Name:________________________

Lab Section:________________________

Exam Code: Math 251 L06/13 Bauer

**Calculators, books, notes and collaboration are *not permitted.***

50 MINUTES

UNIVERSITY OF CALGARY

# MIDTERM EXAM

**SESSION 2003**
**FALL SEMESTER**

**DATE: 31 October 2003**
**TIME: 4:00-4:50**

Please show all your work clearly.
Incorrect answer with work shown may receive partial credit, while correct answers with no work shown may receive no credit.
Please write your name on every page, beginning with this one.
Good luck!

| For Markers Only | |
|---|---|
| 1 | /20 |
| 2 | /30 |
| 3 | /10 |
| 4 | /15 |
| 5 | /15 |
| 6 | /10 |
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1 Evaluate the following limits. Justify your answers carefully. (10 points each)

(a) $\lim_{x\to\infty} \frac{3x+5}{x-4}$
Multiply through by $1/x$ to get

$$\lim_{x\to\infty} \frac{3+5/x}{1-4/x} = \frac{3+0}{1-0} = 3.$$

(b) $\lim_{x\to 0} \frac{\cos 3x-1}{\sin 2x}$
We use the limits $\lim_{x\to 0} \frac{\cos(x)-1}{x} = 0$ and $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. Then

$$\lim_{x\to 0} \frac{\cos 3x - 1}{\sin 2x} = \lim_{x\to 0} \frac{\cos(3x)-1}{3x} \frac{3x}{2x} \frac{2x}{\sin(2x)} = (0)(3/2)(1) = 0.$$

2 Find $\frac{dy}{dx}$ for the following: (10 points each)

(a) $y = (x^2 + e^x)^{10}$

$$\frac{dy}{dx} = \sec\left((x^2+1)^5\right)\tan\left((x^2+1)^5\right)5(x^2+1)^4 2x$$

(b) $y = |\csc x|^x$

Since $\ln y = x \ln|\csc x|$, we have

$$(1/y)\frac{dy}{dx} = (1)\ln|\csc x| + x\frac{1}{\csc x}(-\csc x \cot x).$$

Simplify and solve for $\frac{dy}{dx}$ to get

$$\frac{dy}{dx} = |\csc x|^x(\ln|\csc x| - x\cot x).$$

(c) $e^{xy} = x^2 + y^2$

Differentiate implicitly to get:

$$e^{xy}(x\frac{dy}{dx} + (1)y) = 2x + 2y\frac{dy}{dx}.$$

Expand the left hand side into $e^{xy}(x\frac{dy}{dx}) + e^{xy}(y)$. Then solve for $\frac{dy}{dx} = (2x - ye^{xy})/(xe^{xy} - 2y)$.

3 Use the $\epsilon$-$\delta$ *definition of the limit* to show that

$$\lim_{x\to 2} 5 - (1/2)x = 4.$$

(10 points)

Let $\epsilon$ be any positive number and let $\delta = 2\epsilon$. (N.B.: $\delta$ is ALSO a positive number whose value DEPENDS ON $\epsilon$... you can think of $\delta$ as a dependent variable). If $|x - 2| < \delta$ then

$$|(5 - (1/2)x) - 4| = |1 - (1/2)x| = |-1/2||x - 2| < (1/2)\delta.$$

The last inequality is true because we have assumed that $|x - 2| < \delta$. Since $\delta + 2\epsilon$, we have

$$(1/2)\delta = (1/2)(2\epsilon) = \epsilon.$$

Thus we have actually shown that $|(5 - (1/2)x) - 4| < \epsilon$. Hence 4 satisfies the definition of the limit, so we have proved the equation above.

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4 Show that there is a value of $x$ for which

$$\sin \frac{\pi}{2} x = 10^{-x^2}.$$

(15 points)

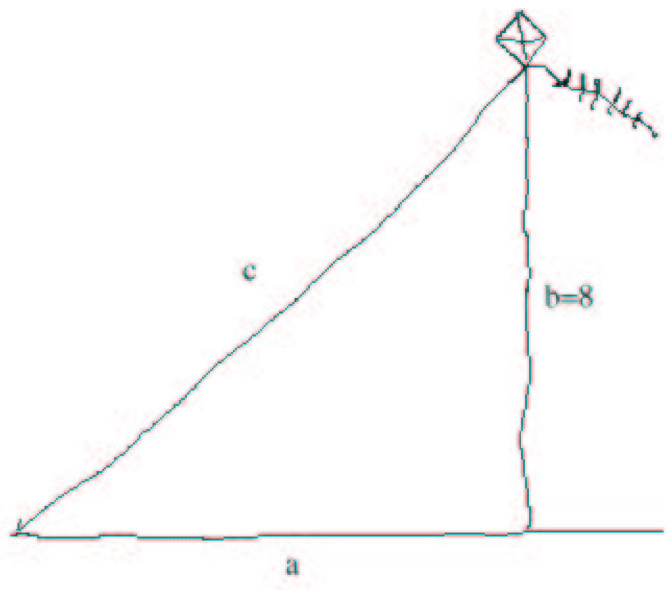
Let $f(x) = \sin\left(\frac{\pi}{2}x\right) - 10^{-x^2}$. Note that $f(x)$ is continuous since it is composed of the functions $\frac{\pi}{2}x$, $-x^2$, $\sin x$, and $10^x$ which are each continuous because they are polynomial, trigonometric or exponential functions.

Since the function is continuous, we attempt to use the Intermediate Value Theorem. We have:

$$f(0) = \sin(0) - 10^0 = -1 < 0$$

$$f(1) = \sin \frac{\pi}{2} - 10^{-1} = 1 - (1/10) > 0$$

Since $f(0) < 0$ and $f(1) > 0$, there must be a value $c$ in between 0 and 1 with $f(c) = 0$. But when $f(c) = 0$, we have $\sin\left(\frac{\pi c}{2}\right) = 10^{-c^2}$.




5 A kite 8 m above the ground moves horizontally at a speed of 2 m/s. At what rate is string being let out when 10 m of string have been let out?
(15 points)

Let $a$ be the distance along the ground denoted by the bottom of the triangle in the picture. Let $c$ be the length of the string. We know that $\frac{da}{dt} = 2$ and we are looking for $\frac{dc}{dt}$ when $c = 10$. We have that

$$a^2 + 8^2 = c^2$$

Implicitly differentiate to find that $2a\frac{da}{dt} + 0 = 2c\frac{dc}{dt}$. So,

$$\frac{dc}{dt} = \frac{a}{c}\frac{da}{dt}.$$

TO solve the problem, we must find out what $a$ is when $c = 10$. To do this, go back to the original equation and use $c = 10$:

$$a^2 + 64 = 100$$

Solving, we find that $a = 6$. That is, $\frac{dc}{dt} = \frac{6}{10}(2) = 6/5$. The string is being let out at a rate of 6/5 m/s.

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6 Let $g^{-1}(x)$ be the *inverse* of a differentiable function $g(x)$. Show that if $y = g^{-1}(x)$, then $\frac{dy}{dx} = \frac{1}{g'(g^{-1}(x))}$.

(Hint: this generalizes the formulas for e.g. $\frac{d}{dx}\arcsin x$ and $\frac{d}{dx}\ln x$.)
(10 points)

Here is an alternate solution to the one found on the noon exam solutions.

If $y = g^{-1}(x)$, then $g(g^{-1}(x)) = x$ by the definition of the inverse. Differentiating implicitly, and using the chain rule, we have

$$g'(g^{-1}(x))\frac{d}{dx}(g^{-1}(x)) = 1$$

That is,

$$\frac{dy}{dx} = \frac{d}{dx}(g^{-1}(x)) = \frac{1}{g'(g^{-1}(x))}.$$

**END OF EXAM**