graph of this function lies above the x-axis on this interval, the area must be positive. You do not know how to integrate this, so trying to do this by evaluating the integral would be very difficult (to say the least!).

Good luck on Monday.

END OF EXAM