The data we have seen show us that we should be cautious when using the lognormal model - there are situations in which it fails to capture the true behaviour of asset price returns. However, for liquid markets (and when market conditions are not too stressful!) it is quite adequate.

The data we have seen show us that we should be cautious when using the lognormal model - there are situations in which it fails to capture the true behaviour of asset price returns. However, for liquid markets (and when market conditions are not too stressful!) it is quite adequate.

The lognormal model says that, for any times $t_1 < t_2$,

$$X(t_2) - X(t_1) = \log \frac{S(t_2)}{S(t_1)} = \hat{\mu}(t_2 - t_1) + \sigma \sqrt{t_2 - t_1} Z,$$

where $Z \sim N(0, 1)$.

In particular, given a discrete set of times $t_n = n\Delta t$, $n = 0, \dots, N$, with $t = N\Delta t$, and, for each n, letting $Z_n \sim N(0, 1)$, we have

$$X(t) = X(0) + \sum_{n=0}^{N-1} \hat{\mu}\Delta t + \sigma \sqrt{\Delta t} Z_n, \quad \text{or} \quad S(t) = S(0) \prod_{n=0}^{N-1} \mathrm{e}^{\hat{\mu}\Delta t + \sigma \sqrt{\Delta t} Z_n}.$$

From the properties of normal random variables, with $Z \sim N(0, 1)$,

$$X(t) = X(0) + \hat{\mu} t + \sigma \sqrt{t} Z, \quad \text{or} \quad S(t) = S(0)\mathrm{e}^{\hat{\mu} t + \sigma \sqrt{t} Z}.$$

Asset price modelling: the lognormal model

The relationship between returns and log-returns

Instead of modelling log-returns as normal random variables, we could take the approach in the text and model returns (over short time intervals) as normal random variables.

$$\frac{S(t+\Delta t) - S(t)}{S(t)} = \mu\Delta t + \sigma\sqrt{\Delta t}Z.$$

Again, if we choose $t_n = n\Delta t$ as before, we can write

$$S(t) = S(0)\prod_{n=0}^{N-1}\left(1 + \mu\Delta t + \sigma\sqrt{\Delta t}Z_n\right), \quad \text{or} \quad \log\frac{S(t)}{S(0)} = \sum_{n=0}^{N-1}\log\left(1 + \mu\Delta t + \sigma\sqrt{\Delta t}Z_n\right).$$

The right-hand side does not sum up quite as nicely as before. If we expand[a] the logarithms in powers of $\hat{\mu}\Delta t + \sigma\sqrt{\Delta t}Z_n$, we get

$$\log\frac{S(t)}{S(0)} \approx \sum_{n=0}^{N-1}\mu\Delta t + \sigma\sqrt{\Delta t}Z_n - \frac{1}{2}\sigma^2\Delta t Z_n^2,$$

where we have neglected anything involving higher powers of Δt.

[a] This is justified under mild assumptions on Z_n.

almost same as before if $\hat{\mu} = \mu - \frac{\sigma^2}{2}$

We have

$$\mathbb{E}[\mu\Delta t + \sigma\sqrt{\Delta t}Z_n - \frac{1}{2}\sigma^2\Delta t Z_n^2] = \mu\Delta t - \frac{1}{2}\sigma^2\Delta t$$
$$\text{Var}[\mu\Delta t + \sigma\sqrt{\Delta t}Z_n - \frac{1}{2}\sigma^2\Delta t Z_n^2] = \sigma^2\Delta t + O(\Delta t)^2.$$

From the Central Limit Theorem, we can write

$$\sum_{n=0}^{N-1}\mu\Delta t + \sigma\sqrt{\Delta t}Z_n - \frac{1}{2}\sigma^2\Delta t Z_n^2 \approx \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}Z.$$

Both this approximation and the previous one become better as $\Delta t \to 0$. Thus, in the limit, the two approaches yield the same result (with $\hat{\mu} = \mu - \frac{1}{2}\sigma^2$):

$$\log\frac{S(t)}{S(0)} = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}Z, \quad \text{or} \quad S(t) = S(0)\mathrm{e}^{(\mu-\frac{1}{2}\sigma^2)t+\sigma\sqrt{t}Z}.$$

The lognormal distribution

If x is normally-distributed, then $y = e^x$ is *lognormally*-distributed. While x can take any real value, y takes only positive values.

The lognormal distribution

If x is normally-distributed, then $y = e^x$ is *lognormally*-distributed. While x can take any real value, y takes only positive values.

If $x \sim N(\alpha, \beta^2)$, then the density function for y is

$$g(y; \alpha, \beta^2) = \frac{1}{y\sqrt{2\pi\beta^2}} e^{-\frac{(\ln y - \alpha)^2}{2\beta^2}}.$$

We can use this to calculate the mean and variance of y:

$$\mathbb{E}[y] = e^{\alpha + \frac{1}{2}\beta^2}, \qquad \mathrm{Var}[y] = e^{2\alpha+\beta^2}\left(e^{\beta^2} - 1\right).$$

The lognormal distribution

If x is normally-distributed, then $y = \mathrm{e}^x$ is *lognormally*-distributed. While x can take any real value, y takes only positive values.

If $x \sim N(\alpha, \beta^2)$, then the density function for y is

$$g(y; \alpha, \beta^2) = \frac{1}{y\sqrt{2\pi\beta^2}} \mathrm{e}^{-\frac{(\ln y - \alpha)^2}{2\beta^2}}.$$

We can use this to calculate the mean and variance of y:

$$\mathbb{E}[y] = \mathrm{e}^{\alpha + \frac{1}{2}\beta^2}, \qquad \mathrm{Var}[y] = \mathrm{e}^{2\alpha+\beta^2}\left(\mathrm{e}^{\beta^2} - 1\right).$$

This means that, given $S(0)$, $S(t)$ is modelled as a random variable with density $g(y; \log S(0) + \hat{\mu}t; \sigma^2 t)$, and that

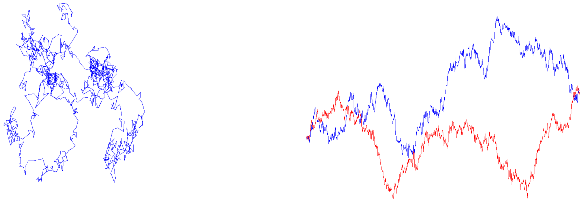
$$\mathbb{E}[S(t)] = S(0)\mathrm{e}^{\mu t} \quad \text{and} \quad \mathrm{Var}[S(t)] = S(0)^2\mathrm{e}^{2\mu t}(\mathrm{e}^{\sigma^2 t} - 1).$$

- We have seen that $S(t)$ is lognormally distributed for each t, which means that $X(t) = \log S(t)$ is normally distributed.
- But that is not the whole story: not only is $X(t)$ normally distributed (given what we know at time 0), but so, for any $\Delta t > 0$, is the increment $X(t + \Delta t) - X(t)$ (given our knowledge at time t).
- In fact $X(t)$ is (almost certainly) a *continuous* curve, athough it is certainly not smooth.
- We can write $X(t) = X(0) + \hat{\mu}t + \sigma W(t)$, where $W(t)$ is a (standard) Brownian motion (or Weiner curve).

Robert Brown

Robert Brown was a Scottish botanist who in the early 19th century observed a continuous jittery motion of tiny particles with pollen grains. He hypothesised that this was because pollen was alive.

A Brownian motion path, with x- and y-coordinates through time

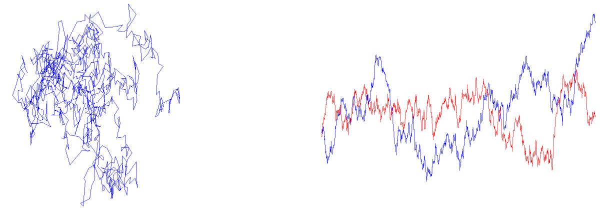


Brownian motion

Robert Brown

Robert Brown was a Scottish botanist who in the early 19th century observed a continuous jittery motion of tiny particles with pollen grains. He hypothesised that this was because pollen was alive.

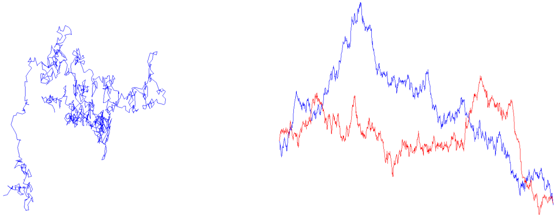
A Brownian motion path, with x- and y-coordinates through time



Robert Brown

Robert Brown was a Scottish botanist who in the early 19th century observed a continuous jittery motion of tiny particles with pollen grains. He hypothesised that this was because pollen was alive.

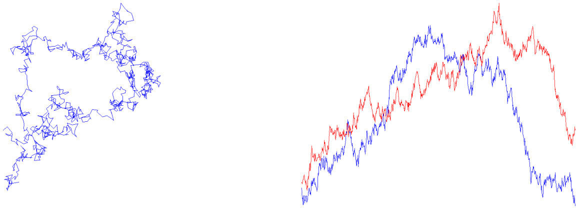
A Brownian motion path, with x- and y-coordinates through time



Robert Brown

Robert Brown was a Scottish botanist who in the early 19th century observed a continuous jittery motion of tiny particles with pollen grains. He hypothesised that this was because pollen was alive.

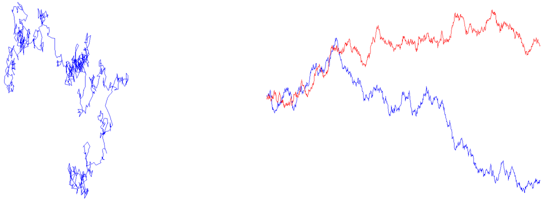
A Brownian motion path, with x- and y-coordinates through time



Robert Brown

Robert Brown was a Scottish botanist who in the early 19th century observed a continuous jittery motion of tiny particles with pollen grains. He hypothesised that this was because pollen was alive.

A Brownian motion path, with x- and y-coordinates through time



Brownian motion - a mathematical description

The mathematical properties of Brownian motion were established by Bachelier, Einstein and Wiener.

Properties of Brownian motion

A (standard) Brownian motion (Wiener process) is a random path $W(t)$ with the following properties:

- $W(t)$ is continuous (with probability 1).
- Nonoverlapping increments are independent.
- $W(t) - W(s) \sim N(0, |t - s|)$.

The mathematical properties of Brownian motion were established by Bachelier, Einstein and Wiener.

Properties of Brownian motion

A (standard) Brownian motion (Wiener process) is a random path $W(t)$ with the following properties:

- $W(t)$ is continuous (with probability 1).
- Nonoverlapping increments are independent.
- $W(t) - W(s) \sim N(0, |t - s|)$.

Scaling and shifting

- If $W(t)$ is a Brownian motion, then so is $W(t + a)$, for any a.
- If $W(t)$ is a Brownian motion, then so is $\frac{1}{\sqrt{b}} W(bt)$, for any $b > 0$.

Simulating asset prices

Given a set of time points

$$0 = t_0 < t_1 < \cdots < t_N = t, \quad \text{with} \quad t_{j+1} - t_j = \Delta_j$$

we can construct a sample of a *discrete* Brownian motion by setting

$$W_0 = 0; \qquad W_{j+1} = W_j + \sqrt{\Delta_j} Z_j, \quad j = 0, \ldots, N-1,$$

where the Z_j are independent standard normal random variables.

Simulating asset prices

Given a set of time points

$$0 = t_0 < t_1 < \cdots < t_N = t, \quad \text{with} \quad t_{j+1} - t_j = \Delta_j$$

we can construct a sample of a *discrete* Brownian motion by setting

$$W_0 = 0; \qquad W_{j+1} = W_j + \sqrt{\Delta_j} Z_j, \quad j = 0, \ldots, N-1,$$

where the Z_j are independent standard normal random variables.

Then we can define, for $j = 0, \ldots, N$, with $X_0 = \log S(0)$,

$$X_j = X_0 + \hat{\mu} t_j + \sigma W_j, \quad \text{and} \quad S_j = \exp(X_j).$$

Simulating asset prices

Given a set of time points

$$0 = t_0 < t_1 < \cdots < t_N = t, \quad \text{with} \quad t_{j+1} - t_j = \Delta_j$$

we can construct a sample of a *discrete* Brownian motion by setting

$$W_0 = 0; \qquad W_{j+1} = W_j + \sqrt{\Delta_j} Z_j, \quad j = 0, \ldots, N-1,$$

where the Z_j are independent standard normal random variables.

Then we can define, for $j = 0, \ldots, N$, with $X_0 = \log S(0)$,

$$X_j = X_0 + \hat{\mu} t_j + \sigma W_j, \quad \text{and} \quad S_j = X_j.$$

We end up with a set of random variables S_j, each of which is lognormally-distributed, so that $X_j = \log S_j \sim N(X_0 + \hat{\mu} t_j, \sigma^2 t_j)$, and, moreover

$$X_{j+1} - X_j \sim N(X_0 + \hat{\mu} \Delta t_j, \sigma^2 \Delta t_j).$$

Simulating asset prices

Simulating S directly

Alternatively, we could simulate using

$$S_0 = S(0); \qquad S_{j+1} = S_j + \mu \Delta t_j S_j + \sigma\sqrt{\Delta t_j} Z_j S_j, \quad j = 0, \ldots, N-1.$$

This will lead to slightly different results to the approach via X_j, even starting with the same samples for Z_j. The difference will depend on $\max_j \Delta t_j$, and will vanish in the limit as this $\to 0$.

This approach corresponds to modelling the returns as normals:

$$r_j := \frac{S_{j+1} - S_j}{S_j} = \mu \Delta t_j + \sigma\sqrt{\Delta t_j} Z_j.$$

This means that

$$\mathbb{E}[r_j^2] = \sigma^2 \Delta t_j + O(\Delta t_j^2), \quad \text{and} \quad \mathrm{Var}[r_j^2] = 2\sigma^4 \Delta t_j^2 + O(\Delta t_j^2).$$

The Central Limit Theorem now tells us that

$$\sum_{j=0}^{N-1} r_j^2 \approx \sigma^2 t \quad \text{when } \max_j \Delta t_j \text{ is small.}$$