Sars in Toronto: Was WHO Mistaken?

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Introduction: According to a report published by the American Social Health Association in 1998, approximately 15 million new cases of sexually transmitted diseases (STD's) are occurring in the United States every year [1]. The cost to the nation runs into billions of dollars. The situation in Canada is perhaps no better. For the propagation of most of STD's there is usually a core group and a non-core group. The members of the core group, though small in number, account for a large percentage of infections in the society. In this paper, we present a model for spread of STD's, and other infectious diseases like SARS, in any society. The model does not allow for a core group, so that it is applicable to the propagation of an infectious disease for which there is no known core group. SARS is such a disease. In this model, the community is divided three ways; namely, those who are susceptible to infection to the disease, those who are infected and those who have recovered.
We apply this model to the recent spread of SARS in Toronto and show, convincingly we hope, that the city of Toronto has been a safe place to visit from April 22, 2003 onwards so that the travel advisory imposed on this city by WHO on April 23, 2003 was perhaps unjustified.

It should be pointed out that in the case of sars, in most cases, the immune system of the body is able to fight the infection quite successfully. However, this cure provides only partial immunity from re-infection (about ten percent of the 'cured' people have been reported to have become infected again in Hong Kong). Because of this, the rate at which people get cured should be different from the one at which the cured people become susceptible again.

We shall assume that, without this disease, the population of our community would increase logistically.

The Model

We write

$x'(t)=A_1 x-A_2 x^2-A_3 x y+c_2 y$,..........(1a)

$y'(t)=A_3 x y-c_1 y-k_1 y$, ...(1b)

and

$z'(t)=c_1 y-c_2 y-k_2 z$...........................(1c)

where
x(t)= the number of susceptible people in the community at time t,
y(t)= the number of infected people in the community at time t,
z(t)= the number of people at time t who have recovered from the disease and are susceptible no more,
$A_1 x-A_2 x^2$ = logistic growth of x(t),
$A_3 xy$ = the rate at which the susceptible people become infected by coming into contact with the infected people,
$c_1 y$ = the rate at which the infected people recover from the disease,
$c_2 y$ = the rate at which the recovered people become susceptible again,

$k_1 y$ = the rate at which the infected people die, and
$k_2 z$ = the rate at which the recovered people die.

The equilibrium points of this dynamic are

$P_1 = (0,0,0)$, $P_2 = (A_1/A_2,0,0)$, and $P_3 = (x_0,y_0,z_0)$ where

$x_0=(c_1 + k_1) / A_3$, $y_0= -((c_1 + k_1) (-A_1 A_3 + A_2 (c_1 + k_1))) / (A_3{}^\wedge 2 (c_1 - c_2 + k_1))$ and $z_0 = -((c_1 - c_2) (c_1 + k_1) (-A_1 A_3 + A_2 (c_1 + k_1))) / (A_3{}^\wedge 2 (c_1 - c_2 + k_1) k_2)$.

For (x_0,y_0,z_0) to be in the first octant, we need $(A_1 A_3 - A_2 (c_1 + k_1))>0$ and $c_1>c_2$. So that we need $A_3>A_2(c_1+k_1)/A_1$ for (x_0,y_0,z_0) to be in the first octant. We now notice that

1. As the treatment improves, c_1 goes up, and the y coordinate of P_3 comes down. This is what we would expect.
2.The disease can be eradicated if $A_3 \leq A_{3\,cr} = A_2(c_1+k_1)/A_1$ which is the critical value of the contact parameter A_3.
3. Through public education, c_2 can be reduced which brings the y coordinate of P_3 down. Again, this is what we would expect.
4. If c_2 is a constant fraction of c_1 (90% say), this y coordinate still comes down as c_1, the cure factor, goes up.
5. If $c_1 = c_2$, which occurs if people are apathetic, then the y coordinate of P_3 is at its maximum.
6. Quite often, a sexual disease is asymptotic, so that c_1 is small (because not too many people come forward for cure). With c_2 as a constant percentage of c_1, this increases the y coordinate of P_3.
7. With public awareness, or with more thorough and routine checking of asymptotic women, (as happened in the U.S. in the eighties for gonorrhea), c_1 goes up, and y_0 of P_3 comes down as actually happened in the case of gonorrhea.

Positivity of the solution: We shall show that if $x(0)\geq 0$, $y(0)\geq 0$ and $z(0)\geq 0$, then the solution of our equations stays in the first octant of the (x,y,z) space. We notice that if y = 0 then y' = 0. This implies that if the moving point (x,y,z) hits the y = 0 plane, then it cannot leave this plane. Combined with $y(0)\geq 0$, this proves the positivity (rather the non negativity) of the y coordinate of the solution of our equations. Since $c_1\geq c_2$, we get $z'(t)\geq 0$ if z = 0, which proves the non negativity of z(t) in $t\geq 0$. Similarly for x(t).

Boundedness of the solution: We notice that $(x+y+z)' = A_1 x - A_2 x^2 - k_1 y - k_2 z$. If we write u = y+z, and k = min(k_1, k_2), it follows that $(x+u)'<A_1 x - A_2 x^2 - ku$. If we now draw the parabola $ku = A_1 x - A_2 x^2$ in the (x,u) plane, then $(x+u)'<0$ at any point which lies outside of this parabola in the (x,u) plane in u > 0. Combined with the positivity of the solution, this proves the boundedness of the solution in the (x,y,z) space.

Stability of the equilibrium points: The stability of the various equilibrium points depends upon the roots of a cubic equation which may be written as $\lambda^3+a_2\lambda^2+a_1\lambda+a_0 = 0$, where the coefficients of this cubic depend upon the point under consideration. A necessary and sufficient condition for stability of the point is $a_0>0$, $a_1>0$, $a_2>0$ and $a_1 a_2-a_0>0$ [2].

The equilibrium point $P_1(0,0,0)$ is found to be always unstable. $P_2(A_1/A_2,0,0)$ is found to be stable if and only if $P_3(x_0,y_0,z_0)$ is not in the first octant and P_3 is found to be always stable provided $x_0>0, y_0>0$ and $z_0>0$. This requires $A_3 > A_{3\,cr} = A_2(c_1+k_1)/A_1$ and $c_1>c_2$.

These considerations make our model suitable for propagation of an infectious disease. Non-negativity of solutions is important in such a model because, if the number of susceptible people, or the number of infected people, or the number of cured people, become negative, it would not make any practical sense at all. Same about boundedness. The model would be meaningless if, at any time, the number of people in any of these categories became infinitely large in the community, say more than a hundred billion people, and more, in the city of Toronto. Stability of any two equilibrium points for the same given values of the parameters is unrealistic in such a model. Also, the result says that if the disease is not endemic, then the society will reach a disease free state and if the disease is endemic then, in the long run, it will reach an equilibrium state. The critical value of the parameter A_3 below which the disease will be wiped out is $A_2(c_1+k_1)/A_1$. If you consider the development of a disease over a short period of time, five to ten years say, then the increase in population can be effectively modeled with $A_2 = 0$, so that $A_{3\,cr} = 0$. This says that an infectious disease, without medical intervention, takes a very long time to be eradicated. This is, in fact, true of most of STD's and we painfully know it to be true in the case of HIV/AIDS. This result also says that if the cure provides very little immunity, then the disease is harder to control. Also, the less the death rate from the disease, the harder it is to control. Common cold is still with us!

This model is also suitable for infectious diseases like SARS, where (1) the incubation period is long, (2) the period during which the patient is not infectious is relatively short and (3) the cure provides partial immunity to the disease, and (4) there is no known core group for the propagation of the disease. We shall now apply this model to the spread of sars in Toronto in the year 2003.

SARS: This disease appeared in Toronto on March 7, 2003 when a patient arrived at the emergency department of the Scarborough Grace Hospital. The disease raged in Toronto, affecting practically every aspect of life in the city. It infected more than two hundred and fifty people killing twenty three of them. Thousands of people were isolated. On April 22, the city of Toronto was declared to be an unsafe area to visit by WHO. The travel advisory itself cost the city hundreds of millions of dollars and adversely affected the economy of the whole country.

Quarantine: In the case of a successful quarantine of a person, his contribution to the A_3xy term is clearly zero. It follows from equation (1b) that if all the infectious people are successfully quarantined, then the current number of infectious people (those quarantined) should come down exponentially. If we consider the expected amount of time during which a person is successfully cured (or he/she dies) to be 20 days (Vietnam was declared to be free of sars by WHO after no new cases appeared in that country for twent days [3]), and take twenty days to be the unit of time, then we get $c_1+k_1 = 1$ and $A_3 = 0$ in equation (1b). We take one day as the unit of time and, in the following figure, compare the curve exp(-t/20) with the actual number of hospitalised sars probables in Toronto from April 22 to May 6, 2003. Sars suspects, those people who showed no physical signs of having contacted the disease but were only suspected of having done so because of their past associations as also those sars probables who were still under investigation, have been left out of our count. The close agreement in this figure is a strong indication that as of April 22, 2003, the infection coefficient A_3 was zero in Toronto and that there was no danger to anybody either living in or travelling to the city. WHO gave its travel advisory for Toronto on April 23, which, in retrospect, was quite unnecessary. But there was no way to know this on April 23. Toronto was declared to be a 'safe' place to visit on April 30.

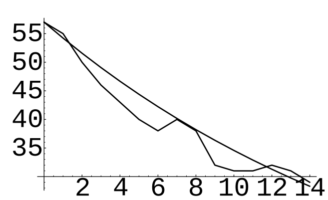

Fig 1: The numbers exp[-i/20] from i = 0 to i = 14 compared with the actual number of sars probables in Toronto between April 22 and May 6. The close agreement between the two sets of numbers strongly indicates that the coefficient A_3 was zero in Toronto from April 22 onwards and that it was a safe place to live in.

References:

1. American Social Health Association, "Sexually Transmitted Diseaes, How many and at what cost", Kaiser Family Foundation, Menlo Park, CA, U.S.A.(1998).
2. Hohn, F.E., "Elementary Matrix Algebra", Macmillan (1964).
3. WHO, April 28, 2003.