# Binomial option pricing



## Objectives

- Review the binomial option pricing model.
- Understand the roles of the various parameters in the model.
- Understand the implementation issues.



## The basic idea

- The idea behind the binomial method is to create a discrete version of the set of possible states of the world.
- Time proceeds in 'ticks' (of length $\Delta t$ perhaps).
- ver each tick, asset prices can vary only in a restricted manner: in fact we typically specify two multipliers $u$ and $d$ ($u > d$) and say that an asset whose value is $S$ at one tick can be *either* at $uS$ *or* at $dS$ at the next tick—only two possibilities (hence the name).
- Starting from an asset price of $S(0)$ at time $0$, we thus generate a *tree* of values that spreads out tick after tick, each branch splitting into two at each stage.

## The shape of the tree

- One of the properties of the a tree involving *multiplicative* moves (the same two possibilities at each node of the tree) is that it will be *recombining*. An up-step followed by a down-step is the same as a down-step followed by an up-step.
- We can label the time steps $t^m = m\Delta t$. We can also label the possible asset-price levels at time $t^m$ by $S_j^m = S(0)u^j d^{m-j}$, so that $j$ denotes the number of up-steps.
- The tree spreads out exponentially at time progresses - between $S_m^m = u^m S(0)$ and $S_0^m = d^m S(0)$.
- If we are at $S_j^m$ at time $t^m$, then at time $t^{m+1}$ we can move to $uS_j^m = S_{j+1}^{m+1}$ or to $dS_j^m = S_j^{m+1}$.
- We can get to $S_j^m$ from the root of the tree by taking $j$ up-steps and $m-j$ down-steps *in any order*.
- Recall: the tree is supposed to model all possible price paths over time. Typically, we are interested in an interval $[0,T]$, with $T = M\Delta t$.



## Derivative assets

Suppose now that we have some derivative contract $V$ whose value $V_j^m$ at each node is to be computed, and we suppose that there is available on the market a bond $B$ which provides a guaranteed rate of return of $e^{r\Delta t}$ over each tick. (Here $r$ is the risk-free interest rate.)

We further suppose that the contract $V$ has a known value (as a function of the asset price level) at some future time $T = M\Delta t$:

$$V_j^M = \Lambda(S_j^M).$$

We want to find a way of deducing what would be a fair price at all other times.

We shall do this by performing the usual balancing act and constructing a portfolio $\Pi$ out of $V$ and $S$.



## Balancing act

- Set $\Pi = V - \Delta S$.
- Suppose we are at some node $(m,j)$ on the tree, and examine what are the possible values of the portfolio a time $\Delta t$ later.
- We have two possibilities:
  1. we move up to $(m+1, j+1)$, getting $\Pi_{j+1}^{m+1} = V_{j+1}^{m+1} - \Delta S_{j+1}^{m+1}$.
  2. we move down to $(m+1, j)$ getting $\Pi_j^{m+1} = V_j^{m+1} - \Delta S_j^{m+1}$.
- But notice now one thing. If we take

$$\Delta = \frac{V_{j+1}^{m+1} - V_j^{m+1}}{S_{j+1}^{m+1} - S_j^{m+1}}$$

then the value of the portfolio at time $m+1$ is exactly the same in both cases.

## What's fair is fair

- If the outcome over the time step is certain, the return must be the same as that from the bond: i.e., we must have

$$\Pi_{j+1}^{m+1} = \mathrm{e}^{r\Delta t}\Pi_j^m.$$

- This equation yields a relationship between the values of $V$ at time $m+1$ and the value at time $m$:

$$V_j^m = \mathrm{e}^{-r\Delta t}\left[pV_{j+1}^{m+1} + (1-p)V_j^{m+1}\right], \quad \text{with} \quad p = \frac{\mathrm{e}^{r\Delta t} - d}{u-d}.$$

- This looks like a discounted expectation, and, so long as $d \le \mathrm{e}^{r\Delta t} \le u$, $p$ has the properties of a probability.
- Under $p$, the expected value of $S$ after one time step is $\mathrm{e}^{r\Delta t}S$.



## Pricing the future

Consider a futures contract, with futures price $K$. The payoff at time $T$ of this contract will be of the form $S-K$, and we are interested in the value now of such a contract. We have $V_j^M = S_j^M - K$, so that

$$V_j^{M-1} = \mathrm{e}^{-r\Delta t}\left[pS_{j+1}^M + (1-p)S_j^M - K\right] = S^{M-1} - K\mathrm{e}^{-r\Delta t}.$$

Continuing in this manner we obtain $V_j^m = S_j^m - K\mathrm{e}^{-r(M-m)\Delta t}$ at each node $(m,j)$.

If we want the value of the contract to be zero at the node $(m,j)$, we need to take $K = S_j^m\mathrm{e}^{r(M-m)\Delta t}$. Now this is precisely what happens in practice: at each tick $m$, futures contracts are available that expire at some point $T$ in the future, and the strike price of those contracts is set to be such that the value at $m$ of the contract is zero.



## Calibrating the tree

- We have not yet completely specified the tree—we need to determine values for $u$ and $d$.
- We also need to choose probabilities on the tree too. The risk-neutral probability $p$ is one choice. But we may want to choose a 'real-world' probability and calibrate the tree to observed asset price movements.
- In order to do this, we compute the mean and variance of movements over one timestep, starting from a value of $S$.
- If $q$ is the probability of an upstep, and writing $S$ for the value of the asset at time $t$, we have the expectations at time $t+\Delta t$:

$$\begin{aligned}
\mathbb{E}[S(t+\Delta t)|S] &= quS + (1-q)dS = S(qu + (1-q)d) \\
\mathbb{E}[(S(t+\Delta t))^2|S] &= qu^2S^2 + (1-q)d^2S^2 \\
\mathrm{Var}[S(t+\Delta t)|S] &= S^2q(1-q)(u-d)^2.
\end{aligned}$$

## Matching reality

- If the observed (conditional) mean and variance of $S(t + \Delta t)$ are $\overline{\mu}S(t)$ and $\overline{\sigma}^2 S(t)^2$ respectively, we can match these by choosing $q$, $u$ and $d$ to satisfy

$$\overline{\mu} = qu + (1-q)d$$
$$\overline{\sigma}^2 = q(1-q)(u-d)^2$$

so that

$$q = \frac{\overline{\mu} - d}{u - d}$$

(implying that we want $d \le q \le u$, with $u \ne d$), and

$$\overline{\sigma}^2 = (\overline{\mu} - d)(u - \overline{\mu}).$$

- The problem is that this is only two equations, and we have three variables to determine.



## Choices, choices

Of the many possible choices, these three are typical.

- $u = \overline{\mu} + \overline{\sigma}$ and $d = \overline{\mu} - \overline{\sigma}$. In this case $q = \frac{1}{2}$.
- $ud = 1$. This leads to

$$u + \frac{1}{u} = 2A, \quad \text{where} \quad A = \frac{\overline{\sigma}^2 + \overline{\mu}^2 + 1}{2\overline{\mu}}$$

so that

$$u = A + \sqrt{A^2 - 1} \quad \text{and} \quad d = A - \sqrt{A^2 - 1}.$$

- $ud = \overline{\mu}^2$. This time we get

$$A = 1 + \frac{\overline{\sigma}^2}{2\overline{\mu}^2}$$

and then similar formulae for $u$ and $d$ apply:

$$u = \overline{\mu}(A + \sqrt{A^2 - 1}) \quad \text{and} \quad d = \overline{\mu}(A - \sqrt{A^2 - 1}).$$



## Matching log-returns

- Writing $x \ln S$, $U = \ln u$ and $D = \ln d$ we can re-run the above exercise. We find:
- $\mathbb{E}[x(t + \Delta t) - x(t)] = qU + (1-q)D.$
- $\text{Var}[x(t + \Delta t) - x(t)] = q(1-q)(U-D)^2.$
- If the observed values for these quantities are $\hat{\mu}\Delta t$ and $\sigma^2 \Delta t$ respectively, then we have similar choices to those already outlined.
- With $q = \frac{1}{2}$ we end up with

$$U = \hat{\mu}\Delta t + \sigma\sqrt{\Delta t} \quad \text{and} \quad D = \hat{\mu}\Delta t - \sigma\sqrt{\Delta t}.$$

- With $U + D = 0$ we get

$$U = \sqrt{\sigma^2 \Delta t + \hat{\mu}^2 \Delta t^2} \quad \text{and} \quad q = \frac{1}{2} + \frac{\hat{\mu}\Delta t}{2U}.$$

## Derivative pricing

- Once we have set our time interval $[0, T]$ and decided on the number of time steps $M$, and have created a calibrated tree (defining $u$, $d$ and $p = \frac{e^{r\Delta t} - d}{u - d}$), we are ready to start.
- If the payoff function for the derivative at time $Y$ is $\Lambda(S)$, we set

$$V_j^M = \Lambda(S_j^M), \qquad j = 0, \ldots, M.$$

- Then, for $m = M - 1, \ldots, 0$, given $\{V_j^{m+1}\}_{j=0}^{m+1}$, we compute

$$\boxed{V_j^m = e^{-r\Delta t}\left[pV_{j+1}^{m+1} + (1-p)V_j^{m+1}\right].}$$



## Binomial formula

- In this setting, we can obtain a quasi-analytical formula for the option price.
- Note that the probability (under the risk-neutral measure $p$) of reaching the node $(m, j)$ is simply $2^{-m}\binom{m}{j}p^j(1-p)^{m-j}$—the number of paths through the tree that reach the node $(m, j)$ times the probability of following such a path, divided by the total number of different paths up to time $t^m$.
- The option price can be written as a discounted expectation of the payoff function:

$$\begin{aligned} V_0^0 &= e^{-rT}\sum_{j=0}^{M}\Lambda(S_j^M)\mathbb{P}[(M, j)] \\ &= e^{-rT}2^{-M}\sum_{j=0}^{M}\binom{M}{j}p^j(1-p)^{M-j}\Lambda(S_j^M). \end{aligned}$$



## How does it shape up?

- We can compare the accuracy by computing an option value from the Black-Scholes formula and seeing how the binomial result varies with $M$.
- We can examine how the design of the tree (the choice of $u$ and $d$) affects the result.
- We can compare the speed of execution.

## American options

- The holder can exercise at any time. We find the fair value of the option by assuming that the holder exercises optimally: at each point the holder will exercise if and only if the payoff exceeds the 'continuation value' of the option.
- The recursive valuation formula does not change much:

$$\boxed{V_j^m = \max\Big(\Lambda(S_j^m), \mathrm{e}^{-r\Delta t}\left[pV_{j+1}^{m+1} + (1-p)V_j^{m+1}\right]\Big).}$$

- At each node $(m, j)$ we can record the choice that is tp be made:

$$C_j^m = \begin{cases} 1 & \text{if} \quad V_j^m = \Lambda(S_j^m) \\ 0 & \text{otherwise.} \end{cases}$$

  Thus the solution tells the holder of the option what the optimal exercise strategy is.