Name:

Exam 2

Directions: You must show all of your work for full credit. A correct answer with no work shown is worth no points, but an incorrect or partial answer with some work shown MAY be worth partial credit. No calculators. Good luck!

1. 10 points. Evaluate the following integrals:

   a. $\int \frac{1}{x^3} + 2x + 10\ dx$

$\int \frac{1}{x^3} + 2x + 10\ dx = \int x^{-3} + 2x + 10\ dx = (-1/2)x^{-2} + x^2 + 10x + C$

   b. $\int_1^3\ x(x^2-1)^{2/3}\ dx$

Let $u = x^2 - 1$. Then $du = 2x\ dx$ or $\frac{1}{2}du = dx$. $u(1) = 0$ and $u(3) = 8$. So:

$$\int_1^3 x(x^2-1)^{2/3}\ dx = \frac{1}{2}\int_0^8 u^{2/3}du = (\frac{1}{2})(\frac{3}{5})u^{5/3}|_0^8 = \frac{3}{10}(8^{5/3} - 0)$$

2. 10 points. Find $F'(x)$ for the functions

   a. $F(x) = \int_a^x t^2 \sin t\ dt$

$F'(x) = x^2 \sin x$ by the fundamental theorem of calculus.

   b. $F(x) = \int_a^{\tan x}\ \sqrt{1+t}\ dt$

$F'(x) = (\sqrt{1 + \tan x})(\sec x^2)$ by the fundamental theorem of calculus plus the chain rule.

3. 20 points. Find two numbers whose difference is 10 and whose product is a minimum.

Let $x - y = 10$ and set $P = xy$. We wish to minimize P (the **p**roduct). Note that $x = 10 + y$ so that $P(y) = (10 + y)y = 10y + y^2$ (another common choice is $y = x - 10$ and this will also work). Then

$$P'(y) = 10 + 2y$$

and $P'(y) = 0$ when $y = -5$. So the candidate for a pair of numbers satisfying the hypotheses is $y = -5$ and $x = 5$.

Now, to finish the problem we have to check that $x = 5$, $y = -5$ is a minimum (it could have been a maximum). There's more than one way to do this. One way is to note that $P''(y) = 2 > 0$ so that the critical point must be a minimum because the curve is always concave up.

4. 20 points. Show that the polynomial $f(x) = x^{13} + 10x^5 + 3x - 2$ has exacly one real root.

Here is a solution by a fellow student which received full credit:

Polynomials are continuous and differentiable, $f(-1)$ is $-1 - 10 - 3 - 2 = -16$ and $f(1)$ is $1 + 10 + 3 - 2 = 11$ so by the IVT there must be a c with $f(c) = 0$.

Roots are places where the function equals zero, and according to Rolle's theorem if there are two points, $f(a)$ and $f(b)$, that both equal zero, there exists a c with $f'(c) = 0$. But:

$$f(x) = x^13 + 10x^5 + 3x - 2$$
$$f'(x) = 13x^{12} + 50x^4 + 3$$

and $13x^{12} + 50x^4 + 3$ is **always** > 0, because all the terms in it are positive (numbers raised to an even power will always be positive). Adding all positive terms can **never** give you zero.

Since there's no c with $f'(c) = 0$, the conditions for Rolle's theorem aren't met. It's a continuous differentiable function because it's a polynomial so that isn't the problem, the only other condition for Rolle's theorem is that there are at least two points $f(a)$ and $f(b)$ that equal zero, so we must not have that. So we know there are less than 2 roots. We know there's at least one root by IVT as shown above, so there's only one real root to this polynomial.

Another full credit solution was to note that since $f'(x) > 0$ everwhere, the function is always increasing so it only takes on each value in its range once. Since odd polynomials have the reals as their range, 0 must be in the range so there must be at least one root (or use IVT here instead).

5. 20 points. Consider the region in the **first quadrant** bounded by the curves $y = x^3$ and $y = 9x$.

   a. Set up but DO NOT EVALUATE an integral computing the area of this region.

Note that the equation $x^3 = 9x$ has three solutions: $x = -3$, $x = 0$, and $x = 3$. Since we are only interested in the region in the first quadrant, we'll discard $x = -3$. All that's left to do is check which function is "on top" between $x = 0$ and $x = 3$. An accurate sketch is an acceptable way to do this, or note that $1^3 < 9(1)$ so $y = 9x$ must be on top. So the area of the region is

$$\int_0^1 9x - x^3 \ dx.$$

   b. Set up but DO NOT EVALUATE an integral computing volume of the solid obtained by revolving this region about the x-axis.

We already checked which function is "on top". So all that's left is to remember the formula for volumes:

$$\int_0^1 \pi(R^2 - r^2)dx = \int_0^1 \pi((9x)^2 - (x^3)^2) \ dx.$$

6. 20 points. An aquarium 2m long, 1m wide and 1m deep is full of water. Set up but DO NOT EVALUATE an integral computing the work needed to pump half of the water out of the aquarium (the density of water is 1000 kg/m^3 and gravity is 9.8 m/s^2)

The first thing you probably want to do is draw the aquarium (my meager computer skills mean that there is no drawing here). But, if this explanation doesn't make sense to you then you can look at a solution on paper that I've taped outside my door.

On my drawing there's a block labelled 2m long, 1m wide and 1m high. The important thing is how you label the height, x. I've chosen to label the bottom layer of the aquarium $x = 1$ since the water on the bottom of the aquarium would have to travel a distance of 1 meter to be pumped out the top of the aquarium. So the top of the aquarium is $x = 0$ (that water is already at the top, it doesn't travel any vertical distance when it is pumped out) and everything in between is... in between.

Ok. Now,

$$\text{Work} = \text{Force} \times \text{Distance}.$$

In this case, the force is the weight of the water and the work is

$$W = \text{Volume} \times \text{Density} \times \text{Gravity} \times \text{Distance}.$$

The density is a constant, gravity is a constant. So we only need to figure out the volume and the distance. Here's the trick: since the distance depends on what horizontal slice of water we're talking about, we look at the volume of a slice of thickness dx and the distance that slice has to travel to get to the top of the tank.

The volume of a slice is

$$V_i = (1)(2)(dx) = 2dx.$$

The distance the i-th slice has to travel is x_i (since my coordinate system has $x = 0$ at the top). So the work required to pump the i-th slice out of the aquarium is

$$W_i = (2dx)(1000)(9.8)(x_i) = 19600x_i dx$$

To find the total work, integrate over all possible values of x_i. Since we only want to pump out half of the water, the right integral is

$$\int_0^{1/2} 19600x \; dx.$$