AMAT 483 Midterm Examination

23th March, 2011

Basic calculators are allowed. No other computing devices or formula sheets may be used.
Time: 50 minutes.

1. The function $\phi(x, y) = N\left(\frac{\ln x}{y} + \frac{y}{2}\right) - \frac{1}{x}N\left(\frac{\ln x}{y} - \frac{y}{2}\right)$, where $N(\cdot)$ is the standard normal cumulative density function, enables the value of a European call option on an asset S with dividend yield δ, strike price K and expiry T, in a market with a risk-free interest rate r to be written in the form

$$C(t, S, K; T, r, \delta, \sigma) = S\,\mathrm{e}^{-\delta(T-t)}\phi\left(\frac{S\,\mathrm{e}^{-\delta(T-t)}}{K\mathrm{e}^{-r(T-t)}}, \sigma\sqrt{T-t}\right).$$

(a) **[2]** Using this fact, or otherwise, verify that C satisfies

$$S\frac{\partial C}{\partial S} + K\frac{\partial C}{\partial K} = C.$$

Solution:
With $x(S, K) := \frac{S\,\mathrm{e}^{-\delta(T-t)}}{K\mathrm{e}^{-r(T-t)}}$,

$$S\frac{\partial C}{\partial S} = C + S\,\mathrm{e}^{-\delta(T-t)}S\frac{\partial x}{\partial S}\frac{\partial \phi}{\partial x},$$

and

$$K\frac{\partial C}{\partial K} = S\,\mathrm{e}^{-\delta(T-t)}K\frac{\partial x}{\partial K}\frac{\partial \phi}{\partial x}.$$

All we need to do to establish the result is to show that

$$S\frac{\partial x}{\partial S} = -K\frac{\partial x}{\partial K},$$

and this is straightforward.

(b) **[1]** Suppose you were an option trader who did not believe the Black-Scholes formula could be used, but you did believe that the relation in part (a) held. How might this relation help you to determine how to hedge a written call option using option prices on the market?

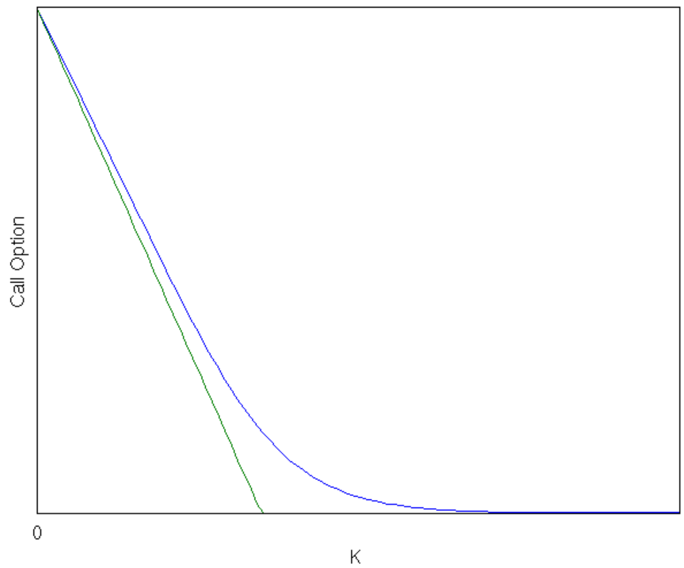
Solution:
In this setting, I would deduce that

$$\frac{\partial C}{\partial S} = \frac{1}{S}\left(C - K\frac{\partial C}{\partial K}\right).$$

I would then be in a position to estimate the value of the Delta $\Delta = \frac{\partial C}{\partial S}$ by estimating $\frac{\partial C}{\partial K}$ from option values for various values of K available in the market.

(c) **[3]** Assuming no dividends, sketch the payoff of a call option $(S - K)_+$ as a function of the **strike price** K, and on the same graph sketch of the value of C (at some time prior to expiry), also as a function of K.

Solution:
The solution is shown in the figure below. Note that, when $K = 0$, the call option price is the same as the asset price (assuming no dividends). If there are dividends, it will be less.



2. Consider the following MATLAB code snippet

```
sigma = 0.5; r = 0.01; K = 1; S0 = 1; T = 0.5; N = 1000;
W = randn(N,1);
S = S0*exp( (r-sigma^2/2)*T+sigma*T*W );
V = mean( exp(-r*T)*max(0,S-K) );
```

   (a) **[3]** Describe what this code is trying to compute, and point out the mistake that will lead it to get the answer wrong (and how it should be corrected).

   Solution:
   The code is trying to compute the value of a European call option on a log-normal asset, with time to expiry $T = 0.5$, current asset value 1, volatility 0.5, strike price 1, etc. There is a mistake in the third line - the random increments W should be multiplied by `sqrt(T)`.

   (b) **[1]** Describe how you would estimate the accuracy of your answer.

   Solution:
   I would compute the sample standard deviation $\hat{\sigma}$ of the discounted payoffs `exp(-r*T)*max(0,S-K)`, and assume that `V` was approximately normal with variance $\hat{\sigma}^2/N$ to deduce a confidence interval for my sample `V`.

   (c) **[3]** Describe a way (other than increasing N) in which the corrected code might be improved in order to improve the accuracy; explain why you think it would help.

   Solution:
   I would add the line `SM = S0*exp( (r-sigma^2/2)*T-sigma*sqrt(T)*W ); S = (S+SM)/2` before the fourth line of code. This should give me samples `V` with a significantly lower variance. This will be the case if the function generating the samples is a monotone function of `W`, which is the case here.

3. The mean and variance of the daily log-returns of a certain (non dividend-paying) stock are estimated to be 0.0004 and 0.0016, respectively. The daily interest rate is 0.0001. (These are not percentages.)

Construct a (one-stage) binomial tree such that the log-price x moves from x at the root of the tree to either $x + u$ (with probability p) or $x - d$ (with probability $1 - p$) after one day.

(a) **[1]** What constraint involving the interest rate must u and d satisfy?

Solution:
$d \leq 0.0001 \leq u$.

(b) **[2]** Find values u, d and p satisfying this constraint such that the mean and variance of the log-returns on the tree match those estimated for the stock.

Solution:
The mean of the log-returns on the tree is $pu + (1 - p)d$, and the variance is $p(1 - p)(u - d)^2$. We can match these to the observed values by choosing $p = 1/2$, $u = 0.0002 + 0.02 = 0.0202$ and $d = 0.0002 - 0.02 = -0.0198$. These values satisfy the constraint from the first part of the question.

(c) **[2]** Find the risk-neutral probability q under which the expected rate of growth in the stock is the same as the risk-free interest rate.

Solution:
With the parameters above we need $q = \frac{e^r - e^d}{e^u - e^d} = \frac{e^{0.0001} - e^{-0.0198}}{e^{0.0202} - e^{-0.0198}}$.