Take-home Quiz 1 Solutions
Math 251 Lecture 01

1 Problem 1

Let $f(x) = x^3$, $a = 0$ and $L = 0$. Find a number $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - L| < 0.2$.

Suppose that $|x^3 - 0| = |x|^3 < 0.2$. Taking the cube root of each side, we find that $|x| < (0.2)^{1/3}$. These two equations are *equivalent*. That is, if we assume that $|x| < (0.2)^{1/3}$ then we are guaranteed that $|x^3| = |x|^3 < (0.2^{1/3})^3 = 0.2$. Thus, if we assume that $\delta = 0.2^{1/3}$ and $|x - a| < 0.2^{1/3}$, we are guaranteed that $|f(x) - L| < 0.2$.

1.1 Comment: Explain your work using sentences

Most of you lost points for not explaining your work using sentences. The point of this exercise in epsilon-delta proofs is to assure that you learn the basic skills involved in writing proofs. The key word here is "writing". For this type of problem, your entire solution should be written in clear, concise, and logical sentences. Scratch work and computations should be minimized. In particular, your mathematics should appear as sentences. For this, you need to use proper mathematical grammar. Do not forget that "=" stands in place of a verb ("is" or "is equal to"). Similarly, "$\leq$" and "$<$" stand in place of phrases containing verbs ("is less than or equal to" and "is less than", respectively). One major difference between mathematical grammar and English grammar is that we are allowed to string some of these verbs together. Thus, the sentence:
Let $f(x) = |x^3| = |x|^3$.
makes sense mathematically, but is awkward to translate into a sentence in English.

1.2 Comment: What does "=" mean?

One last comment about these sentences. Remember that "=" compares two *quantities* and does not compare two *equations* or *inequalities*. Thus a "sentence" like

$$|f(x) - L| > 0.2 = |x^3| > 0.2$$

is nonsensical. In this sentence what one probably means is that the two inequalities $|f(x) - L > 0.2$ and $|x^3| > 0.2$ are equivalent. However, an implication of what is actually said is that $0.2 > 0.2$, which is nonsense. A better way of expressing these ideas is to write something like "The inequality $|x|^3 > 0.2$ is true whenever $|f(x) - L| > 0.2$ since $f(x) = x^3$ and $L = 0$." But don't take my word(s) for it, make up your own, being careful to respect grammar.

See also the comments following Problem 2.

2 Problem 2

Use the formal definition of the limit to verify $\lim_{x \to 1} 3x - 1 = 2$.

Let ϵ be any small positive number. Let $\delta < \epsilon/3$. If $|x - 1| < \delta$, then

$$|f(x) - L| = |3x - 1 - 2| = 3|x - 1| < 3\delta = 3(\epsilon/3) < \epsilon.$$

This verifies the definition of the limit, hence $\lim_{x \to 1} 3x - 1 = 2$.

2.1 Comment: Remove scratch work

This proof should be elegant and self-contained, written in clear prose with a minimal of mathematical calculation. It is not necessary to include the computation of δ. If your proof is written correctly, I will be able to deduce the computation. You CAN incorporate the computation into your proof, but you must do so using prose, complete sentences, and proper grammar (see the comments in the previous problem). I saw this done correctly in only one or two cases. Although my mantra is usually to SHOW ALL WORK, and you should certainly do so on quizzes and exams, the scratch work is not something that belongs in the final draft of this kind of proof.

2.2 The goal of this problem is not to find δ

You can see from the solution I've written that finding δ is only a very small part of the proof. You must verify the definition of a limit, and this involves going back and showing that the δ that you found actually works.

2.3 Let $\epsilon > 0$ or let $\delta > 0$

If you don't introduce variables, they don't exist. It's sort of like what I imagine it must be like to be a debutante at a society party who doesn't know anyone ... you might get mentioned, but nobody knows who you are.

2.4 Another comment about math sentences

Many math sentences are of the form "If .... then ....". For example, " If $|x-1| < \delta$, then $|f(x) - L| = |3x-1-2| = 3|x-1| < 3\delta = 3(\epsilon/3) < \epsilon$." is a pretty common sentence for the kind of proof you are writing. Where's the subject? The object? The verb? These are contained in the inequalities. You should get used to using the words "if" and "then" instead of the arrows that sometimes appear in the text book or on the chalkboard. The arrows are a shortcut, but as with most shortcuts, they are best utilized after you've had enough practice with the "long way".

3 Problem 3

If I haven't mentioned it yet, you guys ROCK. Well, most of you rock - a (very) few of you still have quite a bit of work to do (you can tell if you are in this group if your score is less than about 5/15). All of these problems - and especially this third one - are designed to push your limits so that you can learn new skills. Even though most people did not get a perfect score, I was VERY happy to see the progress you made in a very short time.

Ok, so here is a hint for what to do about Problem 3. Now, most people understood that you have to bound the quantities $|f(x) - L|$ and $|g(x) - M|$. In your write up, you must be very careful to explain WHY you can bound this. For example:

Let $\epsilon > 0$. Since $\lim_{x \to a} f(x) = L$, for any positive number c there must be a $\delta > 0$ with $|f(x) - L| < c$ whenever $|x - a| < c$.

This bounds $|f(x) - L|$. Now the problem is to pick a clever choice for c, one which might make things cancel out later on. However, your choice of c MUST be a constant. $\epsilon/g(x)$ is not a valid choice, because it is not a constant.

For this problem, you can use without proof a result from an earlier problem in the book. It is this: There exists a $\delta > 0$ such that $|g(x)| < 1 + |M|$ whenever $|x - a| < \delta$. Many of you proved this fact, and I am very impressed!