SPECTRAL SEQUENCES OF OPERAD ALGEBRAS

K. BAUER AND L. SCULL

ABSTRACT. We identify conditions under which it is guaranteed that an action of an operad on the E_2 page of a spectral sequence passes to E_r for $r \geq 2$ and hence to the E_∞ page. We consider this question in both the purely algebraic and topological settings.

1. Introduction

The existence of a product structure can greatly aid in spectral sequence computations, and incorporating this structure into a spectral sequence is a standard tool. What makes this possible is that it is well-understood when a spectral sequence is compatible with the algebraic structure present – when a product structure on an E_2 page of a spectral sequence will result in a product on the E_∞ page. This was done, for example, by Massey [7], who defined an infinite sequence of conditions which need to be satisfied, one for each page of the spectral sequence. Other such conditions are summarized by McCleary in [9] and presented nicely for topological spectral sequences by Dugger in [5].

In this paper, we generalize to consider algebraic operations described by operads. An operad is designed to encode a set of operations and their composition laws. This gives a framework for considering many different kinds of algebraic structures from a common veiwpoint, as actions of various operads. Operads and spectral sequences have both become ubiquitous in algebra and topology. We present conditions under which an action of an operad on the E_2 page of a spectral sequence passes to the the E_∞ page. These conditions generalize the conditions for algebras, and open the door for new applications. It seems to be a folk theorem that spectral sequences exist for operad algebras (see, e.g. Section 1.4 of [2]). However, no explicit conditions for such appear in the literature. Our goal is to fill this gap.

In this paper, we explore three circumstances in which a spectral sequence can arise from operad algebras. The first situation is purely algebraic. In Section 3 we consider an exact couple arising from a short exact sequence of chain complexes, each of which is an algebra over a differential graded operad. The other two situations are topological. In

Section 4, we consider exact couples arising from a tower of fibrations. In this case, the entire tower is an algebra over the operad. Finally, in Section 5 we explore the stable setting.

2. Preliminaries

In this section, we briefly review the definitions of operads and operad algebras, and set up the context in which we will be working. Two good references for this material are [4] and [8].

For concreteness, we will consider operads and algebras in one of two underlying categories C. In algebraic settings, $\mathsf{C} = \mathsf{dg} - \mathsf{Mod}_k$, the category of differential graded k-modules (or dg modules), where k is a commutative ground ring. For topological considerations, we use $\mathsf{C} = \mathsf{Tops}$, the category of pointed topological spaces. Many of the results discussed here can be extended without difficulty to an arbitrary symmetric monoidal category.

We will denote a dg module by (A, d) where d is the differential of the module A, or just by A. The tensor product $\otimes$ will denote $\otimes_k$, unless otherwise specified. The degree of an element u of a differential graded module is denoted $|u|$.

2.1. **Operads.**

An operad is designed to encode a system of algebraic operations and their compositions. In our category C, we have an object (which is either a dg module or a space) of n-ary operations for each n. One way to encode this information is via a symmetric sequence. In the category $\mathsf{dg} - \mathsf{Mod}_k$ this is a sequence $\{(P(n), \delta_n)\}_{n\geq 1}$ of dg-modules $P(n)$ with differential δ_n such that each $P(n)$ is equipped with an action of the n-th symmetric group Σ_n (which we can think of as changing the order on the input variables). In Tops a symmetric sequence would simply be a sequence of spaces X_n equipped with an action of Σ_n. For simplicity, we will denote the differential of $P(n)$ by δ when the context makes it clear which n is involved. An operad in C is a symmetric sequence which is a monoid with respect to the product:

$$(P \circ P)(n) = \bigoplus_{k\geq 1} P(k) \otimes \Big(\bigoplus P(j_1) \otimes \cdots \otimes P(j_k) \Big)$$

where the second direct sum is taken over all ways of writing $n = j_1 + \cdots + j_k$. That is, P is an operad if there exists a map $\gamma : P \circ P \to P$ which is unital, associative and equivariant (see for example [4], [8]). Since we think of this map as the composition of operations, it is convenient to denote the image of $(p, q_1, \ldots, q_k) \in P(k) \otimes P(j_1) \otimes \cdots \otimes P(j_k)$ by $p(q_1, \ldots, q_k) \in P(j_1 + \cdots + j_k) = P(n)$.

Note that when $\mathsf{C} = \mathsf{dg} - \mathsf{Mod}_k$, this is a map of dg modules. Therefore, it must also satisfy the derivation relation

$$\delta(p(q_1, \ldots, q_k)) = \delta(p)(q_1, \ldots, q_k) + \sum_{i=1}^{k} \pm(q_1, \ldots, \delta(q_i), \ldots, q_k)$$

where the sign in the sum is given by $(-1)^{|p|+|q_1|+\cdots+|q_{i-1}|}$ for each i. On first reading, the reader may want to simplify to the case where each $P(n)$ is concentrated in degree 0, so that P is just an operad of k-modules.

2.2. **Operad algebras.** An algebra over the operad P is an object which has the operations described by P. Precisely, it is an object A of C equipped with maps $\Gamma : P(k) \otimes A^{\otimes k} \to A$ satisfying the usual associativity and equivariance axioms [8]. We denote the image of $(p, x_1, \ldots, x_k)$ under this map by $p(x_1, \ldots, x_k)$. When $\mathsf{C} = \mathsf{dg} - \mathsf{Mod}_k$, $p(x_1, \ldots, x_k)$ has degree $|x_1| + \cdots + |x_k|$. Again, the structure maps are maps of dg modules and so satisfy a derivation relation:

$$d(p(x_1, \ldots, x_k)) = \delta(p)(x_1, \ldots, x_k) + \sum_{i=1}^{k} \pm p(x_1, \ldots, d(x_i), \ldots, x_k)$$

with sign determined as before: for the i-th term, it is $(-1)^{|p|+|x_1|+\cdots+|x_{i-1}|}$. In addition, the operad maps are in particular morphisms of k-modules, and so we have the following additive properties:

$$p(x_1, \ldots, x_i + y, \ldots, x_n) = p(x_1, \ldots, x_i, \ldots, x_n) + p(x_1, \ldots, y, \ldots, x_n)$$

and

$$(p + q)(x_1, \ldots, x_n) = p(x_1, \ldots, x_n) + q(x_1, \ldots, x_n).$$

Under these assumptions, operad actions are compatible with the differential structure, as the next result shows.

Lemma 2.1. *Let (A, d) be an algebra over the operad (P, δ). Then the homology $H(A, d)$ is an algebra over the homology operad $H(P, \delta)$.*

Proof. Let $[p]$ be a homology class of $H(P, \delta)$, with representative $p \in P(k)$ Let $[x_i]$ be homology classes of $H(A, d)$ for $1 \leq i \leq k$. Then we define $[p]([x_1], \ldots, [x_k]) = [p(x_1, \ldots, x_k)]$ where x_i is a representative in A for each $[x_i]$. The additive properties of the action mentioned above can easily be used to show that varying the representatives of p or the x_i's by a boundary results in the same homology class $[p(x_1, \ldots, x_k)]$. □

2.3. **Examples.** The graded commutative associative operad $Comm(n)$ is created from equivalence classes of connected planar binary trees with one root and n labelled leaves; two such trees are equivalent if they are related by a sequence of adjacent moves on the leaves (corresponding to switching brackets using associativity) or by switching the order of adjacent leaves with appropriate sign (corresponding to graded commutativity). An algebra over this operad is exactly a graded commutative associative algebra

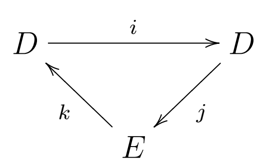
Many other algebraic structures can be encoded by similar operads; for example, associative algebras are encoded by the operation of the operad $Assoc$ created in an analogous way to $Comm$ (but with a suitibly altered equivalence relation); various other algebraic structures have their corresponding operads, such as Lie algebras encoded by the operad Lie. Many examples of operads, their algebras and their uses can be found in [6].

3. Spectral sequences in algebra

In this section we consider spectral sequences of dg modules which are also algebras over a dg operad P. We give conditions which ensure that we produce a spectral sequence of operad algebras. We will be considering operad algebras whose underlying structure is that of a bigraded module with one (vertical) differential. In addition to satisfying the derivation relation from the previous section, therefore, we will also require the operad action to respect both gradings: if $x_i \in A_{p_i,q_i}$, $1 \leq i \leq k$, then $p(x_1, \dots, x_k)$ has degree $(p_1 + \cdots + p_k, q_1 + \cdots + q_k)$.

Our approach to spectral sequences will be through exact couples $\langle E, D, i, j, k \rangle$:

(1)



where E and D are k-modules (possibly graded or bigraded) with k-module maps between them, and the diagram is exact at each corner. If in addition E is a dg operad algebra under the differential jk, we will call this an exact couple of operad algebras; then Lemma 2.1 will ensure that the derived couple inherits an operad action on E' as well. Note that this does not automatically mean that the derived couple will again be an exact couple of operad algebras; we need additional conditions so that the derived map $j'k'$ will again satisfy the Leibniz condition to be a map of dg operad algebras.

One typical way in which an exact couple arises is from short exact sequences of chain complexes. A short exact sequence of the form

$$0 \longrightarrow A \xrightarrow{i} A \xrightarrow{j} C \longrightarrow 0 \tag{2}$$

gives rise to a long exact sequence on homology which in turn becomes an exact couple with $D = H(A)$ and $E = H(C)$, and the map k given by the connecting homomorphism. There are various choices of grading for this; our conventions will be that the bigraded modules $A_{*,*}$ and $C_{*,*}$ have vertical differential d_A and d_C (that is, these are both maps of degree $(+1, 0)$), the injection i has degree $(0, 1)$, and the surjection j has degree $(0, 0)$. This leads to a second quadrant spectral sequence; other choices of degrees and gradings lead to spectral sequences in the other quadrants.

We can introduce an operad action into this picture as follows. Suppose that (P, δ) is an operad of of bigraded modules with one vertical differential, and that $A_{*,*}$ and $C_{*,*}$ are operad algebras over P. Moreover, suppose that the chain complex maps i and j in the short exact sequence are compatible with the operad action, in the following sense:

(i) The surjection j satisfies $j(p(x_1, \ldots, x_k)) = p(j(x_1), \ldots, j(x_k))$.
(ii) The injection i satisfies $i(p(x_1, \ldots, x_k)) = p(x_1, \ldots, i(x_h), \ldots, x_k)$ for any $0 \leq h \leq k$

At first, the condition on the map i may seem strange; it becomes more natural when we remember that the spectral sequence associated to this situation is graded by images of i. For example, if A comes with a filtration $F_0 \subseteq F_1 \subseteq F_2 \cdots \subseteq A$ and i is an inclusion map then this condition is satisfied: raising the filtration degree of one of the inputs of the operation will raise the filtration degree of the result. Another example where this condition is naturally satisfied comes from the Bockstein spectral sequence: here, $A = \oplus_n Q$ is a coproduct of a p-torsion free dg operad algebra Q, and $i : A \to A$ is the "multiply by p" map of algebras. Again, we think of filtering by powers of p, and i adds one more power of p regardless of where it is applied.

The main goal of this section is the following,

Theorem 3.1. *If a spectral sequence arises from a short exact sequence of chain complexes* $0 \to A \xrightarrow{i} A \xrightarrow{j} C \to 0$ *such that* i *and* j *satisfy properties (i) and (ii) as above, then we get a spectral sequence of operad algebras.*

Proof. Recall that to pass from the short exact sequence to the (first) exact couple, we begin with

$$\langle E^1, D^1 \rangle = \langle H(C, d_C), H(A, d_A), i^{(1)}, j^{(1)}, k^{(1)} \rangle,$$

where $i^{(1)}$, $j^{(1)}$ are the maps on homology induced by the maps i and j and $k^{(1)}$ is the connecting homomorphism $k = i^{-1}dj^{-1}$. This exact couple then gives rise to a spectral sequence by successively defining the derived couples: $\langle E^2, D^2, i^{(2)}, j^{(2)}, k^{(2)}\rangle$ is given by $E^2 = H(E^1, d)$ and $D^2 = i(D)$, with $i^{(2)}$ and $k^{(2)}$ induced by i and k with little change, while $j^{(2)}(a) = [j(i^{-1}(a))]$. Iterating this process results in a further derived couple for each $r > 1$ and in general, only the derived maps $j^{(r)}$ change significantly: $j^{(r)}(a) = [j(i^{-(r-1)}(a))]$. The kernel and image of $d^{(r)}$ can be identified at each stage in terms of the initial maps i, j and k:

$$0 \subseteq B^2 \subseteq \cdots \subseteq B^r \subseteq \cdots \subseteq Z^r \subseteq \cdots \subseteq Z^2 \subseteq E^1$$

where $Z^r = k^{-1}(i^{r-1}D)$ and $B^r = j(\ker(i^{r-1}))$, and $E^r = Z^r/B^r$.

We will show that each derivation $d^{(r)} = j^{(r)}k^{(r)}$ satisfies the Leibniz condition; then an easy induction with Lemma 2.1 will show that each $\langle E^r, D^r\rangle$ is an exact couple of operad algebras.

Let $[p] \in H(P(k))$ and let $[x_h]$, $1 \leq i \leq \ell$, be elements of E^r. Then each $[x_h]$ has a representative x_h in C, since $Z^r \subset E^1 = H(C)$. Choose $a_h \in A$ with $j(a_h) = x_h$ for each h. By the definition of Z^r, we can find $a'_h \in A$ such that $i^r(a'_h) = d(a_h)$; so that $k[x_h] = i^{r-1}[a'_h]$. We compute, using the properties (i) and (ii) of i and j:

$$k(p(x_1, \ldots, x_\ell)) = i^{-1}dj^{-1}(p(x_1, \ldots, x_\ell)) \tag{3}$$

$$= i^{-1}d(p(a_1, \ldots, a_\ell)) \tag{4}$$

$$= \sum_{h=1}^{\ell} \pm i^{-1}p(a_1, \ldots, d(a_h), \ldots, a_\ell) \tag{5}$$

$$= \sum_{h=1}^{\ell} \pm p(a_1, \ldots, i^{-1}d(a_h), \ldots, a_\ell) \tag{6}$$

$$= \sum_{h=1}^{\ell} \pm p(a_1, \ldots, i^{r-1}(a'_h), \ldots, a_\ell) \tag{7}$$

$$= i^{r-1}\sum_{h=1}^{\ell} \pm p(a_1, \ldots, a'_h, \ldots, a_\ell) \tag{8}$$

where the sign for the i-th summand is given by $(-1)^{|p|+|x_1|+\cdots+|x_{i-1}|}$.

By the definition of $j^{(r)}$ and $k^{(r)}$, we compute that $j^{(r)}k^{(r)}p(x_1, \ldots, x_\ell)$ is equal to

$$\sum_{i=1}^{\ell} \pm p(j(a_1), \ldots, j(a'_h), \ldots, j(a_\ell)) = \sum_{i=1}^{\ell} \pm p([x_1], \ldots, j^{(r)}k^{(r)}[x_i], \ldots, [x_\ell])$$

as desired.

□

3.1. **Examples.** Specializing to commutative algebras via the *Comm* operad, Theorem 3.1 exactly states that the product structure on the E_r term descends to the E_{r+1}-term, a standard result about algebraic structures. Using another operad such as *Assoc* or *Lie* would give analogous results for the corresponding associative or Lie algebra structures.

By taking the operad approach, Theorem 3.1 has identified conditions for a general algebraic structure present to be compatible with the spectral sequence, and we do not have to consider each algebraic context seperately.

3.2. **Convergence.** Spectral sequences are useful precisely because they relate something computable to some desirable quantity. In the case of spectral sequences arising from short exact sequences of chain complexes $0 \to A \to A \to C \to 0$ which we have been considering, the spectral sequence converges to a limit or colimit of the graded complexes $A_{p,q}$. We will describe the situation for colimits; both cases are well studied in the literature and the interested reader might pursue [11], [5] for further information.

Suppose that $H_p(A_q)$ is the p-th homology of the q-th chain complex: $H(A_{*,q}, d_A)$. Let $H_p(A) = colim_i H_p(A_q)$, where the colimit is taken along a tower with maps $i : H_p(A_q) \to H_p(A_{q+1})$. Define a filtration on $H_p(A)$ by letting $F^q(H_p(A))$ (or just F^q) be the image of $H_p(A_q)$ in $H_p(A)$ under the standard map ω_q into the colimit (this map is more simply denoted ω when we don't want to specify q). The associated graded filtration is

$$Gr^q(H_p(A)) = F^q/F^{q-1}.$$

Under some well-known conditions, we get a map $\Gamma : Gr^q(H_p(A)) \to E_\infty^{p,q}$ which is an isomorphism; in this case the spectral sequence converges to $Gr^q(H_p(A))$.

To be useful, we need this to be a convergence of operad algebras. That is, we need to know that the isomorphism Γ commutes with the structure maps of the operad algebras involved.

Proposition 3.2. *The spectral sequence from Theorem 3.1 converges as an operad algebra to $Gr^q(H_p(A))$.*

Proof. First we examine the operad action on $\oplus Gr^q$. Choose $a_i \in F^{q_i}$, $1 \leq i \leq k$. Then for each i, $a_i = \omega_{q_i}(\alpha_i)$ for some $\alpha_i \in H_p(A_{q_i})$. If

$H(A_{*,*}, d_A)$ is an algebra over the operad P, then so is $F = \oplus F^q$ via the operad action which commutes with ω:

$$p(a_1, \ldots, a_k) := \omega_q(p(\alpha_1, \ldots, \alpha_k))$$

This passes to a well-defined action of the operad P on $\oplus Gr^q$.

Now we want to show that Γ is a map of operad algebras with respect to this action. As is standard with any exact couple, we define the map $\Gamma' : F^q \to E_\infty^{p,q}$ by sending a to $j(\alpha)$ for any choice of preimage α of $a \in F^q$. That is, $j(\alpha)$ is an infinite cycle. This extends to a well-defined injection $\Gamma : Gr^q(H_p(A)) \to E_\infty^{p,q}$. The map Γ is induced by j, which is a map of operad algebras.

□

4. Spectral sequences in topology

We shift our focus now to spectral sequences whose inputs are based topological spaces. A subspace A of a space B gives rise to a long exact sequence

$$\cdots \longrightarrow \pi_n(A) \longrightarrow \pi_n(B) \longrightarrow \pi_n(B, A) \longrightarrow \cdots \tag{9}$$

where $\pi_*(B, A)$ are the relative homotopy groups. Elements in $\pi_n(B, A)$ are homotopy classes of commuting diagrams

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & A \\ \big\downarrow & & \big\downarrow \\ D^n & \longrightarrow & B \end{array} .$$

A sequence of inclusions

$$\cdots \subset A_n \subset A_{n-1} \subset \cdots \subset A_1 \subset A_0 = B$$

gives rise to long exact sequences for each $A_n \subset A_{n-1}$; these fit together to produce an exact couple.

In topology, spectral sequences often arise more generally from a tower of pointed spaces and maps W:

$$\cdots \longrightarrow W_{m+1} \xrightarrow{q_{m+1}} W_m \xrightarrow{q_m} W_{m-1} \longrightarrow \cdots$$

We can generalize the relative homotopy groups by defining $\pi_n(W_m, W_{m+1})$ to be the set of homotopy classes of commuting diagrams

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & W_{m+1} \\ \big\downarrow & & \big\downarrow{\scriptstyle q_{m+1}} \\ D^n & \longrightarrow & W_m \end{array} .$$

It is not hard to see that these relative homotopy groups are isomorphic to the homotopy groups of the homotopy fiber of the map q_{m+1}, which is defined to be the homotopy pull-back of the diagram

$$\begin{array}{ccc} & & \star \\ & & \downarrow \\ W_{m+1} & \xrightarrow{q_{m+1}} & W_m. \end{array}$$

Thus, these are groups for $n \geq 1$, abelian groups for $n \geq 2$ and fit into a long exact sequence of homotopy groups similar to (10). Let W_{+1} be the tower W with grading shifted by one. Putting these together results in an exact couple of homotopy groups for any tower W:

$$\begin{array}{ccc} \pi_*(W) & \xrightarrow{\quad i \quad} & \pi_*(W) \\ {\scriptstyle k}\nwarrow & & \swarrow{\scriptstyle j} \\ & \pi_*(W, W_{+1}) & \end{array}$$

with $i = \pi_*(q)$. After indexing, this often gives rise to a spectral sequence, though there may be problems with exactness coming from the fact that π_0 is not a group and π_1 is not an abelian group. We will assume that we are in a situation where these problems do not cause any concern (e.g. all spaces are connected enough).

Note: throughout what follows in this section, it will be extremely important to keep track of the orientation of various spaces. To aid in this, we suppose at the outset that we have chosen orientation preserving homeomorphisms $D^n/S^{n-1} \cong S^n$ for all n.

Suppose now that P is an operad of pointed spaces.

Definition 4.1. *We say that P acts on the tower W_* if there are suitably associative and equivariant structure maps*

$$P(k) \wedge W_{i_1} \wedge \cdots \wedge W_{i_k} \to W_i$$

where $i = i_1 + \cdots + i_k$, which strictly commute with maps q of the tower W.

Our goal is to show that when an operad P acts on a tower W, this leads naturally to a spectral sequence of operad algebras. The first step is to understand how P acts on the exact couple.

Proposition 4.2. *If P acts on the tower W, then $\pi_*(W, W_{+1})$ is an algebra over the operad $\pi_*(P)$ of graded groups.*

Proof. The graded groups $\pi_*(P(k))$ naturally form an operad since π_* is a product preserving functor.

We need to identify the action of the graded operad $\pi_*(P)$ on the bigraded abelian group $\pi_*(W, W_{+1})$. Let $p \in \pi_{n_0}(P(k))$. Using the homeomorphism $S^{n_0} \cong D^{n_0}/S^{n_0-1}$ we represent p as a diagram

$$\begin{CD} S^{n_0-1} @>{\delta p}>> \star \\ @VVV @VVV \\ D^{n_0} @>{p}>> P(k) \end{CD}$$

Elements $w_{i_1}, \ldots, w_{i_k}$ of $\pi_{n_1}(W_{i_1}, W_{i_1+1}), \ldots, \pi_{n_k}(W_{i_k}, W_{i_k+1})$, respectively, are represented by diagrams

$$\begin{CD} S^{n_t-1} @>{\delta w_{i_t}}>> W_{i_t+1} \\ @VVV @VV{q_{i_t}}V \\ D^{n_t} @>{w_{i_t}}>> W_{i_t} \end{CD}$$

where $1 \leq t \leq k$. We produce $p(w_{i_1}, \ldots, w_{i_k})$ as an element of $\pi_n(W_i, W_{i+1})$ where $n = n_0 + n_1 + \cdots + n_k$ and $i = i_1 + \cdots + i_k$. We will do this by producing two horizontal maps for the diagram

$$\begin{CD} S^{n-1} @>>> W_{i+1} \\ @VVV @VV{q_i}V \\ D^{n} @>>> W_i \end{CD}$$

To define these maps, we first write D^n as a product of smaller disks via a choice of homeomorphism

$$D^n \cong D^{n_0} \wedge D^{n_1} \wedge \cdots \wedge D^{n_k}$$

(any two such homeomorphisms are homotopic, so the choice doesn't matter). The boundary S^{n-1} of D^n can be written as a union of the boundary pieces of each of these smaller disks. That is, S^{n-1} is homeomorphic to a quotient of

$$\coprod_{0 \leq i \leq k} D^{n_0} \wedge \cdots \wedge S^{n_i} \wedge \cdots \wedge D^{n_k}.$$

The quotient data is obtained by understanding how to attach the boundary pieces in this sum to one another. Formally, this can be done by a colimit construction. Let $[k]$ be the set $\{0, \ldots, k\}$ and let $P([k])$ be the power set of *proper* subsets of $[k]$. Then define a functor $F : P([k]) \to Top_*$ by $\chi(S) = X_0(S) \wedge \cdots X_k(S)$ and

$$X_i(S) = \begin{cases} D^{n_i} & \text{if } i \in S \text{ and} \\ S^{n_i-1} & \text{if } i \notin S. \end{cases}$$

The functor χ takes an inclusion of subsets to the map of topological spaces obtained by inclusion of the boundary. Then the boundary of D^n is homeomorphic to the colimit

$$colim_{S \subset P([k])} \chi.$$

For example, if $D^n \simeq D^a \wedge D^b \wedge D^c$ then the boundary S^{n-1} can be described by the colimit of the diagram

$$\begin{array}{ccccc} & & D^a \wedge D^b \wedge S^{c-1} & & \\ & \nearrow & & \nwarrow & \\ D^a \wedge S^{b-1} \wedge S^{c-1} & \longleftarrow & S^{a-1} \wedge S^{b-1} \wedge S^{c-1} & \longrightarrow & S^{a-1} \wedge D^b \wedge S^{c-1} \\ \downarrow & & \downarrow & & \downarrow \\ D^a \wedge S^{b-1} \wedge D^c & \longleftarrow & S^{a-1} \wedge S^{b-1} \wedge D^c & \longrightarrow & S^{a-1} \wedge D^b \wedge D^c \end{array}$$

where each map is given by including the boundary of a disk.

For each $S \subset [k]$, define a map $\chi(S) \to P \wedge W \wedge \cdots \wedge W$ by

$$X_0(S) \wedge X_1(S) \wedge \cdots \wedge X_k(S) \xrightarrow{f_S^0 \wedge f_S^1 \wedge \cdots \wedge f_S^k} P \wedge W \wedge \cdots \wedge W$$

where f_S^i is either δw_{n_i} or w_{n_i} according to whether $X_i(S)$ is S^{n_i-1} or D^{n_i}. This defines a map $f : S^{n-1} \to P \wedge W \wedge \cdots \wedge W$.

When $i = 0$, the image of $f_0^0 \wedge f_0^1 \wedge \cdots \wedge f_0^k$ is the basepoint $\star$, since $\delta p = \star$. For all positive values of i, the image of $f_i^0 \wedge f_i^1 \wedge \cdots \wedge f_i^k$ is

$$P(k) \wedge W_{i_1} \wedge \cdots \wedge W_{i_t+1} \wedge \cdots \wedge W_{i_k}.$$

If we apply the operad structure map to this summand, we produce a map into W_{i+1} where $i = i_1 + \cdots + i_k$. This is independent of the choice of t, so the universal property of the coproduct produces the desired map from $S^{n-1} \to W_{i+1}$.

The map $D^n \to W_i$ is defined more simply by composing $p \wedge w_{i_1} \wedge w_{i_2} \wedge \cdots \wedge w_{i_k}$ with the operad structure maps. The resulting square diagram defining the desired homotopy element $\pi_n(W_i, W_{i+1})$ commutes precisely because the operad structure maps must commute with the maps q of the tower W.

□

We know that the spectral sequence associated to this exact couple will be a spectral sequence of $\pi_*(P)$-algebras precisely when jk is a derivation, that is, when the Leibniz rule holds:

$$jk(p(w_1, \ldots, w_k)) = \sum_{i=1}^{k} \pm p(w_1, \ldots, jkw_i, \ldots, w_k).$$

Note that $jk(p)$ is the null element, so the term contributed by $jk(p)(w_1, \dots, w_k)$ does not appear.

In order to verify this condition, we first recall some of the basic homotopy theory involved. Consider the long exact sequence on homotopy associated to a map $f : Y \to Z$. Let

$$\begin{array}{ccc} S^{m-1} & \xrightarrow{\alpha_1} & Y \\ \downarrow & & \downarrow r \\ D^m & \xrightarrow{\alpha_2} & Z \end{array}$$

represent an element $\alpha \in \pi_m(Z, Y)$. The connecting homomorphism $k : \pi_m(Z, Y) \to \pi_{m-1}(Y)$ is defined by $k(\alpha) = \alpha_1$. If we have an additional map $g : X \to Y$, then we can apply the map $j : \pi_{m-1}(Y) \to \pi_{m-1}(Y, X)$ from the long exact homotopy sequence associated to g to the element $k(\alpha) \in \pi_{m-1}(Y)$. To do this, we use our prespecified homeomorphism $S^{m-1} \cong D^{m-1}/S^{m-2}$. Precomposing with this homeomorphism, $k(\alpha) : S^{m-1} \to X$ is now represented as a map $k(\alpha) : D^{m-1}/S^{m-2} \to X$. This, in turn, can be thought of as a diagram

$$\begin{array}{ccc} S^{m-2} & \xrightarrow{\star} & X \\ \downarrow & & \downarrow r' \\ D^{m-1} & \xrightarrow{k(\alpha)} & Y \end{array} .$$

We want to know that the maps j and k behave nicely with respect to sums. To define the sum in homotopy groups, we use the *pinch* and *fold* maps. Given a disk D^{m+1} (resp. a sphere S^m) the pinch map is the quotient map which identifies the equatorial disk D^m (resp. the equatorial sphere S^{m-1}) to a point, resulting in $D^{m+1} \vee D^{m+1}$ (resp. $S^m \vee S^m$). The folding map $+ : X \vee X \to X$ is the map whose existence is guaranteed by the fact that the one point sum $\vee$ is also the coproduct in the category of pointed topological spaces. Then we can add two elements $\alpha, \beta \in \pi_m(Z, Y)$ producing the sum $\alpha + \beta$:

$$\begin{array}{ccccccc} S^{m-1} & \longrightarrow & S^{m-1} \vee S^{m-1} & \xrightarrow{\alpha_1 \vee \beta_1} & Y \vee Y & \xrightarrow{+} & Y \\ \downarrow & & \downarrow & & \downarrow r \vee r & & \downarrow r \\ D^m & \longrightarrow & D^m \vee D^m & \xrightarrow{\alpha_2 \vee \beta_2} & Z \vee Z & \xrightarrow{+} & Z \end{array}$$

Each of the small squares in this diagram commutes; the largest rectangle is the sum.

From the definition of k, it is clear that $k(\alpha + \beta) = k(\alpha) + k(\beta) = \alpha_1 + \beta_1$. If we are in a situation in which we can apply j, then since the homeomorphism $D^{m-1}/S^{m-2} \cong S^{m-1}$ preserves orientations and respects the sum (i.e. $(D^{m-1} \vee D^{m-1})/(S^{m-2} \vee S^{m-2}) \cong S^{m-1} \vee S^{m-1}$) it is also true that j preserves sums. Thus $j(k(\alpha+\beta) = jk(\alpha) + jk(\beta)$.

With these observations, we can now show:

Theorem 4.3. *If P acts on a tower of fibrations W_*, then the associated spectral sequence is a sequence of $\pi_*(P)$-algebras.*

Proof. As in Proposition 4.2, we represent $p(w_1, \ldots, w_k)$ as

$$\begin{array}{ccccc} S^{n-1} & \longrightarrow & \coprod D^{n_0-1} \wedge \cdots \wedge D^{n_t} \wedge \cdots S^{n_k-1} & \longrightarrow & W_{i+1} \\ \downarrow & & \downarrow & & \downarrow \\ D^n & \longrightarrow & D^{n_0} \wedge \cdots \wedge D^{n_k} & \longrightarrow & W_i \end{array}$$

where the coproduct is taken over all values of $0 \le t \le k$ and over the inclusion of $S^{n_0-1} \wedge \cdots \wedge S^{n_k-1}$ into each summand.

The proof now follows immediately by recognizing that the top line of this diagram is the sum $\sum \pm p(w_1, \ldots, \delta w_t, \ldots w_k)$; the sign is induced by the orientation of the spheres in the coproduct.

∎

Note that our results recover the results of [5] on pairings of multiplicative towers when the *Comm* operad is used.

4.1. **Rewrite:** Now consider spectral sequences of basepointed topological spaces. A basepointed subspace A of a space B gives rise to a long exact sequence

$$\cdots \longrightarrow \pi_n(A) \longrightarrow \pi_n(B) \longrightarrow \pi_n(B, A) \longrightarrow \cdots \tag{10}$$

where $\pi_*(B, A)$ are the relative homotopy groups. Elements in $\pi_n(B, A)$ are homotopy classes of commuting diagrams

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & A \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & B \end{array}.$$

A sequence of inclusions

$$\cdots \subset A_n \subset A_{n-1} \subset \cdots \subset A_1 \subset A_0 = B$$

gives rise to long exact sequences for each $A_n \subset A_{n-1}$; these fit together to produce an exact couple.

In topology, spectral sequences often arise more generally from a tower of pointed spaces and maps W:

$$\cdots \longrightarrow W_{m+1} \xrightarrow{q_{m+1}} W_m \xrightarrow{q_m} W_{m-1} \longrightarrow \cdots$$

We can generalize the relative homotopy groups by defining $\pi_n(W_m, W_{m+1})$ to be the set of homotopy classes of commuting diagrams

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & W_{m+1} \\ \big\downarrow & & \big\downarrow{\scriptstyle q_m} \\ D^n & \longrightarrow & W_m \end{array} .$$

It is not hard to see that these relative homotopy groups are isomorphic to the homotopy groups of the homotopy fiber of the map q_m, which is defined to be the homotopy pull-back of the diagram

$$\begin{array}{ccc} & & \star \\ & & \big\downarrow \\ W_{m+1} & \xrightarrow{q_m} & W_m. \end{array}$$

Thus, these are groups for $n \geq 1$, abelian groups for $n \geq 2$ and fit into a long exact sequence of homotopy groups similar to the one in equation (10). Let W_{+1} be the tower W with grading shifted by one. Assembling these long exact sequences results in an exact couple of homotopy groups for any tower W:

$$\begin{array}{ccc} \pi_*(W) & \xrightarrow{\quad i \quad} & \pi_*(W) \\ {\scriptstyle k}\nwarrow & & \swarrow{\scriptstyle j} \\ & \pi_*(W, W_{+1}) & \end{array}$$

with $i = \pi_*(q)$. After indexing, this often gives rise to a spectral sequence, though there may be problems with exactness coming from the fact that π_0 is not a group and π_1 is not an abelian group. We will assume that we are in a situation where these problems do not cause any concern (e.g. all spaces are connected enough).

We wish to be more precise about this process. Select orientation preserving homeomorphisms $S^n \cong D^n/S^{n-1}$ for all n. An element $[\alpha] \in \pi_n(W_k, W_{k+1})$ represented by a diagram

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\alpha_0} & W_{k+1} \\ \big\downarrow & & \big\downarrow{\scriptstyle q_k} \\ D^n & \xrightarrow{\alpha_1} & W_k \end{array} .$$

Then $k\alpha \in \pi_{n-1}(W_{k+1})$ is represented by α_0. It may be convenient sometimes to think of this as an element of the relative homotopy group $\pi_{n-1}(W_{k+1}, *)$ represented by

$$\begin{array}{ccc} S^{n-2} & \longrightarrow & \star \\ \downarrow & & \downarrow \\ D^{n-1} & \xrightarrow{\alpha_0'} & W_{k+1} \end{array}$$

where α_0' is the map α_0 composed with the quotient map $D^{n-1} \to D^{n-1}/S^{n-2}$ and our preselected homeomorphism $D^{n-1}/S^{n-2} \cong S^{n-1}$. This is the boundary map of the long exact sequence for the pair (W_k, W_{k+1}). Similarly, an element β of $\pi_n(W_k)$ represented as a square diagram

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & \star \\ \downarrow & & \downarrow \\ D^{n} & \xrightarrow{\beta_1} & W_{k} \end{array}$$

(in $\pi_n(W_k, *)$, of course) has $j\beta$ represented by

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\star} & W_{k+1} \\ \downarrow & & \downarrow{\scriptstyle q_k} \\ D^{n} & \xrightarrow{\beta_1} & W_{k} \end{array}$$

in $\pi_n(W_k, W_{k+1})$. The final map in the exact triangle is $i = q_*$, and with these definitions, i, j and k are easily seen to satisfy exactness.

Now suppose that $P = \{P(i)\}_{i \geq 1}$ is an operad of basepointed spaces.

Definition 4.4. *We say that P acts ont he tower W if, for each $i \geq 1$ and each partition $i = i_1 + \ldots + i_k$, there are maps*

$$P \wedge W_{i_1} \wedge \cdots \wedge W_{i_k} \to W_i$$

which strictly commute with the structure maps q of the tower W, and which satisfy

(1) *Associativity:*
(2) *Equivariance:*

In this section, we will show that given a tower W which is acted on by an operad P, the spectral sequence arising from the exact couple 4 is a spectral sequence of P-algebras.

Proposition 4.5. *If the operad P acts on the tower W, then $\pi_*(W, W_+)$ is an algebra over the operad $\pi_*(P)$ of graded groups.*

Proof. First note that the graded groups $\pi_*(P(i))$ form an operad since π_* is a product preserving functor.

To define the maps $\pi_*(P(k)) \otimes \pi_*(W, W_+)^{\otimes k}$, suppose that we are given elements $\rho \in \pi_{n_0}(P(k))$ represented by

$$\begin{array}{ccc} S^{n_0-1} & \longrightarrow & \star \\ \downarrow & & \downarrow \\ D^{n_0} & \xrightarrow{\rho_1} & P(k) \end{array}$$

and elements $\alpha_i \in \pi_{n_i}(W_{m_i}, W_{m_i+1})$, $1 \leq i \leq k$, represented by

$$\begin{array}{ccc} S^{n_i-1} & \xrightarrow{\alpha_{i_0}} & W_{m_i+1} \\ \downarrow & & \downarrow{\scriptstyle q_{m_i}} \\ D^{n_i} & \xrightarrow{\alpha_{i_1}} & W_{m_i} \end{array} .$$

Let $n = n_0 + n_1 + \cdots + n_k$ and $m = m_1 + \cdots + m_k$. Our goal is to produce horizontal maps for the diagram

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & W_{m+1} \\ \downarrow & & \downarrow{\scriptstyle q_m} \\ D^n & \longrightarrow & W_m \end{array} .$$

To do this, first we write D^n as a product of smaller disks via a choice of homeomorphism

$$D^n \cong D^{n_0} \wedge D^{n_1} \wedge \cdots \wedge D^{n_k}$$

(any two such homeomorphisms are homotopic, so the choice doesn't matter). The boundary S^{n-1} of D^n can be written as a union of the boundary pieces of each of these smaller disks, but we must glue the pieces together appropriately. That is, S^{n-1} is homeomorphic to a quotient of

$$\coprod_{0 \leq i \leq k} D^{n_0} \wedge \cdots \wedge S^{n_i} \wedge \cdots \wedge D^{n_k}.$$

The quotient data is obtained by understanding how to attach the boundary pieces in this sum to one another. In the case $k = 1$, this is easily seen to be done by forming the pushout

$$\begin{array}{ccc} S^{n_0-1} \wedge S^{n_1-1} & \longrightarrow & D^{n_0} \wedge S^{n_1-1} \\ \downarrow & & \downarrow \\ S^{n_0-1} \wedge D^{n_1} & \longrightarrow & \partial D^{n_0+n_1}. \end{array}$$

That is, $\partial D^{n_0+n_1} \cong S^{n_0+n_1-1} \cong D^{n_0} \wedge S^{n_1-1} \coprod_{S^{n_0-1} \wedge S^{n_1-1}} S^{n_0-1} \wedge D^{n_1}$. For general $k > 1$, this can be done by a colimit construction. Let $[k]$ be the set $\{0, \ldots, k\}$ and let $\mathcal{P}([k])$ be the power set of *proper* subsets of $[k]$. Then define a functor $F : \mathcal{P}([k]) \to Top_*$ by $\chi(S) = X_0(S) \wedge \cdots X_k(S)$ and

$$X_i(S) = \begin{cases} D^{n_i} & \text{if } i \in S \text{ and} \\ S^{n_i - 1} & \text{if } i \notin S. \end{cases}$$

The functor χ takes an inclusion of subsets to the map of topological spaces obtained by inclusion of the boundary. Each such map carries the assembly data we require, and the boundary of D^n is homeomorphic to the colimit

$$colim_{S \subset \mathcal{P}([k])} \chi.$$

□

5. Spectral sequences in the stable setting

The results of Section 4 can be extended to the stable setting by considering spectral sequences of spectra with operad actions. Here, we only need the old-fashioned notion of a spectrum - that is, a spectrum is a sequence of based spaces $\mathbf{E} = \{E_k\}$ together with structure maps either $\epsilon : \Sigma E_k \to E_{k+1}$ or else $\sigma : E_k \to \Omega E_{k+1}$ (these are of course adjoint).

Modern technology has introduced many improvements to categories of spectra. These advances allow us to work primarily in the homotopy category, rather than with concrete spectra. This is extremely advantageous when working with diagrams of spectra which commute only up to homotopy. Because of the additional delicacy introduced by such considerations, in this exposition, we choose to require more concrete conditions on naive spectra and avoid the homotopy category as much as possible.

The correct notion of homotopy for spectra is the stable homotopy defined to be

$$\pi_n(\mathbf{E}) = \lim_k \pi_{n+k}(E_k)$$

where the limit is taken over the structure maps σ. A map $f : \mathbf{E} \to \mathbf{B}$ of spectra is given by maps $f_k : E_k \to B_k$ which *strictly* commute with the structure maps (or possibly by equivalence classes of such maps of spectra). If we have a commuting ladder of long exact sequences

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \pi_k(E_m) & \longrightarrow & \pi_k(B_m) & \longrightarrow & \pi_k(B_m, E_m) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \pi_{k+1}(E_{m+1}) & \longrightarrow & \pi_{k+1}(B_{m+1}) & \longrightarrow & \pi_{k+1}(B_{m+1}, E_{m+1}) & \longrightarrow & \cdots \end{array}$$

for all k and m, then we obtain a long exact sequence of stable homotopy groups associated to f with

$$\pi_k(\mathbf{B}, \mathbf{E}) = \lim_n \pi_{k+n}(B_n, E_n).$$

As before, we suppose that we have a tower of fibrations

$$\cdots \longrightarrow \mathbf{W_{m+1}} \xrightarrow{q_{m+1}} \mathbf{W_m} \xrightarrow{q_m} \mathbf{W_{m-1}} \longrightarrow \cdots$$

where each $\mathbf{W_i}$ is now a spectrum rather than a pointed space. To avoid the homotopy category and its subtleties, we define our fibrations in a very rigid manner. For us, a fibration of spectra $p : \mathbf{E} \to \mathbf{B}$ is a map of spectra in which each map $p_n : E_n \to B_n$ is a fibration of spaces. The fiber of p_n is then the pullback:

$$\begin{array}{ccc} F_n & \dashrightarrow & PB_n \\ {\scriptstyle d_n}\downarrow & & \downarrow{\scriptstyle ev_1} \\ E_n & \xrightarrow{p_n} & B_n \end{array}$$

Taking fibers of each p_n then produces a spectrum $\mathbf{F} = \{F_n\}$ whose structure maps are constructed via the universal property of the pullback. Let $eval : \Sigma PB_n \to P(\Sigma B_n)$ be the map taking $s \wedge \gamma \in S^1 \wedge PB_n$ to the path $s \wedge \gamma$ in ΣB_n. The the outer square of the diagram

$$\begin{array}{ccccc} \Sigma F_n & \longrightarrow & \Sigma PB_n & \xrightarrow{eval} & P(\Sigma B_n) \\ & \searrow^{\epsilon^F} & & & \downarrow{\scriptstyle P(\epsilon^B)} \\ {\scriptstyle \Sigma d_n}\downarrow & & F_{n+1} & \longrightarrow & PB_{n+1} \\ & & {\scriptstyle d_{n+1}}\downarrow & & \downarrow{\scriptstyle ev_1} \\ \Sigma E_n & \xrightarrow{\epsilon^E} & E_{n+1} & \xrightarrow{p_{n+1}} & B_{n+1} \end{array}$$

is seen to commute after a short diagram chase, ensuring the existence of ϵ^F. We obtain isomorphisms $\pi_k(\mathbf{B}, \mathbf{E}) \cong \pi_k(\mathbf{F})$ in the usual way.

Finally, we require the operad action of $\mathbf{P}$ on the tower $\mathbf{W}_*$ to be strict; that is, the structure maps should produce strictly commuting diagrams.

Theorem 5.1. *If $\mathbf{P}$ acts on a tower of fibrations $\mathbf{W}_*$ of spectra, then the associated spectral sequence is a sequence of $\pi_*(\mathbf{P})$-algebras.*

Under these assumptions, the proof of this theorem is exactly as given for spaces in Section 4. We note, in particular, that this theorem implies that towers of fibrations with multiplicative structure which commute only up to homotopy produce multiplicative spectral sequences, again up to homotopy, since an operad will classify the (e.g. associative) multiplication up to homotopy.

5.1. **Example.** Given a spectrum $\mathbf{E}$, the generalized homology of the spectrum $\mathbf{X}$ with coefficients in $\mathbf{E}$ is defined by

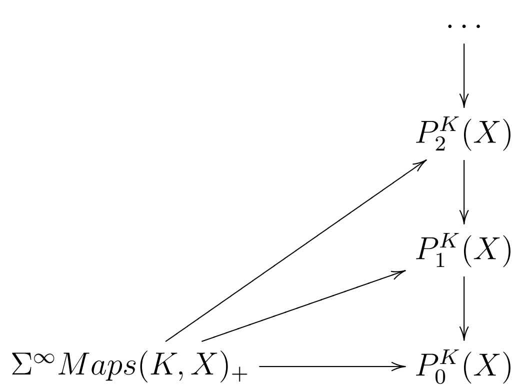
$$\mathbf{E}_n(\mathbf{X}) = \pi_n(\mathbf{E} \wedge \mathbf{X}).$$

When $\mathbf{E}$ is the Eilenberg-MacLane spectrum $\mathbf{H}\mathbb{Z}/\mathbf{p}$, the resulting homology theory is the mod p homology of the spectrum E, which is a homology theory of much study (see, e.g. [1] or [10]). If $\mathbf{E}$ is also a commutative and associative ring spectrum, then an operad action on X passes to an operad action on $\mathbf{E} \wedge \mathbf{X}$ via the maps

$$\mathbf{P}(\mathbf{k}) \wedge \mathbf{E} \wedge \mathbf{X} \wedge \cdots \wedge \mathbf{E} \wedge \mathbf{X} \longrightarrow \mathbf{E}^{\wedge k} \wedge \mathbf{P}(\mathbf{k}) \wedge \mathbf{X}^{\wedge}k \xrightarrow{\mu \wedge \rho} \mathbf{E} \wedge \mathbf{X}$$

where the map μ is the multiplication map for the ring spectrum $\mathbf{E}$ and ρ is the structure map for the operad algebra multiplication for X.

A particularly interesting case is given by the Goodwillie tower of fibrations associated to the functor $\Sigma^\infty Maps(K, X)_+$ as a functor of X and for a fixed finite complex K. The tower of fibrations



was computed by G. Arone [3]. Later, S. Ahearn and N. Kuhn [2] studied the structure of spectral sequences associated to this tower for various values of K. In particular, they showed that the little cubes operad $\mathcal{C}_n$ acts on the tower $P_*^{S^n}(X)$, where $\mathcal{C}_n$ is the operad whose k-th space $\mathcal{C}_n(k)$ consists of an ordered collection of k little n-cubes linearly embedded into the standard n-cube with disjoint interiors and axes parallel to those of the standard n-cube,

Applying generalized mod-p homology $\mathbf{H}_* = \pi_*(\mathbf{H}\mathbb{Z}/\mathbf{p} \wedge -)$ to the tower $P_*^{S_1}(X)$ and using Theorem 5.1, we obtain a spectral sequence of algebras over $\mathbf{H}_*(\mathcal{C}_n)$. Since the Dyer-Lashof operations are induced by the action of the little cubes operad, this explains the authors' note that

"Computationally, this implies that the associated spectral sequences for computing mod p homology admit Dyer-Lashof operations." [2].

References

[1] J. F. Adams, *Stable Homotopy and Generalised Homology*, Chicago Lectures in Mathematics, The University of Chicago Press, Chicago, IL, 1974.

[2] S. Ahearn and N. Kuhn, *Product and other fine structure in polynomial resolutions of mapping spaces*, Algebraic & Geometric Topology, **2** (2002), 591 – 647.

[3] G. Arone, *A generalization of Snaith-type filtration*, Trans. A.M.S. **351** (1999), 1123 – 1250.

[4] B. Fresse, Koszul duality of operads and homology of partition posets, *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*, 115–215, *Contemp. Math.*, 346, Amer. Math. Soc., Providence, RI, 2004.

[5] D. Dugger, *Multiplicative structures on homotopy spectral sequences, Part I*, preprint.

[6] M. Markl, S. Schnider and J. Stasheff, *Operads in Algebra, Topology and Physics*, Mathematical Surveys and Monographs, 96. American Mathematical Society, 2002.

[7] W. S. Massey, *Products in exact couples*, Ann. of Math. (3) **59**, (1954). 558-569.

[8] P. May, *Geometry of Iterated Loop Spaces* , Springer-Verlag 1972.

[9] J. McCleary, *User's guide to spectral sequences*, Publish or Perish, Willmington, DE, 1985.

[10] D. Ravenel, *Complex Cobordism and Stable Homotopy Groups of Spheres*, second edition, AMS Chelsea Publishing, Providence, RI, 2004.

[11] C. Weibel, *An introduction to homological algebra*. Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994.

Department of Mathematics, University of Calgary, Calgary, AB, Canada

E-mail address: kristine@math.ucalgary.ca

Department of Mathematics, Fort Lewis College, Durango, Colorado, USA

E-mail address: scull_l@fortlewis.edu