Report from the Ambassador to Cida-2

Clifton Cunningham

Clifton Cunningham (cunning@math.ucalgary.ca) received his Ph.D. from the University of Toronto under the supervision of James Arthur in 1997. After a postdoc at the University of Massachusetts and visiting positions in Paris at the Ecole Normale Superieure and the Centre National de la Recherche Scientifique, he moved to the University of Calgary where he continues to work. His research uses group representation theory, algebraic geometry, and number theory to work on problems relating to the Langlands Programme.

> *The p-adic numbers are as natural as the reals—indeed it can plausibly be argued that they are logically simpler and that it is only by indoctrination that we feel the reals are simpler.*
>
> —J. W. S. Cassels, *Rational Quadratic Forms*

I should have known better. I had been over this a thousand times, and in every simulation on Earth I had had no difficulty being fastidiously diplomatic. In fact, from hundreds of candidates, I had been chosen to lead the Mathematical First Contact project with Cida-2 mostly because of my ability to discuss mathematics without being rude or offensive—a rare trait, as it turned out. But there I was, faced with a school of angry mathematicians from Cida-2, closing ranks and swimming toward me, their parallel sets of iridescent gills glowing in anger. And they were right to be angry: quoting Neils Henrik Abel might have passed for clever on Earth, but here it was just incendiary. Suddenly I was caught in a swirling mass of sea creatures, and I could feel myself losing consciousness in the claustrophobia of my water suit. Relations between Earth and Cida-2 were threatened—as was my life! I should have known better. How had this happened?

Everything had started so well. Contact with Cida-2 had begun exactly as many had predicted: we detected a radio transmission carrying a list of prime numbers in binary form; and, about the same time, they found the Pioneer 10 space probe, launched by NASA in 1972 and carrying an etched plaque showing the location of our planet on a pulsar map.

With mathematics thus established as the *lingua franca*, it was decided that first contact should be made by mathematicians, and thus the Mathematical First Contact project was born. After a gruelling selection process I was chosen to head the team and charged with the task of studying the development of mathematics on Cida-2 as a means of facilitating a cultural and scientific exchange.

We met on a ship in orbit around Cida-2 specifically designed by the Cidians for first contact between our two species. Our sleep chambers and rest areas were filled with air, while the rest of the ship was filled with the same liquid that completely covered the planet below us. The Cidians are water creatures, so we wore water suits that had been made on earth for this purpose. The Cidians had already constructed a translation machine that rendered their complex clicking and whale-like sounds into English and likewise translated our language into something intelligible to them. We met in the great hall of the ship and, at first, everything went swimmingly.

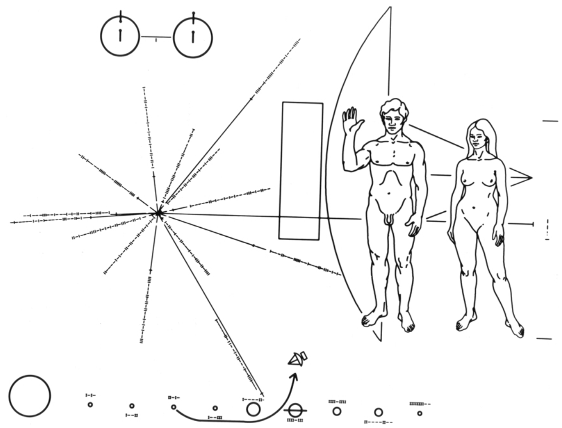

Figure 1. Pioneer 10 plaque.

God made the integers

On the first day, we discussed basic counting systems. A gurgle of agreement rippled through the Cida-2 contingent when I explained that we have an ancient dictum: *God made the integers; all the rest is the work of Mankind*. My counterpart made it clear that this sentiment is actually dogma amongst the inhabitants of Cida-2; but they were perplexed—dare I say troubled—when I explained that this is not considered literal truth by most humans. They intimated that the divine nature of the integers was central to their theology. In retrospect, this was the first sign of trouble, and had I been listening better I would have realized then to tread lightly where mathematics and theology are concerned.

Holding up my hands, I explained the biological basis for our use of base-10 positional notation to represent positive integers. Then I drew a line and populated it with zero and a few elements from the set $\mathbb{N}$ of natural numbers.

Immediately they became very excited and started asking questions. Why had I oriented the line left-to-right and at the same time used right-to-left positional notation? I explained that there are many languages on Earth, and some are written right-to-left, while others are written left-to-right. While I came from a culture that writes left-to-right, our positional notation was developed by people who wrote right-to-left. The oddity they had noticed was a notational vestige of the history of my planet.

A small group of the Cidian mathematicians immediately broke from the main conversation and began making some calculations. Apparently, astronomers from Cida-2 had studied the Pioneer 10 plaque carefully and had understood that it indicated the position of our sun on a pulsar map and that it also gave a brief description of our solar system. However, they had been unable to locate our solar system based on their reading of the map.

The problem, as it turned out, was this oddity concerning the way numbers are written on the Pioneer 10 plaque, which reflects the culture of the authors. For each line drawn from our sun to a pulsar on the Pioneer 10 plaque, there is an associated number, in binary form, indicating the period of the pulsar. When interpreting this diagram, they naturally oriented each radial line from our sun to the pulsar, and read the binary number indicating the period of the pulsar with the matching orientation in positional notation. Likewise, they guessed that the markings above or below each planet in the Pioneer 10 plaque indicated the distance from the sun to that planet, and since the planets were drawn left-to-right from our sun, they read the binary numbers accordingly. For example, the Cidian astronomers had read the symbol $| - - | | |$ (seen below Mars in Figure 2) as $1 \times 2^0 + 0 \times 2^1 + 0 \times 2^2 + 1 \times 2^3 + 1 \times 2^4 + 1 \times 2^5 = 1 + 8 + 16 + 32 = 57$ rather than $1 \times 2^0 + 1 \times 2^1 + 1 \times 2^2 + 0 \times 2^3 + 0 \times 2^4 + 1 \times 2^5 = 1 + 2 + 4 + 32 = 39$, as the makers of the plaque had intended.

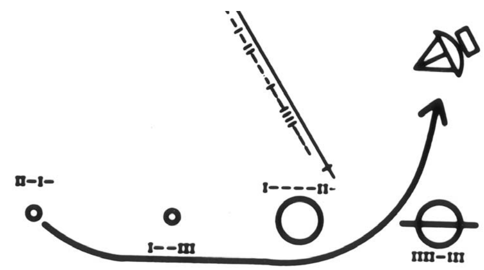

Figure 2. Detail from the Pioneer 10 plaque.

Without meaning to, I had explained to them that the binary numbers on the Pioneer 10 plaque were actually written in such a way that the orientation of the corresponding lines were *opposite* to the orientation of the positional notation of the numbers. This confusion regarding orientation in positional notation explained why they had been unable to locate our solar system, as their calculations quickly confirmed.

With this miscommunication in mind, and in the interests of inter-galactic clarity, we agreed that, if any confusion were possible, we would henceforth write all integers in the form

$$a_0 \times b^0 + a_1 \times b^1 + a_2 \times b^2 + a_3 \times b^3 + \cdots ,$$

where b would indicate the base (typically 10 or 2) and where the coefficients a_i would be subject to the constraint $0 \leq a_i \leq b - 1$. We moved on to a discussion of the field $\mathbb{Q}$ of rational numbers and likewise agreed that we would write rational numbers in the form

$$a_{-n} \times b^{-n} + a_{1-n} \times b^{1-n} + a_{2-n} \times b^{2-n} + a_{3-n} \times b^{3-n} + \cdots$$

to avoid confusion.

The rest of the day was spent discussing algebra and number theory—probably one of the most incredible days of my life—and I discovered that the inhabitants of Cida-2 are far ahead of us in this regard. They have, in fact, made great progress on what we would call Diophantine problems, and I was transfixed by the sophistication of their ideas.

Thus far, the representatives of Cida-2 and Earth were in complete accord. Of course, we had different symbols and definitions, but the idea of counting numbers, integers, and rational numbers had their analogous forms on Cida-2 and we quickly recognized that we understood each other completely.

In time I came to understand that there is a strong relationship between the biological form and the mathematical development of a species. And yet, in spite of this, based on my experience, it appears that different species do not develop different mathematics truths; rather, they organize the subject in different ways and make more progress in some fields than others. I feel that my encounter with the inhabitants of Cida-2 proves what working mathematicians have always known: mathematics is discovered, not invented.

The work of mankind

The second sign of trouble appeared the moment I began describing real numbers on the second day of Mathematical First Contact. The Cidians felt the term 'real number' was necessarily an oxymoron. Remember that numbers are quite literally god-given on Cida-2, with no more—and no less—relation with matter than what is found between the divine and the corporeal worlds. They found the phrase 'real number' ludicrous, and even vaguely insulting. They did not go so far as to accuse me of blasphemy—only, I suspect, because I was still considered an honoured guest. Even after I had constructed the set of real numbers, they thought it was ridiculous that elements of this set should be thought 'real'.

I explained that, at one stage in our mathematical development, we became deeply interested in the order for $\mathbb{Q}$ (as we had discussed during the first day) and the idea of a continuum of numbers. Tremendous progress had been made when mathematicians made this notion precise. The result was what we now call real analysis and, although the notion that space is a continuum in this sense has long been discredited, real analysis lies at the heart of many of our sciences.

In order to explain the development of real analysis as an idea, I needed to be precise about real numbers. And so I decided to explain the concept to my counterparts from Cida-2.

I began by explaining that we write $|\ | : \mathbb{Q} \to \mathbb{Q}$ for the function defined by $|a| = a$ if $a \geq 0$ and $|a| = -a$ if $a \leq 0$; we refer to $|a|$ as the absolute value of a. The most important properties of this function are the following: for all rational numbers a and b,

(i) $|a| \geq 0$ with equality if and only if $a = 0$,
(ii) $|a \times b| = |a| \times |b|$, and
(iii) $|a + b| \leq |a| + |b|$.

As I explained to the mathematicians from Cida-2, only these properties are required to develop real numbers from rational numbers.

Next, I explained that we refer to a sequence $x : \mathbb{N} \to \mathbb{Q}$ as a Cauchy sequence if for every positive rational number ϵ there is some natural number N such that $|x(n) - x(m)| < \epsilon$ for all $n > N$ and $m > N$. They saw immediately that Cauchy sequences form a ring under the obvious operations. Then, I explained that a sequence $x : \mathbb{N} \to \mathbb{Q}$ is said to be null if for every positive rational number ϵ there is some natural number N such that $|x(n)| < \epsilon$ for all $n > N$. Again, it was immediately clear to them that null sequences form a maximal ideal in the ring of Cauchy sequences. The quotient ring (Cauchy sequences modulo null sequences) is thus a field, denoted $\mathbb{R}$, and elements of $\mathbb{R}$ are called real numbers.

Then I presented several propositions concerning the field $\mathbb{R}$, such as the fact that the field $\mathbb{Q}$ of rational numbers embeds into $\mathbb{R}$ in the obvious way, and that $\mathbb{R}$ can be characterized as a completion of $\mathbb{Q}$ with respect to the topology defined by the absolute value function. I also explained that any algebraic closure $\mathbb{C}$ of $\mathbb{R}$ is a degree-2 extension of $\mathbb{R}$, that we call elements of $\mathbb{C}$ complex numbers; then I gave them a primer on complex analysis.

The delegation from Cida-2 was delighted by this constructive definition of the topological fields $\mathbb{R}$ and $\mathbb{C}$, and explained that they had similar notions on their planet. However, as I have already mentioned, they thought it was absurd that we use the word 'real' for the elements of $\mathbb{R}$. They asked me to consider what measurement could possibly return a real number that is not a rational number? They thought it was rather presumptuous that we should suggest that real numbers correspond to space in any 'real' way. They claimed that the Cida-2 number system was no more and no less real than ours—a claim you will be able to evaluate yourself, shortly.

The fateful formula

In the ensuing conversation, the lead mathematician from Cida-2 mentioned the formula

$$-1 = 1 \times 2^0 + 1 \times 2^1 + 1 \times 2^2 + 1 \times 2^3 + 1 \times 2^4 + 1 \times 2^5 + 1 \times 2^6 + \cdots,$$

which we would write

$$-1 = 1 + 2 + 4 + 8 + 16 + 32 + 64 + \cdots.$$

I laughed when I saw this, recognizing it as a one of the expressions often produced by people who study divergent series. "Surely you aren't really using this sort of thing in your serious mathematics," I quipped. If I had been more attuned to the subtle signs of anger exhibited by Cidians, I would have stopped there, but I was feeling glib. "You know, we have another ancient dictum on Earth: *Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever.*"

By quoting Abel in this way, I had gone too far. I had just linked the devil with one of the most revered formulas on a planet where integers were a manifestation of god. I had thus gone beyond ignorance and rudeness, and had walked blindly into blasphemy. I should have known better.

Suddenly I was caught in a swirling mass of angry water creatures, gills glowing in anger, and at the same moment, the claustrophobia of my water suit finally got the better of me and, to my eternal shame, I passed out. The mathematicians swam away in anger, and we were summarily directed to our chambers. The second day had ended on a very low note indeed.

Trial and error

When we reconvened on the third day, it became clear the Cidians had decided to put me to the test before deciding my fate. I was isolated and asked a series of questions. Here is an abridged transcript of my trial, in which I use Earth notation.

Question 1. In what sense is the series $1 + 2 + 4 + 8 + 16 + 32 + 64 + \cdots$ divergent?

Answer. The series is divergent in the sense that it does not define a real number. The series defines a real number if and only if the sequence of partial sums is Cauchy. Consider the sequence of partial sums $x(n) := 1 + 2 + 4 + \cdots + 2^{n-1}$; then $x(n) = 2^n - 1$ for every $n \in \mathbb{N}$. To see that the sequence is not Cauchy, argue as follows. For all natural numbers $n < m$, we have $|x(n) - x(m)| = |2^n - 1 - 2^m + 1| = 2^m - 2^n > 2^n > 2$; taking $\epsilon = 1$, we see that $|x(n) - x(m)| > \epsilon$. Having shown that the sequence is not Cauchy, it follows that the sequence $1 + 2 + 4 + 8 + 16 + 32 + 64 + \cdots$ does not define a real number, and is thus divergent. I should also mention that the sequence cannot possibly converge to -1, since the series is a sum of positive numbers, and therefore cannot possibly equal a negative number.

Question 2. In what sense does the divergent series $1 + 2 + 4 + 8 + 16 + 32 + 64 + \cdots$ equal -1?

Answer. Consider the power series $\sum_{n=0}^{\infty} 2^n z^n = 1 + 2z + 4z^2 + 8z^3 + \cdots$. The radius of convergence of this power series is $\frac{1}{2}$. The complex function $f(z)$ thus defined by this power series is holomorphic in the disc of radius $\frac{1}{2}$ centered at 0. This function admits a unique analytic continuation to $\mathbb{C}$, given by $f(z) = \frac{1}{1-2z}$, which has a pole at $z = \frac{1}{2}$ only; in particular, the analytic continuation of $f(z)$ is defined at $z = 1$. The value of the analytic continuation of $f(z)$ at $z = 1$ is -1. Thus, although it is false that $1 + 2 + 4 + 8 + 16 + 32 + 64 + \cdots$ equals -1 (after all, the series is divergent) it is true that $1 + 2 + 4 + 8 + 16 + 32 + 64 + \cdots$ is the divergent series that corresponds to the value of $f(z)$ at $z = 1$, had we forgotten to replace the power series above with its analytic continuation.

Question 3. Are you aware of other examples of arguments assigning rational numbers to divergent series?
Answer. Yes, the most famous is $-\frac{1}{12} = 1 + 2 + 3 + 4 + 5 + 6 + 7 + \cdots$, and the argument supporting this statement is similar to what I have given above. Consider the Dirichlet series $\sum_{n=1}^{\infty} \frac{1}{n^z} = 1^{-z} + 2^{-z} + 3^{-z} + 4^{-z} + \cdots$. This series defines a holomorphic function $\zeta(z)$ on the set of complex numbers z with real part greater than 1. This function admits a unique analytic continuation to $\mathbb{C}$ (called the Riemann Zeta function) which does not have a pole at $z = -1$. The value of this analytic continuation $\zeta(z)$ at $z = -1$ is $-\frac{1}{12}$ (most easily seen from the functional equation for the Riemann Zeta function). Thus, although it is false that $1 + 2 + 3 + 4 + 5 + 6 + 7 + \cdots$ equals $-\frac{1}{12}$ (since the series is divergent) it is true that $1 + 2 + 3 + 4 + 5 + 6 + 7 + \cdots$ is the divergent series that corresponds to the value of $\zeta(z)$ at $z = -1$, had we forgotten to replace the Dirichlet series with its analytic continuation.

The Cidians withdrew to consider their verdict, and thus ended the third day.

The work of Cida-2

On the fourth day we met the delegates from Cida-2 in the chamber we had used on the first and second days, and it became apparent that the crisis had passed.

"Yours is a narrow-minded species, limited by convention and unable to see your own assumptions. As the design of your Pioneer 10 plaque illustrates, you implicitly assume that your way of doing things is universal, and you find it difficult to see yourselves objectively. Your answer to our first question betrays the same mindset. You have found only one way to interpret the series $1 + 2 + 4 + 8 + 16 + 32 + 64 + \cdots$, and that is by asking if the corresponding sequence of partial sums defines one of your arrogantly-named 'real numbers'. But there is nothing about the series that suggests you try to interpret it as a real number; indeed, you have convincingly demonstrated that it does not define a real number. Thus you have only shown that the series diverges in $\mathbb{R}$. However, the series really does converge, quite literally, though not in $\mathbb{R}$ and, when interpreted in the only sensible way, the series converges to -1. The second part of your answer to our first question lays bare your assumptions. Essentially, you have argued as follows: suppose $1 + 2 + 4 + 8 + 16 + 32 + 64 + \cdots$ is a real number; then $1 + 2 + 4 + 8 + 16 + 32 + 64 + \cdots$ is positive, since it is a sum of positive real numbers; in particular $1 + 2 + 4 + 8 + 16 + 32 + 64 + \cdots$ is not equal to -1, since -1 is negative. But, having first shown that $1 + 2 + 4 + 8 + 16 + 32 + 64 + \cdots$ cannot be interpreted as a real number you have exposed the fallacy of the false premise in your argument."

They were right, of course. In particular, the second part of my answer to their first question was absurd. As one of my colleagues from Earth pointed out, I would hardly argue that the famous formula $e^{\pi\sqrt{-1}} = -1$ is false by first observing that $\sqrt{-1}$ is meaningless as a real number. Rather, I would recognize that the formula relates complex numbers. As she pointed out, it would be impossible to refute this formula without first sensibly defining $\sqrt{-1}$ and π and e. Only once this is done, would it be possible to consider the veracity of the formula.

The mathematicians from Cida-2 had suggested that they had a way to sensibly interpret the series $1 + 2 + 4 + 8 + 16 + 32 + 64 + \cdots$. It was time to put them to the test.

The Cida-2 number system

On the fifth day I demanded that they explain the Cida-2 number system. They began by explaining that their number system is very similar to the real numbers of Earth. In fact, the similarity is quite striking.

On Cida-2 a sequence of rational numbers $x : \mathbb{N} \to \mathbb{Q}$ is called a *Lesneh sequence* if, for every natural number E, there is a natural number N such that 2^{E+1} divides $x(n) - x(m)$ for all integers n and m greater than N.

I decided to take a few minutes to get used to this definition, as I had a feeling that it might be viewed as an analogue of the definition of a Cauchy sequence. In order to do this, I made the following definition. For any non-zero integer n, let $\text{ord}_2(n)$ denote the largest integer such that $2^{\text{ord}_2(n)}$ divides n in $\mathbb{Z}$; in other words, the defining property of $\text{ord}_2(n)$ is the following: $2^{\text{ord}_2(n)}$ divides n and $2^{1+\text{ord}_p(n)}$ does not divide n. This defines a function $\text{ord}_2 : \mathbb{Z}^\times \to \mathbb{Z}$ (where $\mathbb{Z}^\times$ indicates the set of non-zero integers) that extends to the set $\mathbb{Q}^\times$ of non-zero rational numbers by the rule $\text{ord}_2\left(\frac{n}{m}\right) := \text{ord}_2(n) - \text{ord}_2(m)$; I dubbed ord_2 the *Cida-2 order*. I then defined $|\ |_2 : \mathbb{Q} \to \mathbb{Q}$ by the rule $|a|_2 := 2^{-\text{ord}_2(a)}$ if $a \neq 0$ and $|0|_2 = 0$, and called this the *Cida-2 norm*.

With a bit of thought you will see that the Cida-2 norm satisfies the following properties: for all rational numbers a and b,

(i) $|a|_2 \geq 0$ with equality if and only if $a = 0$,
(ii) $|a \times b|_2 = |a|_2 \times |b|_2$,
(iii) $|a + b|_2 \leq |a|_2 + |b|_2$.

In other words, the Cida-2 norm satisfies the same three properties as the absolute value function! In fact, the last property above can be sharpened to $|a + b|_2 \leq \max\{|a|_2, |b|_2\}$.

There is one obvious and important difference between these functions however: the image of the absolute value is the set of non-negative rational numbers, while the image of the Cida-2 norm is the discrete set $\{0\} \cup \{2^k \mid k \in \mathbb{Z}\}$. To get a feel for the behaviour of the Cida-2 norm observe that, although 34 and 2 are far from each other in the absolute value, they are very close in the Cida-2 norm since $|34 - 2|_2 = |32|_2 = \frac{1}{32}$.

Returning to the definition of a Lesneh sequence, notice that 2^{E+1} divides $x(n) - x(m)$ if and only if $|x(n) - x(m)|_2 < \frac{1}{2^E}$. I was thus led to the following proposition:

A sequence of rational numbers is Lesneh if and only if it is Cauchy with respect to the Cida-2 norm. Thus, they had begun as I had, by exhibiting a norm on the field $\mathbb{Q}$ of rational numbers.

Satisfied that I had understood their definition, I asked them to continue. They explained that the most important Lesneh sequence, as far as the formula from Cida-2 is concerned, is precisely the one that I had defined shortly before. This they explained on the sixth day of our Mathematical First Contact.

Finally I was ready for the definition of the Cida-2 number system. The *Cida-2 number system*, denoted by $\mathbb{Q}_2$, is the quotient of the ring of Lesneh sequences by the maximal ideal of sequences which are null with respect to the Cida-2 norm.

I spent the rest of the day studying properties of $\mathbb{Q}_2$. In particular, I learned that every Cida-2 number can be written uniquely in the form

$$a_{-n} \times 2^{-n} + a_{1-n} \times 2^{1-n} + a_{2-n} \times 2^{2-n} + a_{3-n} \times 2^{3-n} + \cdots ,$$

where $-n$ is the Cida-2 order of the Cida-2 number and $0 \leq a_i < 2$ for each i. Of course, this is also true of every rational number, and I saw then that we were back to considering the base-2 expansion (written in inter-galactic standard form) of the rational number.

I also learned that the Cida-2 number system is a topological field, characterized by the property that it is a completion of $\mathbb{Q}$ with respect to the Cida-2 norm. Consequently, real analysis has its counterpart on Cida-2!

Having understood that $\mathbb{Q}_2$ and $\mathbb{R}$ are in many ways analogous fields since they are both completions of $\mathbb{Q}$ (with respect to different topologies, of course), I was interested to learn of one rather dramatic difference: while the algebraic closure of $\mathbb{R}$ is a finite extension of $\mathbb{R}$, the algebraic closure of $\mathbb{Q}_2$ is an infinite extension of $\mathbb{Q}_2$. My full report includes a more complete comparison of the fields $\mathbb{R}$ and $\mathbb{Q}_2$.

Proof of the formula from Cida-2

On the sixth day of Mathematical First Contact I was ready to hear (through the translation machine in my water suit, of course) the proof of the formula from Cida-2 from my extra-terrestrial colleagues: in $\mathbb{Q}_2$,

$$-1 = 1 + 2 + 4 + 8 + 16 + 32 + \cdots .$$

Remember what the right side of the formula means: $1 + 2 + 4 + 8 + 16 + 32 + \cdots$ denotes a sequence of rational numbers $x : \mathbb{N} \to \mathbb{Q}$ defined by $x(n) = 1 + 2 + \cdots + 2^{n-1} = 2^n - 1$. This sequence is not Cauchy—it is Lesneh. Thus, the right side of the formula from Cida-2 is a Cida-2 number. In order to interpret the -1 on the left side as a 2-adic number, recall that each rational number determines a constant sequence of rational numbers; in the case at hand, the rational number -1 corresponds to the sequence of rational numbers $y : \mathbb{N} \to \mathbb{Q}$ defined by $y(n) = -1$ for each natural number n. Notice that the sequence $y : \mathbb{N} \to \mathbb{Q}$ is certainly Cauchy with respect to the Cida-2 norm. Now we see that $-1 = 1 + 2 + 4 + 8 + 16 + 32 + \cdots$ in $\mathbb{Q}_2$ if and only if $x - y$ is null with respect to the Cida-2 norm. To that end, let ϵ be an arbitrary (but henceforth fixed) positive rational number and let N be any integer for which $2^N > \frac{1}{\epsilon}$. Suppose $n > N$. Then $|x(n) - y(n)|_2 = |2^n - 1 + 1|_2 = |2^n|_2 = \frac{1}{2^n} < \frac{1}{2^N} < \epsilon$. Thus $x - y$ is null with respect to the Cida-2 norm, which means x equals y in $\mathbb{Q}_2$. It follows that $-1 = 1 + 2 + 4 + 8 + 16 + 32 + \cdots$ in $\mathbb{Q}_2$, as claimed. The series $1 + 2 + 4 + 8 + 16 + 32 + \cdots$ is the base-2 expansion of -1 and converges in $\mathbb{Q}_2$.

In mathematics we are often faced with expressions that are nonsensical in certain contexts. When this happens, we either move to a context in which the expression is meaningful, or we abandon the train of thought that led to that expression. For example, faced with a solution to $x^2 + 1 = 0$ we might pass from real numbers to complex numbers, or from the ring of integers to the ring of Gaussian integers; likewise, faced with the expression $\frac{1}{0}$ we might pass from the affine line to the projective line, so that we can meaningfully interpret $\frac{1}{0}$ as a point $[1 : 0]$ on the projective line. The series $1 + 2 + 4 + 8 + 16 + \cdots$ does not converge in the real numbers, so the field of real numbers does not provide the right context in which to interpret the expression. However, $1 + 2 + 4 + 8 + 16 + \cdots$ converges in $\mathbb{Q}_2$, and in that field, the series converges to -1.

Numbers on other planets

On the seventh day we rested and a great ceremony marked the end of our meeting. The Mathematical First Contact project had been a great success and we had a bounty of mathematical results to bring back from Cida-2.

Back in our rest quarters, we helped each other out of our suits in a high mood. As I undressed, a small opalescent object fell from my suit. Picking it up, I recognized that it was of Cidian manufacture—it must have been tucked into my suit by one of my extra-terrestrial colleagues.

Back on Earth and after years of work, our scientists eventually discovered that the object was a chemical information storage device. It took many more years of work to learn how to extract information from the gelatinous substance inside the object, but eventually we realized that it was an ancient text suggesting the existence of a planet called s'Nnameir-Zeta in the Seleda system whose inhabitants profess $-\frac{1}{12} = 1 + 2 + 3 + 4 + 5 + 6 + 7 + \cdots$.

In spite of the success of the Mathematical First Contact project with Cida-2, I am always reminded that it was very nearly a complete disaster and I worry that we are not ready ready for another inter-galactic encounter. Nevertheless, astronomers on Earth are now trying to decode the pulsar map contained in the text from Cida-2 and, if we have better luck with the ancient map than the Cidians had with the Pioneer 10 plaque, we will soon be searching for this fabled world.

Further study. Anyone interested in learning more about p-adic numbers would do well to begin with F. Q. Gouvêa's lovely book *p-adic Numbers: An Introduction*, Universitext, Springer-Verlag, 1991.

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