Name: ______________________

Lab Section: ______________________

Exam Code: Math 251 L04/14 Bauer

Calculators, books, notes and collaboration are *not permitted.*

50 MINUTES

UNIVERSITY OF CALGARY

MIDTERM EXAM

SESSION 2003
FALL SEMESTER

DATE: 31 October 2003
TIME: 12:00-12:50

Please show all your work clearly.
Incorrect answer with work shown may receive partial credit, while correct answers with no work shown may receive no credit.
Please write your name on every page, beginning with this one.
Good luck!

For Markers Only	
1	/20
2	/30
3	/10
4	/15
5	/15
6	/10
Σ	/100

Name:________________________

1 Evaluate the following limits. Justify your answers carefully.
(10 points each)

(a) $\lim_{x\to 3} \frac{x^2-5x+6}{x-3}$

$$\lim_{x\to 3} \frac{x^2-5x+6}{x-3} = \lim_{x\to 3} \frac{(x-2)(x-3)}{x-3} = \lim_{x\to 3} x-2 = 1.$$

(b) $\lim_{x\to 0} x \csc 3x$ We use the identity $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

$$\lim_{x\to 0} x \csc 3x = \lim_{x\to 0} \frac{x}{\sin 3x} = (1/3) \lim_{x\to 0} \frac{3x}{\sin 3x} = (1/3)(1) = 1/3.$$

2 Find $\frac{dy}{dx}$ for the following: (10 points each)

(a) $y = \sec\left((x^2+1)^5\right)$

$$\frac{dy}{dx} = \sec\left((x^2+1)^5\right)\tan\left((x^2+1)^5\right)5(x^2+1)^4(2x).$$

(b) $y = \frac{(x-1)^4(x^3-7)^3}{(x-6)^{100}(2x+1)^2}$

(Use logarithmic differentiation for full credit.)

$$\ln y = 4\ln(x-1) + 3\ln(x^3-7) - 100\ln(x-6) - 2\ln(2x+1)$$

$$\frac{1}{y}\frac{dy}{dx} = 4\frac{1}{x-1}(1) + 3\frac{1}{x^3-7}(3x^2) - 100\frac{1}{x-6}(1) - 2\frac{1}{2x+1}(2)$$

$$\frac{dy}{dx} = \frac{(x-1)^4(x^3-7)^3}{(x-6)^{100}(2x+1)^2}\left(\frac{4}{x-1} + \frac{9x^2}{x^3-7} - \frac{100}{x-6} - \frac{4}{2x+1}\right)$$

(c) $xy = x^2y + y^3$

$$x(1)\frac{dy}{dx} + (1)y = x^2(1)\frac{dy}{dx} + 2xy + 3y^2\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2xy - y}{x - x^2 - 3y^2}.$$

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3 Use the ϵ-δ *definition of the limit* to show that

$$\lim_{x \to 4} 15 - 5x = -5.$$

(10 points)

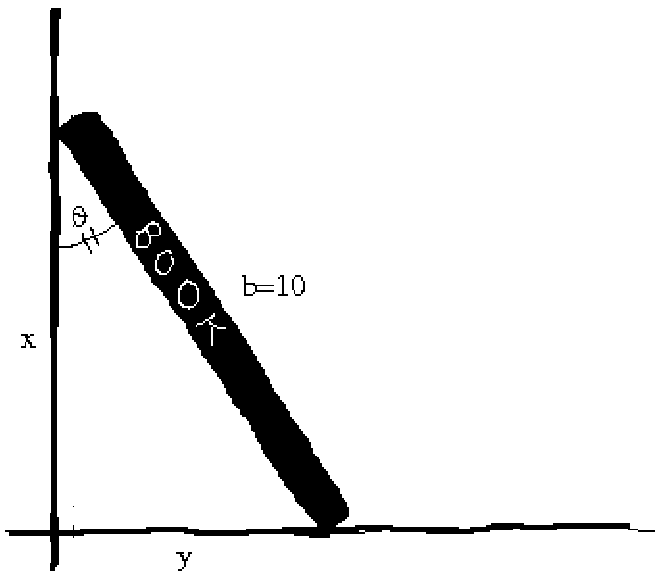
Let ϵ be any positive number and let $\delta = \epsilon/5$. (NB: δ is ALSO a positive number, and it's value DEPENDS ON EPSILON... you can think of δ as a dependent variable.) If $|x - 4| < \delta$ then

$$|(15 - 5x) - (-5)| = |20 - 5x| = |-5||x - 4| = 5|x - 4| < 5\delta.$$

The last inequality is true because we have assumed that $|x - 4| < \delta$. Since $\delta = \epsilon/5$, we have

$$5\delta = 5(\epsilon/5) = \epsilon.$$

Thus we have actually shown that $|(15 - 5x) - (-5)| < \epsilon$. This means that the value $L = -5$ satisfies the definition of the limit, so we have proved the equation above.



4 A book 10 cm tall rests against a vertical wall. If the bottom of the book slides away from the wall at a speed of 2 cm/s, how fast is the angle between the top of the book and the wall changing when that angle is $\pi/4$ radians?
(15 points)

Let x be the length of the wall stretching from the floor to the book. Let y be the length along the floor from the wall to the book. Let θ be the angle made by the book and the wall. We know that $\frac{dy}{dt} = 2$ and we are looking for $\frac{d\theta}{dt}$. We relate y and θ by $\sin\theta = y/10$. Differentiating implicitly, we have:

$$\cos(\theta)\frac{d\theta}{dt} = (1/10)\frac{dy}{dt}.$$

That is, $\frac{d\theta}{dt} = \frac{1}{10\cos(\theta)}\frac{dy}{dt} = \frac{1}{10\cos\pi/4}(2)$. Since $\cos\pi/4 = \frac{1}{\sqrt{2}}$, we have that $\frac{d\theta}{dt} = \sqrt{2}/5$.

5 Use differentials (or, equivalently, a linear approximation) to estimate $\sin(35^o)$. (15 points)

We use the linear approximation to $f(x) = sin(\frac{\pi}{180}x)$ at $x = 30^o$. (NB: The $\frac{\pi}{180}$ just takes measurements in degrees and turns them into radians - we have to do this because we only know the derivative of the function sin in terms of radians.) The formula is:

$$L(x) = f(a) + (x - a)f'(a).$$

So, we have $L(35) = \sin\left(\frac{\pi}{180}30\right) + (35-30)\cos\left(\frac{\pi}{180}30\right) = 1/2 + (5)(\sqrt{3}/2)$.

6 Let $g^{-1}(x)$ be the *inverse* of a differentiable function $g(x)$. Show that if $y = g^{-1}(x)$, then $\frac{dy}{dx} = \frac{1}{g'(g^{-1}(x))}$.

(Hint: this generalizes the formulas for e.g. $\frac{d}{dx} \arcsin x$ and $\frac{d}{dx} \ln x$.)
(10 points)

The equation $y = g^{-1}(x)$ is EXACTLY the same as the equation $g(y) = x$ (this is the definition of the inverse). Implicitly differentiate to get

$$g'(y)\frac{dy}{dx} = 1.$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{1}{g'(y)}.$$

The result follows if you remember that $y = g^{-1}(x)$.

END OF EXAM