On some (more) mathematical models
for development of HIV/AIDS in a community

by

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Abstract: We present three models for the spread of HIV positivity in a community. Two of these are ODE models while the third one is a difference equations model. In all the three models, the predictions of the models are compared with the actual numbers of HIV positive people in both Canada and the United States up to four years in advance. In all the three cases, the two sets of numbers are very close.

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Summary: In this report, we develop three models for spread of HIV/AIDS in a community. In the first two models, the society is divided into three different categories: those people who are less susceptible than others to developing HIV positivity, those who are more susceptible, and those who are HIV positive. In the first part, some of the people who are more susceptible to becoming HIV positive today may become less susceptible tomorrow, (like some people who get cured of their STD (sexually transmitted disease)), while without this disease, the population would grow at an exponential rate. In part two, the growth of the general population is logistic and some people are born into the less susceptible class while others are born into the more susceptible class (like the people with homosexual tendencies), but people do not revert from the more susceptible class to the less susceptible one. In part three, we present a difference equations model, which divides the society into people who are HIV negative, HIV positive, and people with AIDS. In this part we exhibit an interesting phenomenon of the model in which a seemingly chaotic phenomenon in the beginning dies down with time. We also exhibit the existence of what appear to be limit cycle solutions in two of the models.

In all the three parts, we compare the actual number of HIV positive people in both Canada and the United States with the predictions of our model up to four years in advance by estimating the parameters in the model with the help of the actual numbers of HIV positive people in the first two years and then projecting the numbers from the model. In all cases considered, the agreement is very good. We observe that in all the cases, the situation in Canada is not improving with time, while that in the United States is not deteriorating with time. Perhaps it indicates that the HIV prevention programme in United States is working better than the one in Canada.

Here are the models.

PART ONE

An HIV/AIDS model with a
high risk and a low risk population

Introduction: There have been reports in the literature that the so-called bathhouses in the United States may well be the cause of that country leading by far in the number of AIDS cases amongst all industrialized countries. New case rates per hundred thousand in United States are reported to be about ten times those in other industrial countries [1,2]. In recent articles [3,4,5], we have argued that, in spite of all our medical advances, as far as the progress of HIV/AIDS is concerned, the prognosis for North America is not good. While the medical advances are buying us time, our mathematical models predict that based on the present number of HIV infections, we are headed for disaster. Is this because of the bathhouses?

In this article, we develop a mathematical model which takes the effect of bathhouses in the United States into account and arrive at the same conclusion as Thompson [1,2] that frequent sexual (and/or other high risk) activity amongst a small number of people, who are at a relatively higher risk of contacting HIV, may well be the controlling factor in the spread of HIV/AIDS in a society. In this article, we divide the general population into three categories, namely those who are HIV negative and are less susceptible to contacting the disease (i.e. HIV positivity) than others (perhaps because they indulge in safe sex and so on), those who are more likely to contact the disease (perhaps because they frequent the bathhouses, or are injection drug users, or have some sexually transmitted disease (STD)), and those who are infected. This article is a generalization of a previous article [4] where we discussed a similar problem but in that article, a person became a high risk individual only through his contacts with other high risk individuals. This is certainly not true. There is overwhelming evidence in the literature that tendencies like homosexuality are genetically inherited. In this article, we take this into account so that a person may be in a high risk group either because he comes in contact with a high risk person or he is born that way. We also take into account the fact that some people may revert from being in high risk category into a low risk category which happens, for example, when a person gets cured of an STD. These considerations make our model more complicated than before. Our conclusion in this article is again that the controlling factor in the spread of this disease is the contact between people who are at higher risk of contacting the disease with those who are already infected, a factor which describes the use of facilities like bathhouses in the United States. In this article, we shall refer to any sexual (and/or needle sharing or any other activity of this kind) involving an HIV positive individual and an HIV negative individual in a high risk category, as a 'bathhouse activity'.

We apply the results of this article to the HIV positive situation in both Canada and the United States and estimate the number of HIV positive people in each country up to four years in advance. In both cases, our predictions are highly accurate lending credibility to our assumptions here. In Canada, till the year 1999, the total number of HIV positive infections reported was 49800, out of which only 8400 were accounted for by people who are not MSM (men having sex with men) and/or IDU (injection drug users). Of the 8400, 8000 were accounted for by those who contacted HIV through heterosexual contact and the remaining 400 were classified as others [6]. If we classify people who practice heterosexual sex as being in the low risk category and all others in the high risk category, and assume that 10% of the population falls into high risk category, then these 10% of the people account for more than 83% of the HIV infections reported in Canada till 1999. The percentages in the United States are comparable. This leads us to think that we have a rather strong case for dividing the general population into low risk and high risk categories.

1. The Model:

We write

x(t) = number of HIV negative people at low risk of contacting HIV,
y(t) = number of HIV negative people at high risk of contacting HIV (like the people with some sexually transmitted disease (STD), or men having sex with men (MSM), and so on),
z(t) = number of HIV positive people,
cy = Rate at which the number of HIV negative high risk people revert back to being HIV negative low risk people (like the people with STD's who get cured of their particular STD),
$A_3\, xy$ = Rate at which low risk people become high risk people by having contact with other high risk people (for example, a low risk person contacting STD by having sex with a person with STD or an otherwise heterosexual person engaging in MSM due to his contacts with other MSM people),
$A_4\, xz$ = Rate at which HIV negative people at low risk become HIV positive by having sex with (or sharing needles with) people who are HIV positive,
$A_5\, yz$ = Rate at which HIV negative people at high risk (like MSM, injecting drug users (IDU's), people with STD, etc.) become HIV positive by having sex with (or sharing needles with) people who are HIV positive. The parameter A_5 is the parameter which controls what we are calling 'bathhouse activity',
kz = Rate at which HIV positive people die due to AIDS (or for any other reason).

Our model now becomes

$x'(t) = A_1 x - A_2 x^2 - A_3 xy - A_4 xz + cy$..... ...(1.1a)
$y'(t) = B_1 y - B_2 y^2 + A_3 xy - A_5 yz - cy$...........(1.1b)
$z'(t) = A_5 yz + A_4 xz - kz$...................... ......(1.1c)

the terms $A_1 x - A_2 x^2$ describe the logistic growth of the HIV negative low risk population in this model, while the terms $B_1 y - B_2 y^2$ describe the similar growth in the HIV negative high risk population. The quantity A_1 / A_2 is called the limiting value of x. Similarly B_1 / B_2 is the limiting value of y. We shall assume that at t = 0, for every nine people in the low risk category in the general population, there is one person in the high risk category and that for every nine people born in the x group, one person is born in the y group. Thus we shall take $B_1 = A_1$ and $B_2 = 9 A_2$. Unless otherwise stated, all the constants in this model, namely A_i, i = 1, 2, 3, 4, 5, and B_1, B_2, c and k will be considered positive.

Positivity of the solution: We shall prove that if $x(0) \geq 0$, $y(0) \geq 0$ and $z(0) \geq 0$, then x(t), y(t) and z(t) are all nonnegative for all $t \geq 0$. Notice that if z(t) = 0 at any moment t, then z'(t) is also equal to zero at that moment. This shows that if the moving particle (x, y, z) hits the plane z = 0 at any moment, then it cannot move away from this plane. Since $z(0) \geq 0$, this gives $z(t) \geq 0$ in $t \geq 0$. Similarly we get $y(t) \geq 0$. Now in the plane x = 0, we have $x'(t) \geq 0$ which gives $x(t) \geq 0$ in $t \geq 0$. This proves the positivity (rather the non-negativity) of the solution in $t \geq 0$.

Boundedness of the solution: We shall show that if $x(0) \geq 0$, $y(0) \geq 0$, and $z(0) \geq 0$, then x(t), y(t) and z(t) are all bounded in $t \geq 0$. As explained above, we shall assume that $A_1 = B_1 > 0$ and $B_2 > A_2 > 0$. We notice that
$x'+y'+z' = (A_1 x + B_1 y) - (A_2 x^2 + B_2 y^2) - kz$
$< A_1 (x + y) - (A_2 / 2)(x^2 + y^2 + 2 xy) - kz.$
If we now write x+y = u, we get $(u+z)' < A_1 u - (A_2 / 2)u^2 - kz$. If we now consider a point (u,z) in the first quadrant in the (u,z) plane, which point lies outside of the parabola $kz = A_1 u - (A_2 / 2)u^2$ and lies in z > 0, we notice that (u+z)' < 0 at this point. Combined with the non-negativity of the solution proved above, this proves the boundedness of the solution in the (x, y, z) space.

2. The case of $A_2 = 0$:

Usually, A_2 is quite small. If the eventual population of a country is expected to be 400 million for example, then, taking one million as the unit of population, we get $A_2 = A_1 / 400$. If we assume the population of such a country to be 300 million today and increasing at the rate of 1% per year, this gives $A_2 = 10^{-4}$. As a first approximation, we consider $A_2 = 0$. This also implies that $B_2 = 0$. We should also point out that if we consider the development of HIV positive people in a community over a short period of time, over five to ten years say, then the limiting value of the population of the community, the population that the community will approach after a hundred or a thousand or even more years, is rather irrelevant and therefore, our approximation of $A_2 = 0$ is quite appropriate. We shall also assume that $B_1 > c$, i.e. more people are born into the high risk group than revert back from high risk group to low risk group. This assumption appears to be quite reasonable to us.

With $A_2 = 0$, and $B_2 = 0$, the equilibrium points of our model are

$P_1 (0, 0, 0)$, $P_2 (k/A_4, 0, A_1 / A_4)$, and $P_3 (x_0, y_0, z_0)$ where $x_0 = (c k) / \beta$, $y_0 = k \beta_2 / (\beta A_5)$, and $z_0 = (1 / A_5) (B_1 - c + A_3 ck / \beta)$, and where $\beta = A_4 B_1 + A_3 k - A_1 A_5$, and $\beta_2 = \beta - A_4 c$.

For the point P_3 to be in the first octant, we need $\beta_2 > 0$. Notice that at $\beta_2 = 0$, the point P_3 coincides with the point P_2 while for $\beta_2 < 0$, P_3 is not in the first octant, so that if P_1 is unstable, HIV positivity is endemic according to this model. It is easy to see that P_1 is always unstable so that *HIV positivity is endemic according to this model*. It can also be shown that if P_3 is in the first octant, then P_2 is unstable and that if P_3 is not in the first octant, then P_2 is neutrally stable with one eigenvalue negative and the other two complex imaginary. Also we note that there are two kinds of inputs into the high risk group. The first group consists of those people who are born in the high risk category while the second group consists of those who become high risk people through their contacts with other high risk people. People with STD's who get cured of their particular STD and become low risk people again fall into the second group so that we assume that $A_3\, xy > cy$. It follows that $A_3\, x_0 \geq c$ which gives $A_1\, A_5 \geq A_4\, B_1$. Since $B_1 = A_1$ and $A_5 \geq A_4$, this inequality is always satisfied in our model. It is also to be noted that at the equilibrium point (x_0,y_0,z_0), equation 1(a) implies that $A_3\, y_0 + A_4\, z_0 > A_1$.

With these constraints, it can be shown that P_3 is unstable for large values of z_0, the equilibrium value of the HIV positive people. This point is found to be stable for $z_0 < (A_1\, A_3) / (A_4\, (A_3 + A_5 - A_4))$, and unstable otherwise. Because y's are the high risk group, we have assumed that $A_3 + A_5 > A_4$. This result is analogous to a result we established in another model for HIV /AIDS propagation where the endemic state of the disease was also unstable for large values of z_0 [7]. It should be kept in mind however, that for $A_2 = B_2 = 0$, the solution of our equations is not bounded and that the solutions will approach P_3 only from within the basin of attraction of this point. This basin of attraction depends upon the values of the various parameters and we do not delineate it any further. We shall only point out that the two additional equilibrium points $(A_1 / A_2,\ 0,\ 0)$, and $(c / A_3,\ (B_1 - c) / B_2,\ 0)$ which recede to infinity for small values of A_2 and B_2 are both unstable and therefore, the solutions of our equations do not approach these points. Our solutions (with $A_2 = B_2 = 0$), therefore, should be only slightly affected for vanishingly small values of A_2 and B_2.

Because the points at infinity are unstable, we expect the solution to run into a limit cycle for high values of z_0, the equilibrium value of the HIV positive people. We exhibit such a limt cycle below in a case where P_3 is in the first octant but ubstable. Notice how the solution comes repeatedly towards the origin. This is similar to the situation in another model for HIV/AIDS propagation that we developed recently [7].

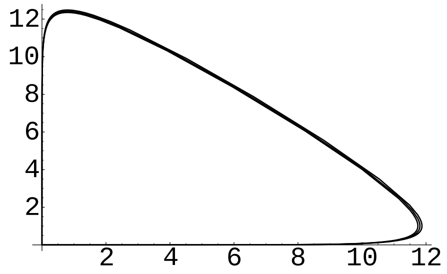

Fig 1: Projection onto y - z plane of a cyclic solution of our equations for $A_1 = .05, A_2 = 0, A_3 = .03, A_4 = .04, A_5 = .05, B_1 = .05, B_2 = 0, c = .001$ and $k = .04$. The equilibrium point P_3 is in the first octant but unstable.

The importance of the parameter A_5: In our model, the constant A_5 models the bathhouse activity between the high risk HIV negative individuals and the HIV positive ones. If either the sexual activity (or any other activity between a high risk individual and an infected one) takes place more often than in other groups (as happens in a bathhouse) or just because such activity involves a high risk individual, this constant will be large. Sufficiently large values of A_5 have the expected effect of annihilating the high risk population. As A_5 goes to its limiting value of $(A_4 B_1 + A_3 k - A_4 c) / A_1$, the equilibrium value of x approaches its limiting value which is k / A_4 in this case, the equilibrium values of y, the high risk population, approaches zero while that of z approaches A_1 / A_4. For appropriately chosen values of other constants (applicable to the situation in Canada), we give below the number of HIV positive people 25 years from now as A_5 increases from .001 to .05.

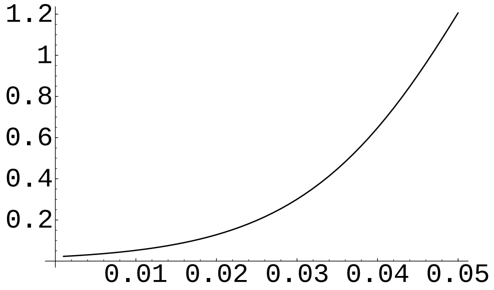

Fig 2: Values of HIV positive individuals 25 years from today for increasing values of A_5 for a community with the current population of 30 million, 10 per cent of whom are in high risk category and 50000 of whom are HIV positive today. In this diagram we have taken $A_1 = .011$, $B_1 = A_1$, $A_3 = .0002$, $A_4 = .0003$, c = .001, and k = .044. We estimated the current value of A_5 in Canada at .033.

3. Application to HIV/AIDS:

As our first application, we apply our results to the data from Canada. In most countries, while there is data on how many people contact HIV through MSM (men having sex with men) or IDU (injecting drug use), there is no data on the *number* of people who are involved in such activities. Therefore, the number of high risk people in our model, the number y, must be roughly estimated.

Till 1999, the total number of HIV positive infections reported in Canada was 49800, out of which only 8400 were accounted for by people who are not MSM and/or IDU [6]. Of the 8400, 8000 were accounted for by those who contacted HIV through heterosexual contact and the remaining 400 were classified as others [6]. In this paper, only the people who practice heterosexual sex are considered to be in the low risk category. All others are considered to be in the high risk category. We realise that this is perhaps not a very precise classification, but this is what we adhered to.

We estimated the constants in our model from the HIV/AIDS data published by Health Canada and the population figures published by Statistics Canada for the years 1995 and 1996 [6, 8]. In the United States, similar data is published by CDC in Atlanta [9]. It was assumed that one out of every ten people in both Canada and the United States in 1996 was in high risk category. It was found that in Canada, a high risk HIV negative person is about 100 times as likely to contact HIV from an HIV positive person as a low risk person. In U.S., this ratio was about 50. Perhaps, it does not mean that high risk people are more in danger (of contacting HIV) in Canada than in the United States, but rather that safe (heterosexual) sex in the United States is not so safe any more and the disease has spread rather widely in the heterosexual population in that country. Based upon these estimates, we calculated the number of HIV positive people in both Canada and the United States for the years 1997, 1998, 1999 and 2000. These estimates, along with the actual figures as published by Health Canada and by CDC, are given in Fig. 3 . Similar results for the United States are given in Fig. 4.

In deriving these estimates for Canada, it was assumed that 1% of the HIV positive people die before developing AIDS, so that the number alive at any moment t is 99% of the total number of HIV positive people diagnosed till time t, minus the total number of AIDS deaths reported till that time. This was not done for the United States, because there the published figures give the number of people living with HIV in any given year. However, HIV positivity is not a reportable condition in many states in the United States. We estimated the number of HIV positive people in U.S. by assuming that, per hundred thousand, the number of HIV positive people is the same in all the states. It is to be noted that these estimates are better than the long term estimates in our predator-prey model but not as good as the ones in our second model [7].

It is to be noted that the maximum error in our estimates is about 3%. This compares favourably with most other researchers in the field. In both these tables, the estimates are derived by estimating the parameters in the model from the HIV/AIDS data and the population figures in 1995 and 1996. It is to be noted that the progression of HIV positive people in both the countries is almost linear (see below).

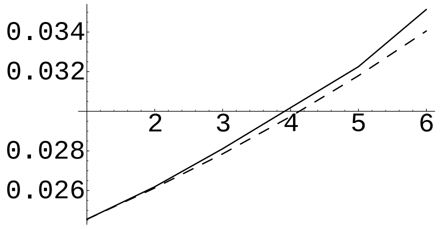

Fig 3; Comparison of HIV positive people in Canada between those predicted by our model and the actual numbers. The constants were calculated from the data from 1995 and 1996 using backward differences. 1995 is the year one. The solid line represents the actual numbers.

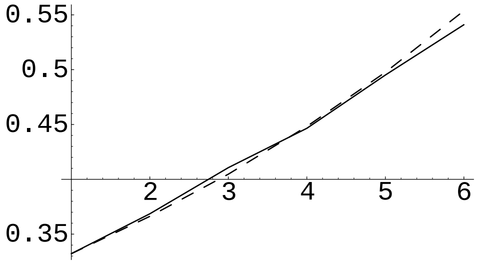

Fig 4; Comparison of HIV positive people in the United States between those predicted by our model and the actual numbers. The constants were calculated from the data from 1995 and 1996 using backward diffrences. 1995 is the year one. The solid line represents the actual numbers.

Note: For appropriately chosen values of the parameters in our model, we give here (in Fig. 5), the development of HIV positive people in a community over a period of fifty years starting with one HIV positive person in the year zero. It is assumed that 10% of the total population of this community falls into high risk category in the year zero. It is also assumed that for every nine persons born in the low risk categoty, one person is born in the high risk category. Notice how the number of HIV positive people goes up exponentially in the beginning but only linearly later on.

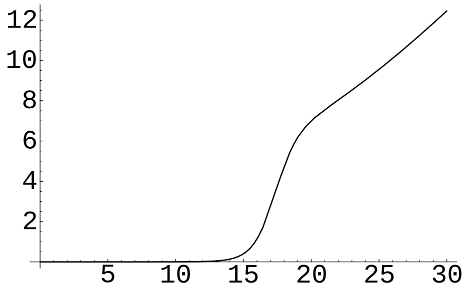

Fig 5: Progression of HIV positive people in a community with $A_1 = .02$, $B_1 = .02$, $A_2 = 0$, $B_2 = 0$, $A_3 = .0007$, $A_4 = .003$, $A_5 = .2$, k = .05 and c = .001. At t = 0, the community has 30 million people, 10 per cent of whom are in the high risk category and one person in the community is HIV positive. Notice how the disease takes about 15 years to take off and the progression of HIV positive people is exponential in the beginning but becomes linear later on. This is what is happening in North America today.

PART TWO

Another HIV/AIDS model with a high risk and a low risk population (the case of c = 0)

1. The case of c = 0: Usually, the value of c in our model is quite small. While it is true that STD's (sexually transmitted diseases) put a patient in a high risk category and some of these STD's, like gonorrhoea, are quite easily treated, it is also true that such a cure provides little or no immunity from re-infection, so that once a person has been treated, he may soon fall back into the high risk group. We may, therefore, take c = 0 as a first approximation. But with c = 0, we may look into the effect of $A_2 > 0$ in our model which result, surprisingly, turns out to be quite profound. We shall now take c = 0 and $A_2 > 0$. With this approximation, our model becomes

The Model:

$x'(t) = A_1 x - A_2 x^2 - A_3 xy - A_4 xz$.................(2.1a)

$y'(t) = B_1 y - B_2 y^2 + A_3 xy - A_5 yz$..................(2.1b)

$z'(t) = A_5 yz + A_4 xz - kz$...............................(2.1c)

In this model, we shall assume that at t = 0, y(0) = x(0) / 9 and that for every nine people born in the x group, one person is born in the y group. Thus we shall take $B_1 = A_1$ and assume that the limiting value of y is equal to one ninth of the limiting value of x, so that $B_2 = 9 A_2$. Unless otherwise stated, all the constants in this model, namely A_i, i = 1,2,3,4,5, and B_1, B_2, and k will be considered positive.

Positivity and Boundedness of the solution: As shown in part one, with x(0) ≥ 0, y(0) ≥ 0 and z(0) ≥ 0, the solutions of our model stay in the first octant of the (x,y,z) space and are bounded in t ≥ 0.

The Equilibrium Points: There are seven equilibrium points this time, namely $P_1(0, 0, 0)$, $P_2(A_1 / A_2, 0, 0)$, $P_3(0, B_1 / B_2, 0)$, $P_4(0, y_4, z_4)$, $P_5(x_5, 0, z_5)$, $P_6(x_6, y_6, 0)$, and $P_7(x_7, y_7, z_7)$, where,

$y_4 = k/A_5$, and $z_4 = (A_5\,B_1 - B_2\,k)\,/\,A_5{}^\wedge 2$;
$x_5 = k/A_4$, and $z_5 = (A_1\,A_4 - A_2\,k)\,/\,A_4{}^\wedge 2$;
$x_6 = (A_1\,B_2 - A_3\,B_1)\,/\,(A_3{}^\wedge 2 + A_2\,B_2)$, and $y_6 = (A_1\,A_3 + A_2\,B_1)\,/\,(A_3{}^\wedge 2 + A_2\,B_2)$; and

$x_7 = -(-A1\,A5{}^\wedge 2 + A4\,A5\,B1 + A3\,A5\,k - A4\,B2\,k)\,/\,(A2\,A5{}^\wedge 2 + A4{}^\wedge 2\,B2)$,
$y_7 = -(A1\,A4\,A5 - A4{}^\wedge 2\,B1 - A3\,A4\,k - A2\,A5\,k)\,/\,(A2\,A5{}^\wedge 2 + A4{}^\wedge 2\,B2)$, and
$z_7 = -(-A1\,A3\,A5 + A3\,A4\,B1 - A2\,A5\,B1 - A1\,A4\,B2 + A3{}^\wedge 2\,k + A2\,B2\,k)\,/\,(A2\,A5{}^\wedge 2 + A4{}^\wedge 2\,B2)$.

Stability of the Equilibrium points: The points P_1 and P_2 are found to be always unstable and will not be considered anymore. Because of the non-negativity of the solution proved above, we shall call a point 'irrelevant' if its x, y or z coordinate is negative and 'relevant' otherwise. Notice however that if, for example, $z_4 = 0$, then P_4 must coincide with either P_1 or P_3. Similarly for the points P_5 and P_6. We shall not consider such degenerate cases. We shall now show that if any one of the points P_i, i = 3, 4, 5, 6, 7 is relevant and stable, then all other points are irrelevant and/or unstable. Also notice that a solution of our equations can go to an equilibrium point if and only if the point is both relevant and stable. This implies an unambiguous behaviour of the solution for large values of t for *any* given values of the parameters. In the following, we shall write

$x_{00} = (A_1\,A_5^2 - A_4\,A_5\,B_1 - A_3\,A_5\,k + A_4\,B_2\,k)$,
$y_{00} = (-A_1\,A_4\,A_5 + A_4^2\,B_1 + A_3\,A_4\,k + A_2\,A_5\,k)$, and
$z_{00} = (A_1\,A_3\,A_5 - A_3\,A_4\,B_1 + A_2\,A_5\,B_1 + A_1\,A_4\,B_2\; - A_3^2\,k - A_2 B_2 k)$.

It is to be noted that at each one of these points, the stability of the point is decided by the roots of an equation $\lambda^3 + a_2\,\lambda^2 + a_1\,\lambda + a_0 = 0$. A necessary and sufficient condition for the stability of the point is that $a_0 > 0, a_1 > 0$, $a_2 > 0$, and $a_1\,a_2 > a_0$ [10].

Stability of the equilibrium point P_3: At this point, we find that
$a_0 = (x_{66}\,z_{44})B_1/(B_2{}^\wedge 2)$,
$a_1 = -((B_1\,B_2)(x_{66} + z_{44}) - x_{66}\,z_{44})/B_2{}^\wedge 2$,
$a_2 = -(x_{66} + z_{44} - B_1\,B_2)/B_2$, so that
$a_1\,a_2 - a_0 = (-1\,/\,B_2{}^\wedge 3)\,((B_1\,B_2 - x_{66})\,(B_1\,B_2 - z_{44})\,(x_{66} + z_{44}))$,
where $x_{66} = A_1\,B_2 - A_3\,B_1$ and $z_{44} = A_5\,B_1 - B_2\,k$.

Notice that if x_{66} and z_{44} are both negative, then the conditions for stability of P_3 are satisfied. If only one of x_{66} and z_{44} is positive (and the other is negative), then a_0 is negative and the point is unstable. If they are both positive but each one is less than (or each one is greater than) $B_1\,B_2$, then $a_1\,a_2 - a_0 < 0$ and the point is unstable, while if they are both positive but only one is less than $B_1\,B_2$, then $a_2 < 0$ and the point is unstable. It follows that P_3 is stable if and only if both x_{66} and z_{44} are negative, i.e. if and only if both P_4 and P_6 are irrelevant. But then P_7 is also irrelevant (because $z_7 < 0$). However $y_7 > 0$ in this case and therefore P_5 is also unstable (see below). It follows that if P_3 is relevant and stable, then all other points are either irrelevant or unstable.

Stability of the equilibrium points P_4, P_5 and P_6: At the point P_4, the coefficients of the stability equation are found to be
$a_0 = -(k\,/\,A_5)\,z_4\,x_{00}$,
$a_1 = -(k\,/\,A_5{}^\wedge 3)\,(B_2\,x_{00} - A_5{}^\wedge 4\,z_4)$, and
$a_2 = -(1\,/\,A_5{}^\wedge 2)\,(x_{00} - A_5\,B_2\,k)$, so that
$a_1\,a_2 - a_0 = (B_2\,k\,(-A_5\,B_2\,k\,x_{00} + x_{00}{}^\wedge 2 + A_5{}^\wedge 5\,k\,z_4))\,/\,A_5{}^\wedge 5$.

It is to be noted that if $x_{00} < 0$ and $z_4 > 0$, then the conditions of stability of P_4 are satisfied. However, if $x_{00} > 0$ and $z_4 > 0$, then $a_0 < 0$ and the point is unstable. If $z_4 < 0$, then the point under consideration is irrelevant. It follows that P_4 is stable and relevant if and only if $x_7 < 0$. Similarly, P_5 is found to be stable and relevant if and only if $y_7 < 0$ and P_6 is found to be relevant and stable if and only if $z_7 < 0$.

This last statement (P_6 is found to be relevant and stable if and only if $z_7 < 0$) is important because it says that the disease (i.e. HIV positivity) can be eradicated if $(A_1 / A_3) < (B_1 / B_2) < (k / A_5)$. This gives some idea about the relation between the eradicability of the disease and the rate at which the high risk population is increasing. This result also implies the desirability of the smallness of A_5, the 'bathhouse activity' indicator in our model.

Stability of the equilibrium point P_7: The stability of P_7 again depends upon the roots λ of the equation $\lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0 = 0$ where after considerable simplification, we find that
$a_0 = (A_2 A_5^2 + A_4^2 B_2) x_7 y_7 z_7$,
$a_1 = A_3^2 x_7 y_7 + A_2 B_2 x_7 y_7 + A_4^2 x_7 z_7 + A_5^2 y_7 z_7$,
$a_2 = A_2 x_7 + B_2 y_7$, and
$a_1 a_2 - a_0 = A_2^2 B_2 x_7^2 y_7 + B_2 y_7^2 (A_3^2 x_7 + A_5^2 z_7) +$
$A_2 x_7 (A_3^2 x_7 y_7 + B_2^2 y_7^2 + A_4^2 x_7 z_7)$.

It follows that P_7 is stable for all positive values of the parameters if and only if its x, y and z coordinates are all positive.

Other stability considerations: It is easy to see that if $y_7 < 0$, then $x_7 > 0$ which implies that if P_5 is relevant and stable, then P_4 is either irrelevant or unstable.
Also, we can see that if both $y_7 < 0$ and P_5 is relevant, then $z_7 > 0$, which implies that if P_5 is relevant and stable, then P_6 is also either irrelevant or unstable.
Also we can see that if both P_4 and P_6 are relevant, then $z_7 > 0$ which implies that P_6 is unstable again.
All this put together says that for given values of the parameters, our solution can only go to one point, so that in each case, barring any periodic solution or a strange attractor, the whole of the first octant acts as the basin of attraction of the solution provided that x(0)>0, y(0)>0 and z(0)>0.
This conclusion is consistent with the similar result we derived in an earlier paper for the case of $B_1 = B_2 = 0$ [4].

The importance of the parameter A_5: In our model, the constant A_5 models the bathhouse activity between the high risk HIV negative individuals and the HIV positive ones. If either the sexual activity (or any other activity between a high risk individual and an infected one) takes place more often than in other groups (as happens in a bathhouse) or just because such activity involves a high risk individual, this constant will be large. It is to be noted that large values of A_5 and relatively small values of A_3 and A_4, tend to 'isolate' the x population from the y and z populations, as is to be expected. As A_5 goes to infinity, the equilibrium value of x approaches its limiting value which is A_1 / A_2, while the equilibrium values of y and z approach zero. As A_5 increases, the value of z, the number of HIV positive people, goes up in the beginning (as is happening in North America today) but then goes down as y, the number of high risk people, goes down. In some societies, because of widespread use of injection drug use amongst the young people, the youth of the society should be considered as being in the high risk group. In such a situation , the youth will die relatively quickly while the young and the old will survive them. This is precisely what is happening in many societies today [11]. Similarly as A_4 goes to infinity, the y population reaches its limiting value which is B_1 / B_2, while the x and z populations approach zero. This is because if A_4 is much larger than A_5, then in our model, x is the high risk population instead of y. Perhaps this is what is happening in some countries in Africa and in Eastern Europe today [11] where because of multiple and across the board sexual partnership and the spread of sexually transmitted diseases, the disease has spread into the homosexual as well as heterosexual population and the population at large, the x's in our model, must be classified as high risk individuals. We would argue that if the community of people at high risk are somehow isolated, then the spread of HIV positivity in a community gets arrested after a while. In the following figure (Fig. 6), we show the number of HIV positive people in our model going up with A_5. The figure shows this number after 25 years starting with one HIV positive person in the year zero, in a situation applicable to Canada. We estimated the current value of A_5 in Canada at .033.

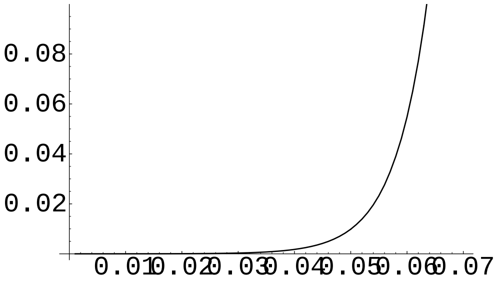

Fig 6: Number of HIV positive people in our model after 25 years, as a function of A_5, starting with one HIV positive person in the year zero, for A_1=.02, A_2=A_1/50, A_3=.003, A_4=.002, B_1=.02, B_2= 9A_2, k = .02, x(0) = 23.4, and y(0) = 2.6 and z(0) = .000001 The figure gives the number in millions of people.

2. Application to HIV/AIDS: As our first application, we apply our results to the data from Canada. In most countries, while there is data on how many people contact HIV through MSM (men having sex with men) or IDU (injecting drug users), there is no data on the *number* of people who are involved in such activities. Therefore, the number of high risk people in our model, the number y, must be roughly estimated.

Till 1999, the total number of HIV positive infections reported in Canada was 49800, out of which only 8400 were accounted for by people who are not MSM and/or IDU [6]. If we assume that about 10% of the population of Canada falls into this latter category, (MSM and/or IDU), then these 10% of the people accounted for 83% of the HIV infections reported. In this paper, only the people who practice heterosexual sex were considered to be in the low risk category. All others were considered to be in the high risk category. We realise that this is perhaps not a very precise classification, but this is what we adhered to.

We estimated the constants in our model from the HIV/AIDS data published by Health Canada and the population figures published by Statistics Canada for the year 1996 [6,8]. In the United States, similar data is published by CDC in Atlanta [9]. It was assumed that 10% of the people in both Canada and the United States in 1996 were in high risk category. It was found that in Canada, a high risk HIV negative person is about 100 times as likely to contact HIV from an HIV positive person as a low risk person. In U.S., this ratio was about 50. Perhaps, it does not mean that high risk people are more in danger (of contacting HIV) in Canada than in the United States, but rather that safe (heterosexual) sex in the United States is not so safe any more and the disease has spread rather widely in the heterosexual population in that country. Based upon these estimates, we calculated the number of HIV positive people in both Canada and the United States for the years 1997, 1998, 1999 and 2000. These estimates, along with the actual numbers as published by Health Canada and by CDC, are given in Figs.7 and 8.
In deriving these estimates for Canada, it was assumed that 1% of the HIV positive people die before developing AIDS, so that the number alive at any moment t is 99% of the total number of HIV positive people diagnosed till time t, minus the total number of AID deaths reported till that time. This was not done for the United States, because there, the published figures give the number of people living with HIV in any given year. However, HIV positivity is not a reportable condition in many states in the United States. We estimated the number of HIV positive people in U.S. by assuming that, per hundred thousand, the number of HIV positive people is the same in all the states. Note the generally excellent agreement of the predictions of our model with the actual numbers. Generally speaking, the actual number of HIV positive people is more than the estimated

number in Canada while it is not so in the United States. Perhaps it says that the HIV prevention programme is working better in U.S. than in Canada. It is also to be noted that the increase in the number of HIV positive people in both countries is almost linear (see below).

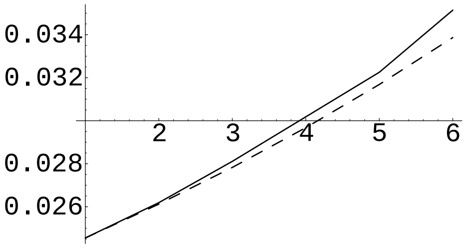

Fig 7; Comparison of HIV positive people in Canada between those predicted by our model and the actual numbers. 1995 is the year one. The solid line represents the actual numbers.

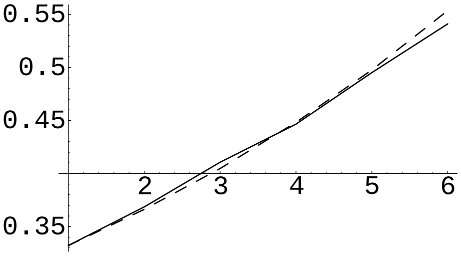

Fig 8; Comparison of HIV positive people in the United States between those predicted by our model and the actual numbers. 1995 is the year one. The solid line represents the actual numbers.

It is to be noted that the maximum error in our estimates is about 3%. This compares favourably with most other researchers in the field [12]. In both these tables, the estimate is derived by estimating the constants from the HIV/AIDS data and the population figures till 1996.

Note: For appropriately chosen values of the parameters in our model, we give here (in Fig. 9), the development of HIV positive people in a community over a period of fifty years starting with one HIV positive person in the year zero. It is assumed that 10% of the total population of this community falls into high risk category in the year zero. It is also assumed that for every nine persons born in the low risk category, one person is born in the high risk category. Notice how the number of HIV positive people goes up exponentially in the beginning but only linearly later on.

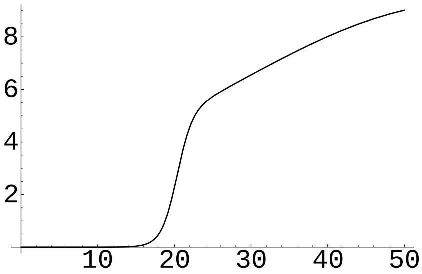

Figure 9: Development of HIV/AIDS over 50 years according to this model with k = .05, A1 = .02, A2 = A1/50, A3 = .001, A4 = .003, A5 = .2, B1 = A1, B2 = 9 A2, x(0) = 23.4, y(0) = 2.6, z(0) = .000001. Notice how the disease took almost

twenty years to take off. Notice also how the increase in the number of HIV positive people becomes linear rather than exponential after a while. This is what is happening in North America today.

PART THREE

On a Difference Equations Predator-Prey Model for HIV/AIDS Development

Introduction: The usual predator-prey model [13] has been successfully applied recently to the spread of HIV/AIDS in both Canada and the United States [3]. In the predator-prey ODE model, the rate of increase of the total population is taken to be equal to the rate at which the HIV negative people would increase without HIV/AIDS minus the rate at which the HIV positive people die. If we take one year to be the unit of time, which is the usual interval at which the data on HIV positive people is published in most countries, including Canada and the United States, then the differential equations become difference equations. In this paper, we present such a difference equations model for spread of HIV/ AIDS in any society and compare the numbers predicted by this model with the actual numbers in both Canada and the United States. In both cases, the agreement is very good.

1. **The Model:** We look upon x[n], u[n], and z[n] as the number of HIV negative people, the number of HIV positive people (whether having AIDS or not) and those people who have developed AIDS respectively, in the year n. We write

$x[n+1] = x[n](1+A_1-A_2x[n])\mathrm{Exp}[-A_3u[n]]$............. ..(3.1a)
$u[n+1] = c_1u[n]\mathrm{Exp}[A_4x[n]]$................................. ..(3.1b)
and
$z[n+1] = c_2z[n] + A_5u[n]$............................... ....(3.1c)

We shall take all the constants A_i, i = 1, 2, 3, 4, 5 and c_1 and c_2 to be positive. The constant A_1 is the rate of growth of HIV negative people near x = 0 in the absence of this disease, A_1 / A_2 is the maximum number of HIV negative people that the society can support in the absence of this disease, $1 - c_1$ is the rate at which the HIV positive people die per unit of time (so that $1/(1-c_1)$ is the expected life span of an HIV positive person). Similarly $1/(1-c_2)$ is the expected life span of a person with AIDS. The constant A_3 determines the rate at which an HIV negative person becomes HIV positive through contact with an HIV positive person. The quantity $\exp(-A_3u_n)$ may also be looked upon as the probability that an HIV negative person does not interact with an HIV positive person during the time n [14, 15]. The constant A_4 determines the rate at which the number of HIV positive people increases because of their contact with HIV negative people and A_5 determines the rate at which HIV negative people develop the AIDS disease. We shall assume all these parameters to be positive and take A_1 to be less than one. If we take one year to be the unit of time, then this last assumption is true in all societies. With $0 < A_1 < 1$, it can be easily shown that if $0 < x_n < A_1 / A_2$, then so is x_{n+1}.

Since A_3 determines the rate at which the number of HIV negative people decreases (due to their contacts with the HIV positive people), and A_4 determines the rate at which the number of HIV positive people increases (due to their contacts with the HIV negative people), these constants cannot be independent of each other in our model. We notice that in the ODE model that we developed previously [7], the quantity $A_1 x - A_2 x^2$ is equal to the rate of change of the total population plus the rate at which the HIV positive people die per unit of time. We put the same restriction on our present model and assume that

$p[n] = (A_1x[n]-A_2 x[n]^2) - (x[n+1]-x[n]) - (u[n+1]-u[n]) > 0$ ....(3.1d)

for any n. This condition puts a restriction on the constants A_3 and A_4. In most societies x_{n+1} / x_n is only slightly greater than one, so that, from equation (3.1a), A_3 must be small compared to one. Condition (3.1d) then implies that A_4 must also be small and linearization shows that this condition is satisfied if $A_3 > A_4\, c_1$.

Boundedness of the solution: We notice that $0 < x_n < A_1 / A_2$ for any n if x_0 is in this interval, and $u_n > 0$ if $u_0 > 0$. Also $x_{n+1} <= \text{or} > x_n$ if $u_n >= \text{or} < \alpha_{n1}$ where $\alpha_{n1} = (1/A_3)\, (\ln(1 + A_1 - A_2\, x_n))$. Also $u_{n+1} <= \text{or} > u_n$ if $x_n <= \text{or} > \alpha_2$ where $\alpha_2 = (1/A_4)\ln(1/c_1)$. Finally $x_{n+1}\, u_{n+1} <= \text{or} > x_n\, u_n$ if $u_n >= \text{or} < \alpha_{n3}$ where $\alpha_{n3} = (1/A_3) (A_4\, x_n + \ln\, (c_1\, (1 + A_1 - A_2\, x_n)))$. We draw the curve $u_n = \alpha_{n1}$, the line $x_n = \alpha_2$, the curve $u_n= \alpha_{n3}$, and the line $x_n = A_1 / A_2$ in the (x_n,u_n) plane. Let the line $x_n=\alpha_2$ intersect the hyperbola $x_n\, u_n= \alpha_4$ where $\alpha_4= (A_1 / (A_2\, A_3))((A_4\, A_1 / A_2) + \ln\, (c_1))$ at the point P_3. The solution of our equations is seen to be bounded in the first quadrant by the lines $x_n = A_1 / A_2$, the hyperbola $x_n\, u_n= \alpha_4$ and the higher of the two horizontal lines, one through the point P_3 and the line $u_n= (1/A_3)(\ln(1+A_1)$. This is easily seen by observing the inequalities stated above.

The equilibrium points and their stability: Just as in the differential equations model that we developed previously [5], the three points of equilibrium in this model are seen to be $P_1\ (0,\ 0,\ 0)$, $P_2\ (A_1 / A_2,\ 0,\ 0)$ and $P_3\ (x_0,\ u_0,\ z_0)$ where

$x_0 = -\ln\, (c_1) / A_4$,
$u_0 = (1/A_3)\ln(1+A_1-A_2 x_0)$, and
$z_0 = A_5\, u_0 / (1 - c_2)$.

Notice that $0 \le x_0 < A_1 / A_2$ implies that $u_0 > 0$ and that $u_0 < 0$ if $x_0 > A_1 / A_2$. The point P_1 is seen to be unstable if $A_1 > 0$, the point P_2 is seen to be unstable (stable) if $u_0 > 0\ (u_0 < 0)$, and the point P_3 is seen to be stable if $u_0\, \text{Exp}[A_3\, u_0] < A_2 / (A_3\, A_4)$ and is seen to be unstable if $u_0\, \text{Exp}[A_3\, u_0] > A_2 / (A_3\, A_4)$ so that a very large value of u_0 tends to make all the three points unstable. It is to be noted that the quantity $A_3\, u_0$ depends only upon A_1, A_2, A_4 and c_1 and is independent of A_3 so that a change in the value of A_3 alone will not affect the stability of P_3. These results are important in establishing our model as a viable one for the spread of HIV/AIDS in any community. The instability of the origin is a universal phenomenon in any society which propagates itself. The instability of P_2 implies that this disease can be eradicated only if the contact parameter A_4 is sufficiently small. If the sick people (HIV positive people in our case) are somehow isolated, then A_4 becomes zero and the disease can be eradicated. The model however says that there is a relation between the longevity of the HIV positive people, the environmental factors which determine the maximum number of HIV negative people that the society can support, the contact parameter A_4, and the eradicability of the disease. This relation is that if the contact parameter A_4 (or c_1, the longevity of the HIV positive people) is so small that $\ln(1/c_1) > A_1\, A_4 / A_2$, then this disease can also be eradicated, so that $\ln(1/c_1)/(A_1 / A_2)$ is the threshold value of the contact parameter A_4, below which value the disease will disappear. We reached a similar conclusion in the ODE model where again P_2 was stable (unstable) if $u_0 < 0\ (u_0 > 0)$ which again required a sufficiently small value of the contact parameter [5]. If the contact parameter A_4 (or c_1, the longevity of the HIV positive people) is sufficiently large, then all the three points are unstable. This result is also analogous to the result in the ODE models that we developed previously [5]. In this last case (when all the three points are unstable), just as in the previous ODE model [5], the solution is seen to run repeatedly towards the origin and further and further away from the origin. However, in the ODE models, the solutions run into a limit cycle [3, 4, 5, 7], while in this model, the solutions are bounded as in that model and run into what would be called a limit cycle for all practical purposes. We give an example of such a 'limit cycle' in Fig.10 below. In a practical situation, if x[n] and u[n] are both sufficiently small (less than one for example) for some n, then this implies the annihilation of the society and we may safely assume that our model is not applicable anymore. In another ODE model that we recently applied to the spread of HIV/AIDS, the solution is also seen to run towards the origin for sufficiently large values of the contact parameter [5]. We would argue that the origin should NOT be an unstable point in epidemiological models and in that sense, the ratio dependent model that we developed before [3] is more realistic than the usual density dependent models in epidemiology. In this case, if all the three points are unstable (and the solution runs repeatedly towards the origin), then the inequality (3.1d) is not satisfied and our model is not applicable anymore (see below).

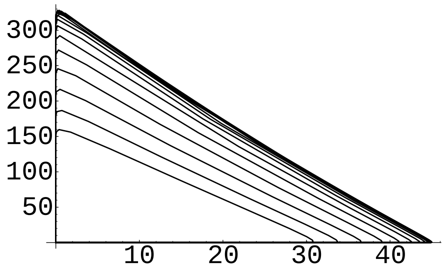

Fig.10: A 'Limit cycle' solution, in the x-u plane, of our equations for $A_1 = .03$; $A_2 = A_1/50$; $A_3 = .01$; $A_4 = .03$; $A_5 = .03$; $c_1 = .97$; $c_2 = .9$; All the equilibrium points are unstable in this case. Notice the large changes in x with u staying small. For vanishingly small values of u, the first and the second terms in p[n] cancel each other while for sufficiently large x[n], the third term is negative. Inequality (3.1d) may not be satisfied for all n. Calculations show that it is not.

2. Application to HIV/AIDS:

The system of equations (3.1,a,b) is the usual predator-prey system and may be applied to study the spread of HIV/AIDS in the same manner as the corresponding differential equations model [3]. We look upon x[n] and u[n] as the number of HIV negative and HIV positive (whether having developed AIDS or not) people respectively. The HIV negative people become HIV positive when they come into contact with HIV positive people, otherwise they stay HIV negative, so that Exp[-A_3u[n]] may be looked upon as the probability that an HIV negative person does NOT come into contact with an HIV positive person in the n-th year. Constantino et al [14] have applied a similar set of difference equations to the dynamics of insect populations with a similar interpretation of the exponential term in their model. With appropriate normalization, the equations (3.1a,b) may also be looked upon as applications of Bayes' Rule in probability.

We applied equations (3.1) to the spread of HIV/AIDS in Canada and the United States. The constants in the model were evaluated from the data published by Health Canada and the CDC in the United States [6, 9]. We took one year as the unit of time. The year zero was 1995. The constants c_1 and c_2 were estimated independently. We took $c_1 = .9$ and $c_2 = .5$ implying that the life expectancy of an HIV positive person, from the time that he/she contacts this positivity, is ten years while that of a person who has developed AIDS is two years in North America. These estimates are in keeping with the estimates of other researchers in the field [16]. The number A_2 was taken to be equal to $A_1 / 50$ for Canada and equal to $A_1 / 400$ for the United States. This corresponds to assuming that without this disease, the limiting population would be 50 million for Canada and 400 million for the United States. We consider these assumptions to be reasonable. The other constants were determined keeping the additional constraint implied by inequality (3.1d) in mind. It was noted that if P_3 is stable, and if the inequality (3.1d) is satisfied for n = 0, then it is satisfied for all n. A proof of this assertion is lacking. However, we give the values of p[n] in one particular case in Fig.11 to support our statement. In the case when all the three points are unstable, and the solution runs into a 'limit cycle', a look at Fig 10 suggests why inequality (3.1d) may not be satisfied for all n. In Fig 10, we see a large change in x with u staying small, so that the first and second terms in condition (3.1d) cancel each other (approximately) while for sufficiently large x[n], the third one may be negative and the inequality is not satisfied. In this case our model is not relevant to the propagation of HIV/AIDS in a community (notice the very large values of u in Fig. 10).

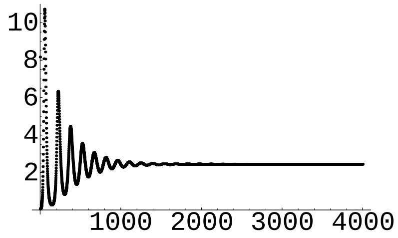

Fig.11: The values of p[n] with p[0]>0 and with A_1=.0272975, $A_2=A_1 / 400$, A_3=.000824 and A_4=.000779, A_5=.363606, C_1=.9, c_2=.5, x(0) = 270, u(0) = 1, and z(0) = 0. P_3 is stable in this case.

If we try to determine the parameters in our model without any consideration for (3.1d), it may happen that some of these constants turn out to be negative. To avoid this possibility, we notice that in another model that we developed [7], the quantity $(A_1 x - A_2 x^2)$ is equal to the rate of change of the combined population of both HIV negative and HIV positive people plus the rate (per unit of time) at which the HIV positive people die. It follows that in an expanding (total) population, which is the case in most countries of the world today, the quantity A_1 must be positive. We applied the same criterion to our difference equations model, hence the restriction (3.1d). We now estimated the constants in our model, and with the help of the model, we estimated the number of HIV positive people and the number of AIDS patients in both Canada and the United States up to five years in advance. We used the year 1995 as the year zero. The numbers x[0], x[1], u[0], u[1], z[0] and z[1], available from the data published by Health Canada and the CDC in the United States were used to estimate the constants in our model and then the model was used to determine the values of x, u, and z for the years 1997, 1998, 1999, 2000. The values thus predicted by the model were compared with the actual numbers also available from the data published by the above named agencies [6, 9]. The results are given in Figures 12 and 13 for Canada and in Figures 14 and 15 for the United States. The solid lines in these figures represent the actual numbers while the dotted lines give the estimates of our model. All numbers are in millions of people. Notice the very good agreement in the case of HIV positive people. The estimates for the number of AIDS patients are also reasonably good in Canada while they are in excellent agreement in the United States. It is to be noted that, according to this model, the actual number of AIDS patients in Canada is more than the estimate from the values of the constants obtained from the data from earlier years. Some of the discrepancy could perhaps be explained due to under reporting of AIDS deaths which is common in both the countries [6, 9].

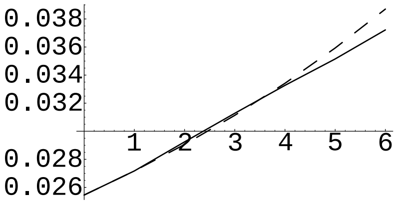

Fig. 12: Comparison of the number of HIV positive people in Canada with estimates of the difference model. 1995 is the year zero. The data for 1995 and 1996 were used to determine the constants in the model. Solid lines represent the actual numbers.

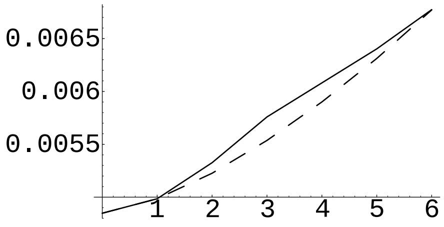

Fig. 13: Comparison of the number of AIDS patients in Canada with estimates of the difference model. The year zero is 1995. The data for 1995 and 1996 were used to determine the constants in the model. Solid lines represent the actual numbers.

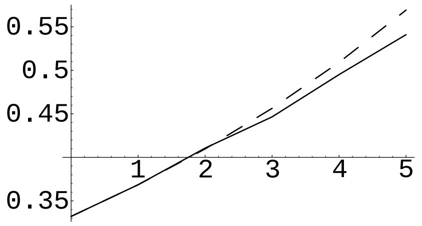

Fig. 14: Comparison of the number of HIV positive people in the U.S. with estimates of the difference model. The year zero is 1995. The data for 1995 and 1996 were used to determine the constants in the model. Solid lines represent the actual numbers.

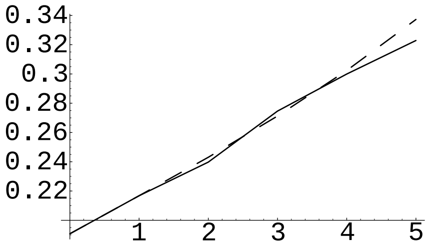

Fig. 15: Comparison of the number of AIDS patients in the U.S. with estimates of the difference model. The year zero is 1995. The data for 1995 and 1996 were used to determine the constants in the model. Solid lines represent the actual numbers.

3. Chaos reversed?

In this section, we point out a property of equations (3.1) which seems to imply that a chaotic situation can be reversed under this process. For large values of A_1 and with u_n= 0, equation (3.1a) will exhibit chaos [17]. With a small value of u[0], however, and a large value of x[0], (and a large value of A_1), the solution of equations (3.1,a,b) shows a chaotic behaviour in the beginning, but this behaviour dies down as u[n] goes up, slowing down the growth of x[n]. In Fig. 16, we show the x values in a solution for A_1=2.93, A_2=A_1/50, A_3=.01, A_4=.001, A_5=.03, c_1=.97, c_2=.9, x[0]=30, u[0]=1, and z[0]=0. The most interesting feature to notice here is the quite prominent near cycle of period three, as also the near cycles of periods 2, 4, 8 and so on. We say *near cycles* because ours is not a logistic equation and the repetition of the values is only approximate. A cycle of three is particularly important because it implies cycles of all other orders in a chaotic process [18]. We also show the progression of x values from n = 0 to n = 150 in Fig 17. Notice the irregular behaviour in the beginning, which becomes almost regular (with a period of three, count 20 cycles from n = 80 to n = 140) and then irregular and chaotic again. The point P_3 is stable in this case.

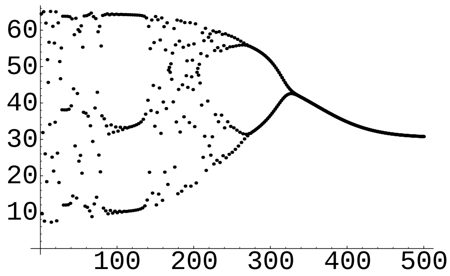

Fig 16: This diagram shows the chaotic behaviour of x values for values of the parameters indicated in the text. The chaotic behaviour in the beginning evolves into a period of three which period disappears again but later on exhibits periods of eight, four and two and then a fixed point. This is a reversal of the chaotic behaviour for the pure logistic equation. This is chaotic behaviour without a chaotic attractor.

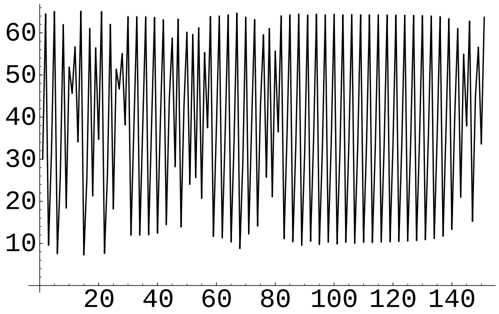

Fig 17: Same as Fig.16. This figure shows the progression of x[n] from n = 0 to n = 150. Notice the near regular behaviour (with a period of three) from n = 80 to n = 140. Count 20 cycles from n = 80 to n = 140. This behaviour dies down and x goes to a limit an n goes up.

Now suppose we treat the data in Fig.17 as a time series and we know only the first 150 terms of this time series. Could we then discern "the method behind the madness"? Could we find out the rule, even approximately, which produced this time series? We tried the method advocated by Sugihara et.al.[19] and plotted x[n+1] against x[n] from n = 0 to n = 149. The result is given in Fig 18 which figure indicates that indeed there is a strong correlation between x[n] and x[n+1] for these values of n. The relation here looks a little bit like the one in the Henon Attractor [20].

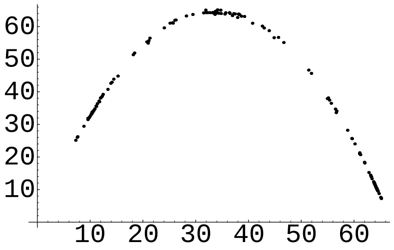

Fig 18: Values of x_{n+1} plotted against x_n from the time series data in Fig.17

Sensitivity to initial conditions is often considered to be a signature of chaos [20]. To see if there is this sensitivity in our model, we tried two initial conditions for the equations (3.1,a,b,c) with the same values of the parameters as the ones that produced Figures 16 and 17. We tried {x(0),u(0),z(0)} = {30,1,0} and {x(0),u(0),z(0)} = {30,1.000001,0}. This initial distance of .000001 in the phase space was seen to increase to 22.6006 giving an amplification of more than 22.6 million in 150 steps, implying sensitivity to initial conditions in a process which goes eventually to an equilibrium point. With the same values of the parameters as in this example, but with $c_1 = .971$, the result is even more dramatic [21].

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