Discrete hedging

(Following Wilmott (1994), "Discrete Charms," Risk Magazine, Vol. 7, No. 3, pp. 48–51.)
We suppose that we are hedging at intervals of δt, and that $S = \mathrm{e}^x$, with $\delta x = \hat{\mu}\delta t + \sigma\phi\sqrt{\delta t}$, where $\phi \sim N(0,1)$, and $\hat{\mu} = \mu - \frac{1}{2}\sigma^2$.
We are going to hedge an option V with Δ units of stock S, so that our portfolio $\Pi = V - \Delta S$. Then

$$\delta\Pi = \delta V - \Delta\delta S,$$

and so

$$\mathrm{var}[\delta\Pi] = \mathrm{var}[\delta V] + \Delta^2\,\mathrm{var}[\delta S] - 2\Delta\,\mathrm{cov}[\delta V, \delta S].$$

We want to minimize this using Δ, and the minimum is attained when

$$\Delta = \frac{\mathrm{cov}[\delta V, \delta S]}{\mathrm{var}[\delta S]}.$$

We will compute this ratio to $O(\epsilon^2)$.
We write $\epsilon = \sqrt{\delta t}$, so that $\delta x = \epsilon\sigma\phi + \epsilon^2\hat{\mu}$.
Now, to $O(\epsilon^4)$,

$$\delta S = S\left(\delta x + \frac{1}{2}\delta x^2 + \frac{1}{6}\delta x^3 + \frac{1}{24}\delta x^4\right),$$

and

$$\begin{aligned}\delta V &= \epsilon^2 V_t + \frac{\epsilon^4}{2}V_{tt} + V_S\delta S + \frac{1}{2}V_{SS}\delta S^2 + \frac{1}{6}V_{SSS}\delta S^3 + \frac{1}{24}V_{SSSS}\delta S^4 + \epsilon^2 V_{tS}\delta S + \frac{\epsilon^2}{2}V_{tSS}\delta S^2 \\ &= \epsilon^2 V_t + \frac{\epsilon^4}{2}V_{tt} + \left(V_S + \epsilon^2 V_{tS}\right)\delta S + \frac{1}{2}\left(V_{SS} + \epsilon^2 V_{tSS}\right)\delta S^2 + \frac{1}{6}V_{SSS}\delta S^3 + \frac{1}{24}V_{SSSS}\delta S^4.\end{aligned}$$

Now, in terms of expectations,

$$\begin{aligned}\overline{\delta x} &= \hat{\mu}\epsilon^2 & \overline{\delta S} &= \epsilon^2 S\left(\mu + \frac{1}{2}\left(\mu^2 + \frac{\sigma^2}{2}\right)\epsilon^2\right) \\ \overline{\delta x^2} &= \sigma^2\epsilon^2 + \hat{\mu}^2\epsilon^4 & \overline{\delta S^2} &= \epsilon^2 S^2\left(\sigma^2 + \left(\mu^2 + 2\sigma^2\mu + \frac{\sigma^4}{2}\right)\epsilon^2\right) \\ \overline{\delta x^3} &= 3\sigma^2\hat{\mu}\epsilon^4 & \overline{\delta S^3} &= 3\epsilon^4\sigma^2 S^3\left(\mu + \sigma^2\right) \\ \overline{\delta x^4} &= 3\sigma^4\epsilon^4 & \overline{\delta S^4} &= 3\sigma^4 S^4\epsilon^4.\end{aligned}$$

It follows from this that

$$\mathrm{var}[\delta S] = \epsilon^2 S^2\sigma^2\left(1 + \left(2\mu + \frac{\sigma^2}{2}\right)\epsilon^2\right).$$

Moreover, we can see too that, still to $O(\epsilon^4)$,

$$\begin{aligned}\mathrm{cov}[\delta V, \delta S] &= \overline{\delta V\delta S} - \overline{\delta V}\,\overline{\delta S} \\ &= \left(V_S + \epsilon^2 V_{tS}\right)\mathrm{var}[\delta S] + \frac{1}{2}\left(V_{SS} + \epsilon^2 V_{tSS}\right)\left(\overline{\delta S^3} - \overline{\delta S^2}\,\overline{\delta S}\right) + \frac{1}{6}V_{SSS}\left(\overline{\delta S^4} - \overline{\delta S^3}\,\overline{\delta S}\right) \\ &= \left(V_S + \epsilon^2 V_{tS}\right)\mathrm{var}[\delta S] + \frac{S^3}{2}\sigma^2\epsilon^4 V_{SS}\left(2\mu + 3\sigma^2\right) + \frac{1}{2}V_{SSS}\sigma^4 S^4\epsilon^4,\end{aligned}$$

so that

$$\Delta = V_S + \epsilon^2 V_{tS} + \epsilon^2 \frac{\frac{S}{2} V_{SS} \left(2\mu + 3\sigma^2\right) + \frac{S^2}{2}\sigma^2 V_{SSS}}{1 + \left(2\mu + \frac{\sigma^2}{2}\right)\epsilon^2} = V_S + \epsilon^2 \left(V_{tS} + \frac{S}{2} V_{SS} \left(2\mu + 3\sigma^2\right) + \frac{S^2}{2}\sigma^2 V_{SSS} \right).$$

Now, if we assume that, to zero'th order in ϵ, V satisfies the Black-Scholes equation

$$V_t = rV - rSV_S - \frac{\sigma^2}{2} S^2 V_{SS},$$

then

$$V_{tS} = -rSV_{SS} - \sigma^2 S V_{SS} - \frac{\sigma^2}{2} S^2 V_{SSS},$$

so that, to $O(\epsilon^2)$,

$$\Delta = V_S + \epsilon^2 \left(\mu - r + \frac{\sigma^2}{2}\right) S V_{SS}.$$

This is the choice that minimizes $\operatorname{var} \delta\Pi$. Wilmott then equates the expected return on $\delta\Pi$ with the riskless rate r to obtain the modified Black-Scholes equation

$$V_t + rSV_S + \frac{\sigma_*^2}{2} S^2 V_{SS} - rV = 0,$$

with

$$\sigma_*^2 = \sigma^2 + \epsilon^2 (\mu - r)\left(3(\mu - r) + \sigma^2\right),$$

or, equivalently (to $O(\epsilon^2)$),

$$\sigma_* = \sigma \left(1 + \frac{\epsilon^2}{2\sigma^2} (\mu - r)\left(3(\mu - r) + \sigma^2\right)\right).$$

To see this, set

$$\overline{\delta\Pi} = \left(\mathrm{e}^{r\delta t} - 1\right)\Pi = \left(r\epsilon^2 + \frac{1}{2} r^2 \epsilon^4\right)\Pi.$$

Working to $O(\epsilon^4)$, we have

$$\begin{aligned}
\epsilon^2 (r + \frac{1}{2} r^2 \epsilon^2)(V - \Delta S) &= \overline{\delta V} - \Delta \overline{\delta S} \\
&= \epsilon^2 V_t + \frac{\epsilon^4}{2} V_{tt} + \left(V_S + \epsilon^2 V_{tS}\right)\overline{\delta S} \\
&\quad + \frac{1}{2}\left(V_{SS} + \epsilon^2 V_{tSS}\right)\overline{\delta S^2} + \frac{1}{6} V_{SSS}\overline{\delta S^3} + \frac{1}{24} V_{SSSS}\overline{\delta S^4} - \Delta\overline{\delta S}
\end{aligned}$$

so that

$$\begin{aligned}
(r + \frac{1}{2} r^2 \epsilon^2)(V - \Delta S) &= V_t + \frac{\epsilon^2}{2} V_{tt} + \left(V_S + \epsilon^2 V_{tS}\right)\left(\mu + \frac{1}{2}\left(\mu^2 + \frac{\sigma^2}{2}\right)\epsilon^2\right) \\
&\quad + \frac{1}{2}\left(V_{SS} + \epsilon^2 V_{tSS}\right) S^2 \left(\sigma^2 + \left(\mu^2 + 2\sigma^2\mu + \frac{\sigma^4}{2}\right)\epsilon^2\right) \\
&\quad + \frac{1}{2} V_{SSS}\epsilon^2 \sigma^2 S^3 \left(\mu + \sigma^2\right) + \frac{1}{8} V_{SSSS}\sigma^4 S^4 \epsilon^2 \\
&= \dots
\end{aligned}$$