BALL-POLYHEDRA

BY

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Abstract. We study two notions. One is that of *spindle convexity*. A set of circumradius not greater than one is spindle convex if, for any pair of its points, it contains every short circular arc of radius at least one, connecting them. The other objects of study are bodies obtained as intersections of finitely many balls of the same radius, called *ball-polyhedra*. We find analogues of several results on convex polyhedral sets for ball-polyhedra.

1. Introduction

The main goal of this paper is to study the geometry of intersections of finitely many congruent balls, say of unit balls, from the viewpoint of discrete geometry in Euclidean space. We call these sets *ball-polyhedra*. Some special classes have been studied in the past; see, e.g. [12], [13] and [42]. For Reuleaux polygons see [34] and [35]. Nevertheless, the name ball-polyhedra seems to be a new terminology for this special class of linearly convex sets. In fact, there is a special kind of convexity entering along with ball-polyhedra which we call *spindle convexity*. We thank the referee for suggesting this name for this notion of convexity that was first introduced by Mayer [42] as "Überkonvexität".

The starting point of our research described in this paper was a sequence of lectures of the first named author on ball-polyhedra given at the University of Calgary in the fall of 2004. Those lectures have been strongly motivated by the following recent papers that proved important new geometric properties of intersections of finitely many congruent balls: a proof of the Borsuk conjecture for finite point sets in three-space based on the combinatorial geometry of "spherical polytopes" ([1], pp. 215); Sallee's theorem [46] claiming that the class of the so-called "Reuleaux polytopes" is dense in the class of sets of constant width in $\mathbb{R}^3$; a proof of the Kneser-Poulsen conjecture in the plane by K. Bezdek and Connelly [8] including

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the claim that under any contraction of the center points of finitely many circular disks of $\mathbb{R}^2$ the area of the intersection cannot decrease, and finally an analogue of Cauchy's rigidity theorem for triangulated ball-polyhedra in $\mathbb{R}^3$ [11]. In addition it should be noticed that ball-polyhedra play an essential role in the proof of Grünbaum-Heppes-Straszewicz theorem on the maximal number of diameters of finite point sets in $\mathbb{R}^3$; see [36].

This paper is not a survey on ball-polyhedra. Instead, it lays a rather broad ground for future study of ball-polyhedra by proving several new properties of them and raising open research problems as well.

The structure of the paper is the following. First, notations and basic results about spindle convex sets are introduced in Sections 2 and 3. Some of these results demonstrate the techniques that are different from the ones applied in the classical theory. It seems natural that a more analytic investigation of spindle convexity might belong to the realm of differential geometry.

In Section 4, we find analogues of the theorem of Kirchberger for separation by spheres. In Section 5, we prove spindle convex analogues of the classical theorems of Carathéodory and Steinitz regarding the linear convex hull.

In Section 6, we make the first steps in understanding the boundary structures of ball-polyhedra. We present examples that show that the face-structure of these objects is not at all obvious to define. Section 7 contains our results on intersections of unit spheres in $\mathbb{R}^n$. The questions discussed there are motivated primarily by a problem of Maehara [38] and are related to the goal of describing faces of ball-polyhedra. Also, we construct a counter-example to a conjecture of Maehara in dimensions at least 4. Then, in Section 8, we discuss variants of the important Kneser–Poulsen problem. In Section 9, we provide a partial characterization of the edge-graphs of ball-polyhedra in $\mathbb{R}^3$, similar to the Theorem of Steinitz regarding convex polyhedra in $\mathbb{R}^3$.

Then, in Section 10, a conjecture of the first named author about convex bodies in $\mathbb{R}^3$ with axially symmetric sections is proved for ball-polyhedra in $\mathbb{R}^3$. We extend an illumination result in $\mathbb{R}^3$ of Lassak [37] and Weissbach [50] in Section 11. In Section 12, we prove various analogues of Dowker–type isoperimetric inequalities for two-dimensional ball-polyhedra based on methods of Fejes-Tóth [27]. Finally, in Section 13, we examine spindle convex variants of Erdős–Szekeres-type questions.

2. Notations and Some Basic Facts

Let $(\mathbb{R}^n, ||\ \ ||)$, where $n \geq 2$, be the standard Euclidean space with the usual norm and denote the origin by o. The Euclidean distance between $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ is $||a - b||$. The closed line segment between two points is denoted by $[a, b]$, the open line segment is denoted by (a, b). For the closed, n-dimensional ball with center $a \in \mathbb{R}^n$ and of radius $r > 0$ we use the notation $\mathbf{B}^n[a, r] := \{x \in \mathbb{R}^n : ||a - x|| \leq r\}$. For the open n-dimensional ball with center $a \in \mathbb{R}^n$ and of radius $r > 0$ we use the notation $\mathbf{B}^n(a, r) := \{x \in \mathbb{R}^n : ||a - x|| < r\}$. The $(n - 1)$-dimensional sphere with center $a \in \mathbb{R}^n$ and of radius $r > 0$ is denoted by $\mathbb{S}^{n-1}(a, r) := \{x \in \mathbb{R}^n : ||a - x|| = r\}$. Any sphere or ball in the paper is of positive radius. When r is omitted, it is assumed to be one. Using the usual conventions, let card, conv, int, bd and diam denote cardinality, convex hull, interior, boundary and diameter of a set, respectively. We note that a 0-dimensional sphere is a pair of distinct points.

We introduce the following additional notations. For a set $X \subset \mathbb{R}^n$ let

$$\mathbf{B}[X] := \bigcap_{x\in X} \mathbf{B}^n[x] \text{ and } \mathbf{B}(X) := \bigcap_{x\in X} \mathbf{B}^n(x). \tag{2.1}$$

Definition 2.1. Let a and b be two points in $\mathbb{R}^n$. If $||a-b|| < 2$, then the *closed spindle* of a and b, denoted by $[a,b]_s$, is defined as the union of circular arcs with end points a and b that are of radii at least one and are shorter than a semicircle. If $||a-b|| = 2$, then $[a,b]_s := \mathbf{B}^n[\frac{a+b}{2}]$. If $||a-b|| > 2$, then we define $[a,b]_s$ to be $\mathbb{R}^n$.

The *open spindle*, denoted as $(a,b)_s$, in all cases is the interior of the closed one.

Remark 2.2. If $||a-b|| \leq 2$, then $[a,b]_s := \mathbf{B}[\mathbf{B}[\{a,b\}]]$, and $(a,b)_s := \mathbf{B}(\mathbf{B}[\{a,b\}])$.

Definition 2.3. The *circumradius* $\mathrm{cr}(X)$ of a bounded set $X \subseteq \mathbb{R}^n$ is defined as the radius of the unique smallest ball that contains X (also known as the circumball of X); that is,

$$\mathrm{cr}(X) := \inf\{r > 0 : X \subseteq \mathbf{B}^n[q,r] \text{ for some } q \in \mathbb{R}^n\}.$$

If X is unbounded, then $\mathrm{cr}(X) = \infty$.

Now, we are ready to introduce two basic notions that are used throughout this paper.

Definition 2.4. A set $C \subset \mathbb{R}^n$ is *spindle convex* if, for any pair of points $a,b \in C$, we have $[a,b]_s \subseteq C$.

Definition 2.5. Let $X \subset \mathbb{R}^n$ be a finite set such that $\mathrm{cr}(X) \leq 1$. Then we call $P := \mathbf{B}[X] \neq \emptyset$ a *ball-polyhedron*. For any $x \in X$ we call $\mathbf{B}^n[x]$ a *generating ball* of P and $\mathbb{S}^{n-1}(x)$ a *generating sphere* of P. If $n = 2$, then we call a ball-polyhedron a *disk-polygon*.

Remark 2.6. A spindle convex set is clearly convex. Moreover, since the spindle of two points has non-empty interior (if it exists), a spindle convex set is either 0-dimensional (if it is one point) or full-dimensional. Also, the intersection of spindle convex sets is again a spindle convex set.

Definition 2.7. The *arc-distance* of $a,b \in \mathbb{R}^n$ is the arc-length of either shorter unit circular arcs connecting a and b, when $||a-b|| \leq 2$; that is,

$$\rho(a,b) := 2\arcsin\left(\frac{\|a-b\|}{2}\right).$$

If $||a-b|| > 2$, then $\rho(a,b)$ is undefined.

Remark 2.8. If $a,b,c \in \mathbb{R}^n$ are points such that $||a-b|| < ||a-c|| \leq 2$, then $\rho(a,b) < \rho(a,c)$.

The proof of the following claim is straightforward.

Claim 2.9 (Euclidean arm-lemma)**.** *Given two triangles with vertices a,c,b and a,c,b', respectively, in $\mathbb{R}^2$ such that $||c-b|| = ||c-b'||$ and the angle at c in the first triangle is less than in the second. Then $||a-b|| < ||a-b'||$.*

In general, the arc-distance is not a metric. The following lemma describes how the triangle-inequality holds or fails in some situations. This lemma and the next corollary are from [10], and they are often applicable, as in Lemma 12.1.

Lemma 2.10. *Let $a, b, c \in \mathbb{R}^2$ be points such that $||a-b||, ||a-c||, ||b-c|| \leq 2$. Then*

$$\begin{array}{rlcll} (i) & \rho(a,b)+\rho(b,c) & > & \rho(a,c) & \iff b \notin [a,c]_s; \\ (ii) & \rho(a,b)+\rho(b,c) & = & \rho(a,c) & \iff b \in \mathrm{bd}[a,c]_s; \\ (iii) & \rho(a,b)+\rho(b,c) & < & \rho(a,c) & \iff b \in (a,c)_s. \end{array}$$

Corollary 2.11. *Let $a, b, c, d \in \mathbb{R}^2$ be vertices of a spindle convex quadrilateral in this cyclic order. Then*

$$\rho(a,c)+\rho(b,d) > \rho(a,b)+\rho(c,d)$$

that is, the total arc-length of the diagonals is greater than the total arc-length of an opposite pair of sides.

3. Separation

This section describes results dealing with the separation of spindle convex sets by unit spheres motivated by the basic facts about separation of convex sets by hyperplanes as they are introduced in standard textbooks; e.g., [16].

Lemma 3.1. *Let a spindle convex set $C \subset \mathbb{R}^n$ be supported by the hyperplane H in $\mathbb{R}^n$ at $x \in \mathrm{bd}\, C$. Then the closed unit ball supported by H at x and lying in the same side as C contains C.*

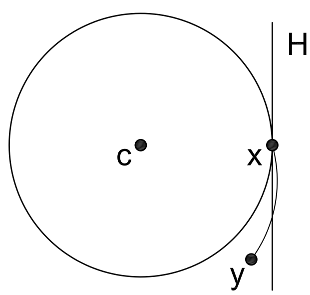

Figure 1

Proof. Let $\mathbf{B}^n[c]$ be the unit sphere that is supported by H at x and is in the same closed half-space bounded by H as C. We show that $\mathbf{B}^n[c]$ is the desired unit ball.

Assume that C is not contained in $\mathbf{B}^n[c]$. So, there is a point $y \in C$, $y \notin \mathbf{B}^n[c]$. Then, by taking the intersection of the configuration with the plane that contains x, y and c, we see that there is a shorter unit circular arc connecting x and y that does not intersect $\mathbf{B}^n(c)$ (Figure 1). Hence, H cannot be a supporting hyperplane of C at x, a contradiction. □

Corollary 3.2. *Let $C \subset \mathbb{R}^n$ be a spindle convex set. If $\mathrm{cr}(C) = 1$ then $C = \mathbf{B}^n[q]$ for some $q \in \mathbb{R}^n$. If $\mathrm{cr}(C) > 1$ then $C = \mathbb{R}^n$.*

Proof. Observe that if C has two distinct support unit balls then $\mathrm{cr}(C) < 1$. Thus, the first assertion follows. The second is clear. $\square$

Definition 3.3. If a ball $\mathbf{B}^n[c]$ contains a set $C \subset \mathbb{R}^n$ and a point $x \in \mathrm{bd}\, C$ is on $\mathbb{S}^{n-1}(c)$, then we say that $\mathbb{S}^{n-1}(c)$ or $\mathbf{B}^n[c]$ *supports* C at x.

The following corollary appears in [36] without proof.

Corollary 3.4. *Let $A \subset \mathbb{R}^n$ be a closed convex set. Then the following are equivalent.*

- *(i)* *A is spindle convex.*
- *(ii)* *A is the intersection of unit balls containing it; that is, $A = \mathbf{B}[\mathbf{B}[A]]$.*
- *(iii)* *For every boundary point of A, there is a unit ball that supports A at that point.*

Theorem 3.5. *Let $C, D \subset \mathbb{R}^n$ be spindle convex sets. Suppose C and D have disjoint relative interiors. Then there is a closed unit ball $\mathbf{B}^n[c]$ such that $C \subseteq \mathbf{B}^n[c]$ and $D \subset \mathbb{R}^n \backslash \mathbf{B}^n(c)$.*

Furthermore, if C and D have disjoint closures and one, say C, is different from a unit ball, then there is a closed unit ball $\mathbf{B}^n[c]$ such that $C \subset \mathbf{B}^n(c)$ and $D \subset \mathbb{R}^n \backslash \mathbf{B}^n[c]$.

Proof. Since C and D are spindle convex, they are convex, bounded sets with disjoint relative interiors. So, their closures are convex, compact sets with disjoint relative interiors. Hence, they can be separated by a hyperplane H that supports C at a point, say x. The closed unit ball $\mathbf{B}^n[c]$ of Lemma 3.1 satisfies the conditions of the first statement.

For the second statement, we assume that C and D have disjoint closures, so $\mathbf{B}^n[c]$ is disjoint from the closure of D and remains so even after a sufficiently small translation. Furthermore, C is a spindle convex set that is different from a unit ball, so $c \notin \mathrm{conv}(C \cap \mathbb{S}^{n-1}(c))$. Hence, there is a sufficiently small translation of $\mathbf{B}^n[c]$ that satisfies the second statement. $\square$

Definition 3.6. Let $C, D \subset \mathbb{R}^n, c \in \mathbb{R}^n, r > 0$. We say that $\mathbb{S}^{n-1}(c, r)$ *separates* C from D if $C \subseteq \mathbf{B}^n[c, r]$ and $D \subseteq \mathbb{R}^n \backslash \mathbf{B}^n(c, r)$, or $D \subseteq \mathbf{B}^n[c, r]$ and $C \subseteq \mathbb{R}^n \backslash \mathbf{B}^n(c, r)$. If $C \subseteq \mathbf{B}^n(c, r)$ and $D \subseteq \mathbb{R}^n \backslash \mathbf{B}^n[c, r]$, or $D \subseteq \mathbf{B}^n(c, r)$ and $C \subseteq \mathbb{R}^n \backslash \mathbf{B}^n[c, r]$, then we say that C and D are *strictly separated by* $\mathbb{S}^{n-1}(c, r)$.

4. A Kirchberger-type Theorem for Ball-polyhedra

The following theorem of Kirchberger is well known (e.g., [5]). If A and B are finite (resp. compact) sets in $\mathbb{R}^n$ with the property that for any set $T \subseteq A \cup B$ of cardinality at most $n+2$ the two sets $A \cap T$ and $B \cap T$ can be strictly separated by a hyperplane, then A and B can be strictly separated by a hyperplane. We show that no similar statement holds for separation by unit spheres.

We construct two sets A and B showing that there is no analogue of Kirchberger's theorem for separation by a unit sphere. Then we prove an analogue for separation by a sphere of radius at most one. Let $A := \{a\} \subset \mathbb{R}^n$ be a singleton set and $b_0 \in \mathbb{R}^n$ be a point with $0 < ||a - b_0|| =: \delta < 1$. Then $\mathbf{B}^n[a] \setminus \mathbf{B}^n(b_0)$ is a non-convex, closed set bounded by two closed spherical caps: an inner one C that belongs to $\mathbb{S}^{n-1}(b_0)$ and an outer one that belongs to $\mathbb{S}^{n-1}(a)$ (Figure 2). Now, we choose points $b_1, b_2, \dots, b_{k-1}$ such that for every i the set $\mathbf{B}^n[b_i] \cap C$ is a spherical cap of radius ε and we have also

$$C \subset \bigcup_{j=1}^{k-1} \mathbf{B}^n[b_j] \quad \text{and} \quad C \not\subset \bigcup_{j=1, j\neq i}^{k-1} \mathbf{B}^n[b_j] \quad \text{for} \quad i = 1, 2, \dots, k-1. \tag{4.1}$$

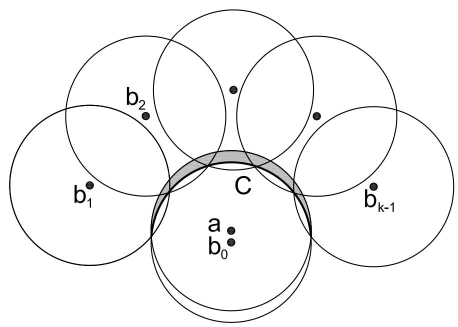

FIGURE 2

Let $B := \{b_0, b_1, \dots, b_{k-1}\}$. From (4.1) it easily follows that

$$\mathbf{B}^n(a) \subset \bigcup_{j=0}^{k-1} \mathbf{B}^n[b_j] \quad \text{and} \quad \mathbf{B}^n(a) \not\subset \bigcup_{j=0, j\neq i}^{k-1} \mathbf{B}^n[b_j] \quad \text{for} \quad i = 0, 1, \dots, k-1. \tag{4.2}$$

From the first part of (4.2) it is clear that there is no $c \in \mathbb{R}^n$ with the property that $a \in \mathbf{B}^n(c)$ and $B \subset \mathbb{R}^n \setminus \mathbf{B}^n[c]$. On the other hand, if ε is sufficiently small, then $a \in \operatorname{conv} B$. Hence, there is no $c \in \mathbb{R}^n$ such that $B \subset \mathbf{B}^n(c)$ and $a \notin \mathbf{B}^n[c]$. So, we have shown that A and B cannot be strictly separated by a unit sphere.

However, by the second part of (4.2), for any $T \subset A \cup B$ of cardinality at most k, there is a $c \in \mathbb{R}^n$ such that $T \cap A \subset \mathbf{B}^n(c)$ and $T \cap B \subset \mathbb{R}^n \setminus \mathbf{B}^n[c]$. This shows that *there is no Kirchberger-type theorem for separation by unit spheres.*

In Theorem 4.4 we provide a weaker analogue of Kirchberger's theorem. For its proof we need the following version of Kirchberger's theorem, which is a special case of Theorem 3.4 of Houle [32], and a lemma.

Theorem 4.1. *Let $A, B \subset \mathbb{R}^n$ be finite sets. Then A and B can be strictly separated by a sphere $\mathbb{S}^{n-1}(c, r)$ such that $A \subset \mathbf{B}^n(c, r)$ if, and only if, for every $T \subset A \cup B$ with $\operatorname{card} T \leq n+2$, $T \cap A$ and $T \cap B$ can be strictly separated by a sphere $\mathbb{S}^{n-1}(c_T, r_T)$ such that $T \cap A \subset \mathbf{B}^n(c_T, r_T)$.*

Lemma 4.2. *Let $A, B \subset \mathbb{R}^n$ be finite sets and suppose that $\mathbb{S}^{n-1}(o)$ is the smallest sphere that separates A from B such that $A \subseteq \mathbf{B}^n[o]$. Then there is a set $T \subseteq A \cup B$ with $\operatorname{card} T \leq n+1$ such that $\mathbb{S}^{n-1}(o)$ is the smallest sphere $\mathbb{S}^{n-1}(c, r)$ that separates $T \cap A$ from $T \cap B$ and satisfies $T \cap A \subset \mathbf{B}^n[c, r]$.*

Proof. First observe that $A \neq \emptyset$. Assume that $\mathbb{S}^{n-1}(o)$ separates A from B such that $A \subset \mathbf{B}^n[o]$. Now, let us note also that $\mathbb{S}^{n-1}(o)$ is the smallest sphere separating A and B such that $A \subset \mathbf{B}^n(o)$ if, and only if, there is no closed spherical cap of radius less than $\pi/2$ that contains $A \cap \mathbb{S}^{n-1}(o)$ and whose interior with respect to $\mathbb{S}^{n-1}(o)$ is disjoint from $B \cap \mathbb{S}^{n-1}(o)$. Indeed, if there is a sphere $\mathbb{S}^{n-1}(x,r)$ that separates A and B and satisfies $r < 1$ and $A \subset \mathbf{B}^n[x,r]$, then we may choose $\mathbb{S}^{n-1}(o) \cap \mathbf{B}^n[x,r]$ as such a spherical cap, a contradiction. On the other hand, if C is such a closed spherical cap then, by the finiteness of A and B, we can move $\mathbb{S}^{n-1}(o)$ to a sphere $\mathbb{S}^{n-1}(x,r)$ that separates A and B such that $\mathbf{B}^n[x,r] \cap \mathbb{S}^{n-1}(o) = C$ and $r < 1$, a contradiction.

We may assume that $A, B \subset \mathbb{S}^{n-1}(o)$. Let us take a point $q \in \mathbf{B}^n[o] \setminus \{o\}$. Observe that the closed half-space that does not contain o and whose boundary contains q and is perpendicular to q intersects $\mathbb{S}^{n-1}(o)$ in a closed spherical cap of radius less than $\pi/2$. Let us denote this spherical cap and its interior with respect to $\mathbb{S}^{n-1}(o)$ by C_q and D_q, respectively. Observe that we have defined a one-to-one mapping between $\mathbf{B}^n[o] \setminus \{o\}$ and the family of closed spherical caps of $\mathbb{S}^{n-1}(o)$ with radius less than $\pi/2$.

Let us consider a point $p \in \mathbb{S}^{n-1}(o)$. Note that $p \in C_q$ for some $q \in \mathbf{B}^n[o] \setminus \{o\}$ if, and only if, the straight line passing through p and q intersects $\mathbf{B}^n[o]$ in a segment of length at least $2\|p - q\|$.

Set

$$F_p := \{q \in \mathbf{B}^n[o] \setminus \{o\} : p \in C_q\} \text{ and} \qquad (4.3)$$

$$G_p := \{q \in \mathbf{B}^n[o] \setminus \{o\} : p \notin D_q\}.$$

It is easy to see that

$$F_p = \mathbf{B}^n[p/2, 1/2] \setminus \{o\} \text{ and } G_p = \mathbf{B}^n[o] \setminus \Big(\mathbf{B}^n(p/2, 1/2) \cup \{o\}\Big). \qquad (4.4)$$

By the first paragraph of this proof, $\mathbb{S}^{n-1}(o)$ is the smallest sphere separating A and B and satisfying $A \subset \mathbf{B}^n[o]$ if, and only if, $\left(\bigcap_{a\in A} F_a\right) \cap \left(\bigcap_{b\in B} G_b\right) = \emptyset$.

Let f denote the inversion with respect to $\mathbb{S}^{n-1}(o)$; for the definition of this transformation, we refer [51] Chapter III. More specifically, let us define $f(x) := x/\|x\|^2$ for $x \in \mathbb{R}^n \setminus \{o\}$. For any $p \in \mathbb{S}^{n-1}(o)$, let $H^+(p)$ (resp., $H^-(p)$), denote the closed half-space bounded by the hyperplane tangent to $\mathbb{S}^{n-1}(o)$ at p that contains (resp., does not contain) $\mathbb{S}^{n-1}(o)$. Using elementary properties of inversions, we see that $f(F_p) = H^-(p)$ and $f(G_p) = H^+(p) \setminus \mathbf{B}^n(o)$. Hence, $\mathbb{S}^{n-1}(o)$ is the smallest sphere separating A and B and satisfying $A \subset \mathbf{B}^n[o]$ if, and only if,

$$I := \Big(\bigcap_{a\in A} H^-(a)\Big) \cap \Big(\bigcap_{b\in B} \big(H^+(b) \setminus \mathbf{B}^n(o)\big)\Big) \qquad (4.5)$$

is empty. Observe that $\mathbf{B}^n(o) \cap H^-(a) = \emptyset$ for any $a \in A$. Since $A \neq \emptyset$, we have

$$I = \Big(\bigcap_{a\in A} H^-(a)\Big) \cap \Big(\bigcap_{b\in B} H^+(b)\Big). \qquad (4.6)$$

As $H^-(p)$ and $H^+(p)$ are convex for any $p \in \mathbb{S}^{n-1}(0)$, Helly's Theorem yields our statement. □

Remark 4.3. There are compact sets $A, B \subset \mathbb{R}^n$ such that $\mathbb{S}^{n-1}(o)$ is the smallest sphere that separates A from B and $A \subseteq \mathbf{B}^n[o]$ but, for any finite $T \subseteq A \cup B$, there is a sphere $\mathbb{S}^{n-1}(x, r)$ that separates $T \cap A$ and $T \cap B$ such that $r < 1$ and $T \cap A \subseteq \mathbf{B}^n[x, r]$.

We show the following 3-dimensional example. Let us consider a circle $\mathbb{S}^1(x, r) \subset \mathbb{S}^2(o)$ with $r < 1$ and a set $A_0 \subset \mathbb{S}^1(x, r)$ that is the vertex set of a regular triangle. Let B be the image of A_0 under the reflection about x. Clearly, $\mathbb{S}^1(x, r)$ is the only circle in its affine hull that separates A_0 and B. Hence, every 2-sphere that separates A_0 and B contains $\mathbb{S}^1(x, r)$. Consider two points $a \in A_0$ and $y \in (o, a)$ and set $A = A_0 \cup \mathbf{B}^2(y, \|a - y\|)$. Then the smallest sphere that separates A and B and contains A in its convex hull is $\mathbb{S}^2(o)$. Nevertheless, it is easy to show that, for any finite set $T \subset A$, there is a sphere $\mathbb{S}^2(c_T, r_T)$ separating T and B such that $r_T < 1$ and $T \subset \mathbf{B}^2[c_T, r_T]$.

Theorem 4.4. *Let $A, B \subset \mathbb{R}^n$ be finite sets. Then A and B can be strictly separated by a sphere $\mathbb{S}^{n-1}(c, r)$ with $r \leq 1$ such that $A \subset \mathbf{B}^n(c, r)$ if, and only if, the following holds. For every $T \subseteq A \cup B$ with $\operatorname{card} T \leq n + 3$, $T \cap A$ and $T \cap B$ can be strictly separated by a sphere $\mathbb{S}^{n-1}(c_T, r_T)$ with $r_T \leq 1$ such that $T \cap A \subset \mathbf{B}^n(c_T, r_T)$.*

Proof. We prove the "if" part of the theorem, the opposite direction is trivial. Proposition 4.1 guarantees the existence of the smallest sphere $\mathbb{S}^{n-1}(c', r')$ that separates A and B such that $A \subseteq \mathbf{B}^n[c', r']$. According to Lemma 4.2, there is a set $T \subseteq A \cup B$ with $\operatorname{card} T \leq n + 1$ such that $\mathbb{S}^{n-1}(c', r')$ is the smallest sphere that separates $T \cap A$ from $T \cap B$ and whose convex hull contains $T \cap A$. By the assumption, we have $r' < r_T \leq 1$. Note that Theorem 4.1 guarantees the existence of a sphere $\mathbb{S}^{n-1}(c^*, r^*)$ that strictly separates A from B and satisfies $A \subset \mathbf{B}^n(c^*, r^*)$. Since $r' < 1$, there is a sphere $\mathbb{S}^{n-1}(c, r)$ with $r \leq 1$ such that $\mathbf{B}^n[c', r'] \cap \mathbf{B}^n(c^*, r^*) \subset \mathbf{B}^n(c, r) \subset \mathbb{R}^n \setminus \left(\mathbf{B}^n(c', r') \cup \mathbf{B}^n[c^*, r^*]\right)$. This sphere clearly satisfies the conditions in Theorem 4.4. □

Problem 4.5. *Prove or disprove that Theorem 4.4 extends to compact sets.*

5. The spindle convex hull: The Theorems of Carathéodory and Steinitz

In this section we study the spindle convex hull of a set and give analogues of the well-known theorems of Carathéodory and Steinitz to spindle convexity. The theorem of Carathéodory states that the convex hull of a set $X \subset \mathbb{R}^n$ is the union of simplices with vertices in X. Steinitz's theorem is that if a point is in the interior of the convex hull of a set $X \subset \mathbb{R}^n$, then it is also in the interior of the convex hull of at most $2n$ points of X. This number $2n$ cannot be reduced as shown by the cross-polytope and its center point. We state the analogues of these two theorems in Theorem 5.7. We note that, unlike in the case of linear convexity, the analogue of the theorem of Kirchberger does not imply the analogue of the theorem of Carathéodory.

Motivated by Lemma 3.1 we make the following definition.

Definition 5.1. Let X be a set in $\mathbb{R}^n$. Then the *spindle convex hull* of X is $\mathrm{conv_s}\, X := \bigcap\{C \subseteq \mathbb{R}^n : X \subseteq C \text{ and } C \text{ is spindle convex in } \mathbb{R}^n\}$.

The straightforward proof of the following elementary property of the spindle convex hull is omitted.

Proposition 5.2. *Let $P \subset H$, where H is an affine subspace of $\mathbb{R}^n$. Assume that A is contained in a closed unit ball. Then the spindle convex hull of P with respect to H coincides with the intersection of H with the spindle convex hull of P in $\mathbb{R}^n$.*

Definition 5.3. Let $\mathbb{S}^k(c,r) \subset \mathbb{R}^n$ be a sphere such that $0 \le k \le n-1$. A set $F \subset \mathbb{S}^k(c,r)$ is *spherically convex* if it is contained in an open hemisphere of $\mathbb{S}^k(c,r)$ and for every $x, y \in F$ the shorter great-circular arc of $\mathbb{S}^k(c,r)$ connecting x with y is in F. The *spherical convex hull* of a set $X \subset \mathbb{S}^k(c,r)$ is defined in the natural way and it exists if, and only if, X is in an open hemisphere of $\mathbb{S}^k(c,r)$. We denote it by $\mathrm{Sconv}\left(X, \mathbb{S}^k(c,r)\right)$.

Remark 5.4. Carathéodory's Theorem can be stated for the sphere in the following way. If $X \subset \mathbb{S}^k(c,r)$ is a set in an open hemisphere of $\mathbb{S}^k(c,r)$, then $\mathrm{Sconv}\left(X, \mathbb{S}^k(c,r)\right)$ is the union of spherical simplices with vertices in X. The proof of this spherical equivalent of the theorem uses the central projection of the open hemisphere to $\mathbb{R}^k$.

Remark 5.5. It follows from Definition 2.1 that if $C \subset \mathbb{R}^n$ is a spindle convex set such that $C \subset \mathbf{B}^n[q]$ and $\mathrm{cr}(C) < 1$ then $C \cap \mathbb{S}^{n-1}(q)$ is spherically convex on $\mathbb{S}^{n-1}(q)$.

The following lemma describes the surface of a spindle convex hull (Figure 3).

Lemma 5.6. *Let $X \subset \mathbb{R}^n$ be a closed set such that $\mathrm{cr}(X) < 1$ and let $\mathbf{B}^n[q]$ be a closed unit ball containing X. Then*

(i) $X \cap \mathbb{S}^{n-1}(q)$ *is contained in an open hemisphere of* $\mathbb{S}^{n-1}(q)$ *and*
(ii) $\mathrm{conv_s}(X) \cap \mathbb{S}^{n-1}(q) = \mathrm{Sconv}\left(X \cap \mathbb{S}^{n-1}(q), \mathbb{S}^{n-1}(q)\right)$.

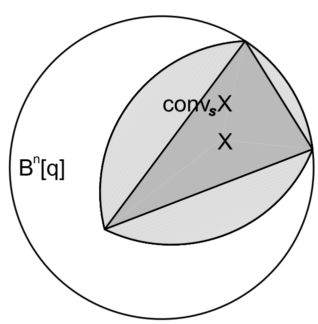

FIGURE 3

Proof. Since $\mathrm{cr}(X) < 1$, we obtain that X is contained in the intersection of two distinct closed unit balls which proves (i). Note that by (i), the right hand side $Z := \mathrm{Sconv}\left(X \cap \mathbb{S}^{n-1}(q), \mathbb{S}^{n-1}(q)\right)$ of (ii) exists. We show that the set on the left hand side is contained in Z; the other containment follows from Remark 5.5.

Suppose that $y \in \mathrm{conv_s}(X) \cap \mathbb{S}^{n-1}(q)$ is not contained in Z. We show that there is a hyperplane H through q that strictly separates Z from y. Consider an open hemisphere of $\mathbb{S}^{n-1}(q)$ that contains Z, call the spherical center of this hemisphere p. If y is an exterior point of the hemisphere, H exists. If y is on the boundary of the hemisphere, then, by moving the hemisphere a little, we find another open hemisphere that contains Z, but with respect to which y is an exterior point.

Assume that y is contained in the open hemisphere. Let L be a hyperplane tangent to $\mathbb{S}^{n-1}(q)$ at p. We project Z and y centrally from q onto L and, by the separation theorem of convex sets in L, we obtain an $(n-2)$-dimensional affine subspace T of L that strictly separates the image of Z from the image of y. Then $H := \mathrm{aff}(T \cup \{q\})$ is the desired hyperplane.

Hence, y is contained in one open hemisphere of $\mathbb{S}^{n-1}(q)$ and Z is in the other. Let v be the unit normal vector of H pointing toward the hemisphere of $\mathbb{S}^{n-1}(q)$ that contains Z. Since X is closed, its distance from the closed hemisphere containing y is positive. Hence, we can move q a little in the direction v to obtain the point q' such that $X \subset \mathbf{B}^n[q] \cap \mathbf{B}^n[q']$ and $y \notin \mathbf{B}^n[q']$. As $\mathbf{B}^n[q']$ separates X from y, the latter is not in $\mathrm{conv_s}\, X$, a contradiction. □

We prove the main result of this section.

Theorem 5.7. *Let $X \subset \mathbb{R}^n$ be a closed set.*

(i) *If $y \in \mathrm{bd}\,\mathrm{conv_s}\, X$ then there is a set $\{x_1, x_2, \dots, x_n\} \subseteq X$ such that $y \in \mathrm{conv_s}\{x_1, x_2, \dots, x_n\}$.*

(ii) *If $y \in \mathrm{int}\,\mathrm{conv_s}\, X$ then there is a set $\{x_0, x_1, \dots, x_n\} \subseteq X$ such that $y \in \mathrm{int}\,\mathrm{conv_s}\{x_0, x_1, \dots, x_n\}$.*

Proof. Assume that $\mathrm{cr}(X) > 1$. Then $\mathbf{B}[X] = \emptyset$ hence, by Helly's theorem, there is a set $\{x_0, x_1, \dots, x_n\} \subseteq X$ such that $B[\{x_0, x_1, \dots, x_n\}] = \emptyset$. By Corollary 3.2, it follows that $\mathrm{conv_s}(\{x_0, x_1, \dots, x_n\}) = \mathbb{R}^n$. Thus, (i) and (ii) follow.

Now we prove (i) for $\mathrm{cr}(X) < 1$. By Lemma 3.1, Remark 5.4 and Lemma 5.6 we obtain that $y \in \mathrm{Sconv}\left(\{x_1, x_2, \dots, x_n\}, \mathbb{S}^{n-1}(q)\right)$ for some $\{x_1, x_2, \dots, x_n\} \subset X$ and some $q \in \mathbb{R}^n$ such that $X \subseteq \mathbf{B}^n[q]$. Hence, $y \in \mathrm{conv_s}\{x_1, x_2, \dots, x_n\}$.

We prove (i) for $\mathrm{cr}(X) = 1$ by a limit argument as follows. Without loss of generality, we may assume that $X \subseteq \mathbf{B}^n[o]$. Let $X^k := (1 - \frac{1}{k})X$ for any $k \in \mathbb{Z}^+$. Let y^k be the point of $\mathrm{bd}\,\mathrm{conv_s}(X^k)$ closest to y. Thus, $\lim_{k\to\infty} y^k = y$. Clearly, $\mathrm{cr}(X^k) < 1$, hence there is a set $\{x_1^k, x_2^k, \dots, x_n^k\} \subseteq X^k$ such that $y^k \in \mathrm{conv_s}\{x_1^k, x_2^k, \dots, x_n^k\}$. By compactness, there is a sequence $0 < i_1 < i_2 < \dots$ of indices such that all the n sequences $\{x_1^{i_j} : j \in \mathbb{Z}^+\}, \{x_2^{i_j} : j \in \mathbb{Z}^+\}, \dots, \{x_n^{i_j} : j \in \mathbb{Z}^+\}$ converge. Let their respective limits be $x_1, x_2, \dots, x_n$. Since X is closed, these n points are contained in X. Clearly, $y \in \mathrm{conv_s}\{x_1, x_2, \dots, x_n\}$.

To prove (ii) for $\mathrm{cr}(X) \le 1$, suppose that $y \in \mathrm{int}\,\mathrm{conv_s}\, X$. Then let $x_0 \in X \cap \mathrm{bd}\,\mathrm{conv_s}\, X$ be arbitrary and let y_1 be the intersection of $\mathrm{bd}\,\mathrm{conv_s}\, X$ with the ray starting from x_0 and passing through y. Now, by (i), $y_1 \in \mathrm{conv_s}\{x_1, x_2, \dots, x_n\}$ for some $\{x_1, x_2, \dots, x_n\} \subseteq X$. Then clearly $y \in \mathrm{int}\,\mathrm{conv_s}\{x_0, x_1, \dots, x_n\}$. □

The same proof with a simple modification provides the analogue of the "Colorful Carathéodory Theorem" ([41] p. 199).

Theorem 5.8. *Consider $n+1$ finite point sets $X_1, \dots, X_{n+1}$ in $\mathbb{R}^n$ such that the spindle convex hull of each contains the origin. Then there is an $(n+1)$-point set $T \subset X_1 \cup \dots \cup X_{n+1}$ with $\operatorname{card}(T \cap X_i) = 1$ for each $i \in \{1, 2, \dots, n+1\}$ such that $o \in \operatorname{conv_s} T$.*

6. The Euler-Poincaré Formula for Standard Ball-polyhedra

The main result of this section is the Euler-Poincaré formula for a certain family of ball-polyhedra. However, before developing that, we present Example 6.1 to show that describing the face lattice of arbitrary ball-polyhedra is a difficult task. The example is as follows.

We construct a 4-dimensional ball-polyhedron P which has a subset F on its boundary that, according to any meaningful definition of a face for ball-polyhedra, is a 2-dimensional face. However, F is homeomorphic to a band, hence it is *not* homeomorphic to a disk. This example demonstrates that even if one finds a satisfactory definition for the face lattice of a ball-polyhedron that models the face lattice of a convex polytope, it will not lead to a CW-decomposition of the boundary of ball-polyhedra.

Example 6.1. Take two unit spheres in $\mathbb{R}^4$, $\mathbb{S}^3(p)$ and $\mathbb{S}^3(-p)$ that intersect in a 2-sphere $\mathbb{S}^2(o,r) := \mathbb{S}^3(p) \cap \mathbb{S}^3(-p)$ of $\mathbb{R}^4$. Now, take a closed unit ball $\mathbf{B}^4[q] \subset \mathbb{R}^4$ that intersects $\mathbb{S}^2(o,r)$ in a spherical cap $\mathbb{S}^2(o,r) \cap \mathbf{B}^4[q]$ of $\mathbb{S}^2(o,r)$ which is greater than a hemisphere of $\mathbb{S}^2(o,r)$, but is *not* $\mathbb{S}^2(o,r)$. Such a unit ball exists, since $r < 1$. Let the ball-polyhedron be $P := \mathbf{B}^4[p] \cap \mathbf{B}^4[-p] \cap \mathbf{B}^4[q] \cap \mathbf{B}^4[-q]$. Now, $F := \mathbb{S}^2(o,r) \cap \mathbf{B}^4[q] \cap \mathbf{B}^4[-q]$ is homeomorphic to a two-dimensional band. Also, F is a subset of the boundary of P that deserves the name of “2-face”.

Definition 6.2. Let $\mathbb{S}^l(p,r)$ be a sphere of $\mathbb{R}^n$. The intersection of $\mathbb{S}^l(p,r)$ with an affine subspace of $\mathbb{R}^n$ that passes through p is called a *great-sphere* of $\mathbb{S}^l(p,r)$. Note that $\mathbb{S}^l(p,r)$ is a great-sphere of itself. Moreover, any great-sphere is itself a sphere.

Definition 6.3. Let $P \subset \mathbb{R}^n$ be a ball-polyhedron with a family of generating balls $\mathbf{B}^n[x_1], \dots, \mathbf{B}^n[x_k]$. This family of generating balls is called *reduced* if removing any of the balls yields that the intersection of the remaining balls becomes a set larger than P. Note that, for any ball-polyhedron, distinct from a singleton, there is a unique reduced family of generating balls. A *supporting sphere* $\mathbb{S}^l(p,r)$ of P is a sphere of dimension l, where $0 \le l \le (n-1)$, which can be obtained as an intersection of some of the generating spheres of P from the reduced family of generating spheres of P such that $P \cap \mathbb{S}^l(p,r) \neq \emptyset$.

Note that the intersection of finitely many spheres in $\mathbb{R}^n$ is either empty, or a sphere, or a point.

In the same way that the faces of a convex polytope can be described in terms of supporting hyperplanes, we describe the faces of a certain class of ball-polyhedra in terms of supporting spheres.

Definition 6.4. Let P be an n-dimensional ball-polyhedron. We say that P is *standard* if for any supporting sphere $\mathbb{S}^l(p,r)$ of P the intersection $F := P \cap \mathbb{S}^l(p,r)$ is homeomorphic to a closed Euclidean ball of some dimension. We call F a *face* of P, the *dimension* of F is the dimension of the ball that F is homeomorphic to.

If the dimension is $0, 1$ or $n-1$, then we call the face a *vertex*, an *edge* or a *facet*, respectively.

Note that the dimension of F is independent of the choice of the supporting sphere containing F.

In Section 9, we present reasons why standard ball-polyhedra are natural, relevant objects of study in $\mathbb{R}^3$. Example 6.1 demonstrates the reason behind studying these objects in higher dimensions.

For the proof of the next theorem we need the following definition.

Definition 6.5. Let K be a convex body in $\mathbb{R}^n$ and $b \in \operatorname{bd} K$. Then the *Gauss image* of b with respect to K is the set of outward unit normal vectors of hyperplanes that support K at b. Clearly, it is a spherically convex subset of $\mathbb{S}^{n-1}(o)$ and its dimension is defined in the natural way.

Theorem 6.6. *Let P be a standard ball-polyhedron. Then the faces of P form the closed cells of a finite CW-decomposition of the boundary of P.*

Proof. Let $\{\mathbb{S}^{n-1}(p_1), \ldots, \mathbb{S}^{n-1}(p_k)\}$ be the reduced family of generating spheres of P. The *relative interior* (resp., the *relative boundary*) of an m-dimensional face F of P is defined as the set of those points of F that are mapped to $\mathbf{B}^m(o)$ (resp., $\mathbb{S}^{m-1}(o)$) under any homeomorphism between F and $\mathbf{B}^m[o]$.

For every $b \in \operatorname{bd} P$ define the following sphere

$$S(b) := \bigcap \{\mathbb{S}^{n-1}(p_i) : p_i \in \mathbb{S}^{n-1}(b), i \in \{1, \ldots, k\}\}. \tag{6.1}$$

Clearly, $S(b)$ is a support sphere of P. Moreover, if $S(b)$ is an m-dimensional sphere, then the face $F := S(b) \cap P$ is also m-dimensional as b has an m-dimensional neighbourhood in $S(b)$ that is contained in F. This also shows that b belongs to the relative interior of F. Hence, the union of the relative interiors of the faces covers $\operatorname{bd} P$.

We claim that every face F of P can be obtained in this way, i.e., for any relative interior point b of F we have $F = S(b) \cap P$. Clearly, $F \supseteq S(b) \cap P$, as the support sphere of P that intersects P in F contains $S(b)$. It is sufficient to show that F is at most m-dimensional. This is so, because the Gauss image of b with respect to P is at least $(n-m-1)$-dimensional, since the Gauss image of b with respect to $\bigcap \{\mathbf{B}^n[p_i] : p_i \in \mathbb{S}^{n-1}(b), i \in \{1, \ldots, k\}\} \supseteq P$ is $(n-m-1)$-dimensional.

The above argument also shows that no point $b \in \operatorname{bd} P$ belongs to the relative interior of more than one face. Moreover, if $b \in \operatorname{bd} P$ is on the relative boundary of the face F then $S(b)$ is clearly of smaller dimension than F. Hence, b belongs to the relative interior of a face of smaller dimension. This concludes the proof of the theorem. $\square$

Corollary 6.7. *The reduced family of generating balls of any standard ball-polyhedron P in $\mathbb{R}^n$ consists of at least $n+1$ unit balls.*

Proof. Since the faces form a CW-decomposition of the boundary of P, it has a vertex v. The Gauss image of v is $(n-1)$-dimensional. So, v belongs to at least n generating spheres from a reduced family. We denote the centers of those spheres by $x_1, x_2, \ldots, x_n$. Let $H := \operatorname{aff}\{x_1, x_2, \ldots, x_n\}$. Then $\mathbf{B}[\{x_1, x_2, \ldots, x_n\}]$

is symmetric about H. Let σ_H be the reflection of $\mathbb{R}^n$ about H. Then $S := \mathbb{S}(x_1) \cap \mathbb{S}(x_2) \cap \cdots \cap \mathbb{S}(x_n)$ contains the points v and $\sigma_H(v)$, hence S is a sphere, not a point. Since P is a standard ball-polyhedron, there is a unit-ball $\mathbf{B}[x_{n+1}]$ in the reduced family of generating balls of P that does not contain S. □

Corollary 6.8. *Let Λ be the set containing all faces of a standard ball-polyhedron $P \subset \mathbb{R}^n$ and the empty set and P itself. Then Λ is a finite bounded lattice with respect to ordering by inclusion. The atoms of Λ are the vertices of P and Λ is atomic, i.e., for every element $a \in \Lambda$ with $a \neq \emptyset$ there is a vertex x of P such that $x \in a$.*

Proof. First, we show that the intersection of two faces F_1 and F_2 is another face (or the empty set). The intersection of the two supporting spheres that intersect P in F_1 and F_2 is another supporting sphere of P, say $\mathbb{S}^l(p, r)$. Then $\mathbb{S}^l(p, r) \cap P = F_1 \cap F_2$ is a face of P. From this the existence of a unique maximum common lower bound (i.e. an *infimum*) for F_1 and F_2 follows.

Moreover, by the finiteness of Λ, the existence of a unique infimum for any two elements of Λ implies the existence of a unique minimum common upper bound (i.e., a *supremum*) for any two elements of Λ, say C and D, as follows. The supremum of C and D is the infimum of all the (finitely many) elements of Λ that are above C and D.

Vertices of P are clearly atoms of Λ. Using Theorem 6.6 and induction on the dimension of the face it is easy to show that every face is the supremum of its vertices. □

Corollary 6.9. *A standard ball-polyhedron P in $\mathbb{R}^n$ has k-dimensional faces for every $0 \leq k \leq n-1$.*

Proof. We use an inductive argument on k, where we go from $k = n-1$ down to $k = 0$. Clearly, P has facets. A k-face F of P is homeomorphic to $\mathbf{B}^k[o]$, hence its relative boundary is homeomorphic to $\mathbb{S}^{k-1}$, if $k > 0$. Since the $(k-1)$-skeleton of P covers the relative boundary of F, P has $(k-1)$-faces. □

Corollary 6.10. *(Euler-Poincaré Formula) For any standard n-dimensional ball-polyhedron P we have:*

$$1 + (-1)^{n+1} = \sum_{i=0}^{n-1} (-1)^i f_i(P),$$

where $f_i(P)$ denotes the number of i-dimensional faces of P.

Proof. It follows from the above theorem and the fact that a ball-polyhedron in $\mathbb{R}^n$ is a convex body, hence its boundary is homeomorphic to $\mathbb{S}^{n-1}(o)$. □

Corollary 6.11. *Let $n \geq 3$. Any standard ball-polyhedron P is the spindle convex hull of its $(n-2)$-dimensional faces. Furthermore, no standard ball-polyhedron is the spindle convex hull of its $(n-3)$-dimensional faces.*

Proof. For the first statement, it is sufficient to show that the spindle convex hull of the $(n-2)$-faces contains the facets. Let p be a point on the facet, $F = P \cap \mathbb{S}^{n-1}(q)$. Take any great circle C of $\mathbb{S}^{n-1}(q)$. Since F is spherically convex on $\mathbb{S}^{n-1}(q)$, $C \cap F$

is a unit circular arc of length less than π. Let $r, s \in \mathbb{S}^{n-1}(q)$ be the two endpoints of $C \cap F$. Then r and s belong to the relative boundary of F. Hence, by Theorem 6.6, r and s belong to an $(n-2)$-face. Clearly, $p \in \mathrm{conv_s}\{r, s\}$.

The proof of the second statement follows. By Corollary 6.9 we can choose a relative interior point p of an $(n-2)$-dimensional face F of P. Let q_1 and q_2 be the centers of the generating balls of P from a reduced family such that $F := \mathbb{S}^{n-1}(q_1) \cap \mathbb{S}^{n-1}(q_2) \cap P$. Clearly, $p \notin \mathrm{conv_s}((\mathbf{B}[q_1] \cap \mathbf{B}[q_2]) \setminus \{p\}) \supseteq \mathrm{conv_s}(P \setminus \{p\})$. □

7. A Counterexample to a Conjecture of Maehara in Dimensions at Least Four

Helly's Theorem, as stated for convex sets, adds nothing to the current theory. However, the following result of Maehara [38] is very suggestive.

Theorem 7.1. *Let $\mathfrak{F}$ be a family of at least $n+3$ distinct $(n-1)$-spheres in $\mathbb{R}^n$. If any $n+1$ of the spheres in $\mathfrak{F}$ have a point in common, then all of the spheres in $\mathfrak{F}$ have a point in common.*

Maehara points out that neither $n+3$ nor $n+1$ can be reduced. First, we prove a variant of Theorem 7.1.

Theorem 7.2. *Let $\mathfrak{F}$ be a family of $(n-1)$-spheres in $\mathbb{R}^n$, and k be an integer such that $0 \leq k \leq n-1$. Suppose that $\mathfrak{F}$ has at least $n-k$ members and that any $n-k$ of them intersect in a sphere of dimension at least $k+1$. Then they all intersect in a sphere of dimension at least $k+1$. Furthermore, $k+1$ cannot be reduced to k.*

Proof. Amongst all the intersections of any $n-k$ spheres from the family, let S be such an intersection of minimal dimension. By assumption, S is a sphere of dimension at least $k+1$. Now, one of the $n-k$ spheres is redundant in the sense that S is contained entirely in this sphere. After discarding this redundant sphere, S is now the intersection of only $(n-k)-1$ members of the family, but any $n-k$ members intersect in a sphere of dimension at least $k+1$. So, the remaining members of the family intersect S. Since the dimension of S is minimal, S is contained in these members. In particular, $\bigcap \mathfrak{F} = S$.

Fixing n and k, $0 \leq k \leq n-1$, we show that $k+1$ cannot be reduced to k by considering a regular n-simplex in $\mathbb{R}^n$, with circumradius one, and a family of $n+1$ unit spheres centered at the vertices of this simplex. The intersection of any $n-k$ of them is a sphere of dimension at least k, but the intersection of all of them is a single point which, as we recall, is not a sphere in the current setting. □

Maehara [38] conjectured the following stronger version of Theorem 7.1.

Conjecture 7.3. *Let $\mathfrak{F}$ be a family of at least $n+2$ distinct $(n-1)$-dimensional unit spheres in $\mathbb{R}^n$, where $n \geq 3$. Suppose that any $n+1$ spheres in $\mathfrak{F}$ have a point in common. Then all the spheres in $\mathfrak{F}$ have a point in common.*

After Proposition 3 in [38], Maehara points out the importance of the condition $n \geq 3$ by showing the following statement, also known as *Ţiţeica's theorem* (sometimes called *Johnson's theorem*). This theorem was found by the Romanian mathematician, G. Ţiţeica in 1908 (for historical details, see also [4], and [33] p. 75).

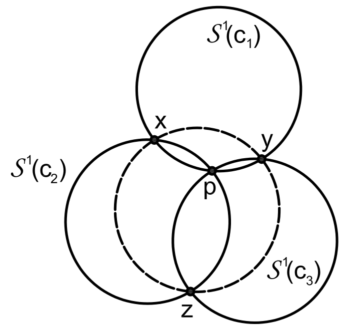

FIGURE 4

Proposition 7.4. *Let $\mathbb{S}^1(c_1)$, $\mathbb{S}^1(c_2)$ and $\mathbb{S}^1(c_3)$ be unit circles in $\mathbb{R}^2$ that intersect in a point p (see Figure 4). Let $\{x,p\} := \mathbb{S}^1(c_1) \cap \mathbb{S}^1(c_2)$, $\{y,p\} := \mathbb{S}^1(c_1) \cap \mathbb{S}^1(c_3)$ and $\{z,p\} := \mathbb{S}^1(c_2) \cap \mathbb{S}^1(c_3)$. Then x, y and z lie on a unit circle.*

In the remaining part of this section, we show that Conjecture 7.3 is false for $n \geq 4$. To construct a suitable family $\mathfrak{F}$ of unit spheres, we need the following lemma.

Lemma 7.5. *The following are equivalent.*

(i) *There is an n-simplex $P \subset \mathbb{R}^n$ with circumsphere $\mathbb{S}^{n-1}(o,R)$ and a sphere $\mathbb{S}^{n-1}(x_1,r)$ tangent to all facet-hyperplanes of P such that either $R^2 - 2rR = d^2$ or $R^2 + 2rR = d^2$ holds, where $d := \|x_1 - o\|$.*

(ii) *There is a family of $n+2$ distinct $(n-1)$-dimensional unit spheres in $\mathbb{R}^n$ such that any $n+1$ of them have a common point but not all of them have a common point.*

Proof. First, we show that (ii) follows from (i). Observe that, from $R^2 - 2rR = d^2$, we have $R > d$, which implies that $x_1 \in \mathbf{B}^n(o,R)$. Similarly, if $R^2 + 2rR = d^2$, then $x_1 \notin \mathbf{B}^n[o,R]$. Thus, $x_1 \notin \mathbb{S}^{n-1}(o,R)$. Since $\mathbb{S}^{n-1}(x_1,r)$ is tangent to every facet-hyperplane of P, x_1 is not contained in any of these hyperplanes.

Consider the inversion f with respect to $\mathbb{S}^{n-1}(x_1,r)$. Let a_i be a vertex of P and H_i denote the facet-hyperplane of P that does not contain a_i, for $i = 2,3,\ldots,n+2$. We set $\mathbb{S}^{n-1}(c_i,r_i) := f(H_i)$, $x_i := f(a_i)$, for $i = 2,3,\ldots,n+2$. Finally, $\mathbb{S}^{n-1}(c_1,r_1) := f(\mathbb{S}^{n-1}(o,R))$.

Let $2 \leq i \leq n+2$. Since H_i is tangent to $\mathbb{S}^{n-1}(x_1,r)$, $S^{n-1}(c_i,r_i)$ is a sphere tangent to $\mathbb{S}^{n-1}(x_1,r)$ and contains x_1. Hence, the radius of $\mathbb{S}^{n-1}(c_i,r_i)$ is $r_i = \frac{r}{2}$. We show that also $r_1 = \frac{r}{2}$. If $x_1 \in \mathbf{B}^n(o,R)$, then, using the definition of inversion and the equations in (i), we have

$$2r_1 = \operatorname{diam} \mathbb{S}^{n-1}(c_1,r_1) = \frac{r^2}{R+d} + \frac{r^2}{R-d} = \frac{2r^2R}{R^2-d^2} = r. \tag{7.1}$$

If $x_1 \notin \mathbf{B}^n[o,R]$, then

$$2r_1 = \operatorname{diam} \mathbb{S}^{n-1}(c_1,r_1) = \frac{r^2}{d-R} - \frac{r^2}{d+R} = \frac{2r^2R}{d^2-R^2} = r. \tag{7.2}$$

Let $\mathfrak{F} := \{\mathbb{S}^{n-1}(c_i, \frac{r}{2}) : i = 1, \ldots, n+2\}$. Observe that $x_1 \in \mathbb{S}^{n-1}(c_i, \frac{r}{2})$, for every $i \neq 1$, and that $x_i \in \mathbb{S}^{n-1}(c_1, \frac{r}{2}) \cap \mathbb{S}^{n-1}(c_j, \frac{r}{2})$ for every $j \neq i$. So, $\mathfrak{F}$ is a family of $n+2$ spheres of radius $\frac{r}{2}$ such that any $n+1$ of them have a common point.

We assume $y \in \bigcap \mathfrak{F}$. Then $y \neq x_1$, since $x_1 \notin \mathbb{S}^{n-1}(o, R)$ and thus $x_1 \notin \mathbb{S}^{n-1}(c_1, \frac{r}{2}) = f(\mathbb{S}^{n-1}(o, R))$. Hence, $z := f(y) = f^{-1}(y)$ exists. The point z, according to our assumption, is contained in every facet-hyperplane of P and also in its circumsphere, a contradiction. So, $\mathfrak{F}' := \{\mathbb{S}^{n-1}(\frac{2}{r} \cdot c_i) : i = 1, \ldots, n+2\}$ is a family of unit sphere that satisfies (ii).

A similar argument shows that (ii) implies (i). □

Theorem 7.6. *For any $n \geq 4$, there exists a family of $n+2$ distinct $(n-1)$-dimensional unit spheres in $\mathbb{R}^n$ such that any $n+1$, but not all, of them have a common point.*

Proof. We use Lemma 7.5 and construct a simplex P and a sphere $\mathbb{S}^{n-1}(x_1, r)$ such that they satisfy Lemma 7.5 (i). We set $m := n-1$.

Consider a line L containing the origin o and a hyperplane H which is orthogonal to L and is at a given distance $t \in (0,1)$ from o. Let u denote the intersection point of L and H. We observe that $t = \|u\|$ and let $b := \frac{1}{t}u$. Then $b \in \mathbb{S}^{n-1}(o, 1)$. Let F be a regular m-simplex in H whose circumsphere is $S^{n-1}(o,1) \cap H$. Thus, u is the center of F and the circumsphere of $P := \mathrm{conv}(F \cup \{b\})$ is $\mathbb{S}^{n-1}(o, 1)$. Clearly, there is a unique sphere $\mathbb{S}^{n-1}(c, r)$ tangent to every facet-hyperplane of P such that $c \in L$ and $c \notin P$. We set $d := \|c\|$.

Our aim is to show that, with a suitable choice of t, P and $\mathbb{S}^{n-1}(c, r)$ satisfy Lemma 7.5 (i). So, in the remaining part of the proof, we calculate $g_m(t) := d(t)^2 + 2r(t) - 1$ as a function of t, and show that this function has a root on the interval $(0,1)$, for $m \geq 3$. We note that if $g_m(t) = 0$, for some t, then P and $\mathbb{S}^{n-1}(c, r)$ satisfy the first equality in Lemma 7.5 (i) for $R = 1$ and $x_1 = c$.

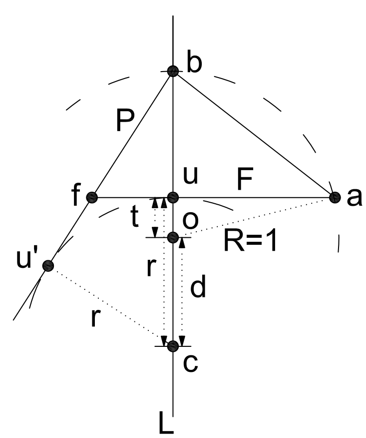

FIGURE 5

Consider a vertex a of F and the center f of the facet of F that does not contain a (see Figure 5). Then $\|b-u\| = 1-t$ and $\|a-u\| = \sqrt{1-t^2}$. Since in an m-dimensional regular simplex the distance of the center from any vertex is m times as large as the distance of the center and any facet-hyperplane, we have $\|u-f\| = \frac{\sqrt{1-t^2}}{m}$. We observe that $\mathbb{S}^{n-1}(c,r)$ is tangent to the facet-hyperplane H_a of P that does not contain a. Let u' denote the intersection point of $\mathbb{S}^{n-1}(c,r)$ and H_a. Clearly, u', f and b are collinear and $\|u-c\| = \|u'-c\| = r$. Furthermore, the two triangles $\operatorname{conv}\{c,u',b\}$ and $\operatorname{conv}\{f,u,b\}$ are co-planar and similar. Hence,

$$\frac{\|b-f\|}{\|b-c\|} = \frac{\|u-f\|}{\|u'-c\|}. \tag{7.3}$$

We have that $\|b-f\| = \sqrt{(1-t)^2 + \frac{1-t^2}{m^2}}$, $\|b-c\| = 1+r-t$, $\|u'-c\| = r$ and $\|u-f\| = \frac{\sqrt{1-t^2}}{m}$. So, we have an equation for r which yields

$$r = \frac{\sqrt{1+t}}{m^2}\left(\sqrt{m^2+1-(m^2-1)t} + \sqrt{1+t}\right). \tag{7.4}$$

Observe that $d = |r-t|$. From this and (7.4), we have

$$g_m(t) = \left(\frac{\sqrt{1+t}}{m^2}\left(\sqrt{m^2+1-(m^2-1)t} + \sqrt{1+t}\right) - t\right)^2 + \tag{7.5}$$

$$+\frac{2\sqrt{1+t}}{m^2}\left(\sqrt{m^2+1-(m^2-1)t} + \sqrt{1+t}\right) - 1.$$

Let us observe that $g_3\left(\frac{1}{2}\right) = 0$, and that $g_m(0) < 0$ and $g_m(1) > 0$, for every $m > 3$. Since g_m is continuous on $[0,1]$, g_m has a root in the interval $(0,1)$, for all $m \geq 3$. Thus, for every $n \geq 4$, we have found a simplex P and a sphere $\mathbb{S}^{n-1}(c,r)$ that satisfy Lemma 7.5 (i). □

8. Monotonicity of the Inradius, the Minimal Width and the Diameter of a Ball-polyhedron under a Contraction of the Centers

One of the best known open problems of discrete geometry is the Kneser–Poulsen conjecture. It involves unions (resp., intersections) of finitely many balls in $\mathbb{R}^n$ and states that, under arbitrary contraction of the center points, the volume of the union (resp., intersection) does not increase (resp., decrease). Recently, the conjecture has been proved in the plane by K. Bezdek and R. Connelly in [8] and it is open for $n \geq 3$. The interested reader is referred to the papers [9], [17], [18] and [19] for further information on this problem. In this section, we investigate similar problems. Namely, we apply an arbitrary contraction to the center points of the generating balls of a ball-polyhedron, and ask whether the inradius (resp., the circumradius, the diameter and the minimum width) can decrease.

Theorem 8.1. *Let $X \subset \mathbb{R}^n$ be a finite point set contained in a closed unit ball of $\mathbb{R}^n$ and let Y be an arbitrary contracted image of X in $\mathbb{R}^n$. Then the inradius of $\mathbf{B}[Y]$ is at least as large as the inradius of $\mathbf{B}[X]$.*

Proof. First, observe the following fact. If r denotes the inradius of $\mathbf{B}[X]$ (that is, the radius of the largest ball contained in $\mathbf{B}[X]$) and R denotes the circumradius of X (that is, the radius of the smallest ball containing X), then $r + R = 1$. Second, we recall the following monotonicity result (see for example [2]). The circumradius of X is at least as large as the circumradius of Y. From these two observations our theorem follows immediately. □

The following construction (see Figure 6) shows that both the diameter and the circumradius of an intersection of unit disks in the plane can decrease under a continuous contraction of the centers. We describe the construction in terms of polar coordinates. The first coordinate of a vector (that is, a point) is the Euclidean distance of the point from the origin, the second is the oriented angle of the vector and the oriented x-axis.

Let $c_1 := (0.5, \frac{\pi}{3}), c_2 := (0.5, -\frac{\pi}{3})$, $c_1' := (0.5, \frac{\pi}{4})$ and $c_2' := (0.5, -\frac{\pi}{4})$. Let X be the set of centers $X := \{o, c_1, c_2\}$, and $Y := \{o, c_1', c_2'\}$. Clearly, Y is a continuous contraction of X. However, a simple computation shows that both the diameter and the circumradius of $\mathbf{B}[Y]$ is smaller than that of $\mathbf{B}[X]$.

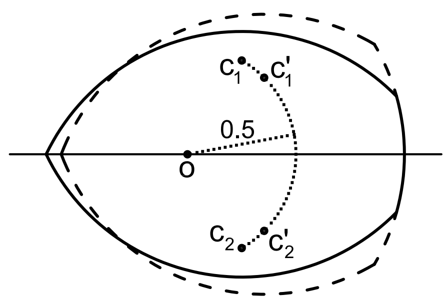

Figure 6

A similar construction shows that the minimal width of an intersection of unit disks on the plane can decrease under a continuous contraction of the centers.

Let $c_1 := (0.8, \frac{\pi}{10}), c_2 := (0.8, -\frac{\pi}{10})$ and $c_1' := c_2' := (0.8, 0)$. Let X be the set of centers $X := \{o, c_1, c_2\}$, and $Y := \{o, c_1'\}$. Clearly, Y is a continuous contraction of X. However, a simple computation shows that the minimum width of $\mathbf{B}[Y]$ is smaller than that of $\mathbf{B}[X]$.

9. The Problem of Finding an Analogue to a Theorem of Steinitz for Ball-polyhedra in $\mathbb{R}^3$

One may define the vertices, edges and faces of *any* ball-polyhedron P in $\mathbb{R}^3$ in the natural way, as in [11]. Henceforth in this section, we assume that P is a ball-polyhedron in $\mathbb{R}^3$ with at least three balls in the reduced family of generating balls of P. Then P has faces that are spherically convex on the generating sphere of P that they belong to. Edges of P are circular arcs of radii less than one (not full circles) ending in vertices. Moreover, every vertex is adjacent to at least three edges and at least three faces of P.

In this paper, a *graph* is non-oriented and has finitely many vertices and edges. A graph is 2-*connected* (resp., 3-*connected*), if it has at least three (resp., four) vertices and deleting any vertex (resp., any two vertices) yields a connected graph. A graph is *simple* if it contains no loop (an edge with identical end-points) and no parallel edges (two edges with the same two end-points).

The edge-graph of P contains no loops, but may contain parallel edges. Moreover, it is 2-connected and planar.

By the construction given in [11], there is a ball-polyhedron P in $\mathbb{R}^3$ with two faces meeting along a series of edges. The family of vertices, edges and faces of P (together with the empty set and P itself) do *not* form an algebraic lattice with respect to containment.

Remark 9.1. Clearly, a ball-polyhedron P in $\mathbb{R}^3$ is standard if, and only if, the vertices, edges and faces of P (together with $\emptyset$ and P) form an algebraic lattice with respect to containment.

It follows that for any two faces F_1 and F_2 of a standard ball-polyhedron P in $\mathbb{R}^3$, the intersection $F_1 \cap F_2$ is either empty or one vertex or one edge of P.

In what follows, we investigate whether an analogue of the famous theorem of Steinitz regarding the edge-graph of convex polyhedra holds for standard ball-polyhedra in $\mathbb{R}^3$. Recall that this theorem states that a graph is the edge-graph of some convex polyhedron in $\mathbb{R}^3$ if, and only if, it is simple, planar and 3-connected.

Claim 9.2. *Let $\bar{P}$ be a convex polyhedron in $\mathbb{R}^3$ with the property that every face of $\bar{P}$ is inscribed in a circle. Let Λ denote the face lattice of $\bar{P}$.*
Then there is a sequence $\{P_1, P_2, \dots\}$ of standard ball-polyhedra in $\mathbb{R}^3$ with face lattices isomorphic to Λ such that $\lim\limits_{k\to\infty} kP_k = \bar{P}$ in the Hausdorff metric.

Proof. Let $\mathcal{F}$ denote the set of the (two-dimensional) faces of $\bar{P}$; let c_F denote the circumcenter, r_F the circumradius, and n_F the inner unit normal vector of the face $F \in \mathcal{F}$. We define P'_k as the following intersection of closed balls of radius k.

$$P'_k := \bigcap_{F\in\mathcal{F}} \mathbf{B}\left[c_F + \left(\sqrt{k^2 - r_F^2}\right) n_F, k\right] \tag{9.1}$$

Clearly, $P_k := \frac{1}{k} P'_k$ is a ball-polyhedron in $\mathbb{R}^3$. The terms face, vertex and edge of P'_k are defined in a natural way, exactly as for ball-polyhedra in $\mathbb{R}^3$. It is easy to see that every vertex of $\bar{P}$ is a vertex of P'_k. Moreover, a simple approximation argument shows that, for sufficiently large k, P'_k is a standard ball-polyhedron in $\mathbb{R}^3$ with a face lattice that is isomorphic to Λ. Clearly, $\lim\limits_{k\to\infty} P'_k = \bar{P}$. Now, we take a $k_0 \in \mathbb{Z}^+$ such that the face lattice of P_{k_0} is isomorphic to Λ and we replace the (finitely many) elements of the sequence $\{P_1, P_2, \dots\}$ that have a face lattice non-isomorphic to Λ by P_{k_0}. The sequence of ball-polyhedra obtained this way satisfies the requirements of the claim. $\square$

Corollary 9.3. *If Λ is a graph that can be realized as the edge-graph of a convex polyhedron $\bar{P}$ in $\mathbb{R}^3$, with the property that every face of $\bar{P}$ is inscribed in a circle, then Λ can be realized as the edge-graph of a standard ball-polyhedron in $\mathbb{R}^3$.*

We note that *not* every 3-connected, simple, planar graph can be realized as the edge-graph of a convex polyhedron in $\mathbb{R}^3$ with all faces having a circumcircle. See pp. 286-287 in [29].

Claim 9.4. *The edge-graph of any standard ball-polyhedron P in $\mathbb{R}^3$ is simple, planar and a 3-connected graph.*

Proof. Let G be the edge-graph of P. It is clearly planar. By a simple case analysis, one obtains that G has at least four vertices. First, we show that G is simple. Clearly, there are no loops in G.

Assume that two vertices v and w are connected by at least two edges e_1 and e_2. From the reduced family of generating spheres of P, let Q be the intersection of those that contain e_1 or e_2. Clearly, $Q = \{v, w\}$ which contradicts Remark 9.1.

Now, we show that G is 3-connected. Let v and w be two arbitrary vertices of G. Take two vertices s and t of G, both different from v and w. We need to show that there is a path between s and t that avoids v and w. We define two subgraphs of G, C_v and C_w as follows. Let C_v (resp., C_w) be the set of vertices of P that lie on the same face as v (resp., w) and are distinct from v (resp., w). Let an edge e of G connecting two points of C_v (resp., C_w) be an edge of C_v (resp., C_w) if, and only if, e is an edge of a face that contains C_v (resp., C_w).

By Remark 9.1, C_v and C_w are cycles. Moreover, v and w are incident to at most two faces in common.

Case 1: v and w are not incident to any common face; that is, $v \notin C_w$ and $w \notin C_v$. Since G is connected, there is a path connecting s and t. We may assume that this path does not pass through any vertex twice. Assume that this path includes v by passing through two edges, say e_1 and e_2 that share v as a vertex. Let the vertex of e_1 (resp., e_2) different from v be v_1 (resp., v_2). Clearly, $v_1, v_2 \neq w$ and they are contained in C_v, which is a cycle. Thus, the edges e_1 and e_2 in the path may be replaced by a sequence of edges of C_v that connects v_1 and v_2. If the path passes through w then it may be modified in the same manner to avoid w, thus we obtain the desired path.

Case 2: v and w are incident to one or two common faces. Let C be the subgraph of $C_v \cup C_w$ spanned by the union of vertices of C_v and C_w erasing v and w. Since P is a standard ball-polyhedron, C is a cycle. Similarly to the preceding argument, any path from s to t may be modified such that it does not pass through v and w using edges of C. □

We pose the following questions.

Problem 9.5. *Prove or disprove that every 2-connected planar graph with no loops is the edge-graph of a ball-polyhedron in $\mathbb{R}^3$.*

Problem 9.6. *Prove or disprove that every 3-connected, simple, planar graph is the edge-graph of a standard ball-polyhedron in $\mathbb{R}^3$.*

10. Ball-polyhedra in $\mathbb{R}^3$ with Symmetric Sections

Let $K \subset \mathbb{R}^3$ be a convex body with the property that any planar section of K is axially symmetric. The first named author conjectured (see [28]) that, in this case, K is either a body of revolution or an ellipsoid. A remarkable result related to the conjecture is due to Montejano [43]. He showed that if $K \subset \mathbb{R}^3$ is a convex body with the property that, for some point $p \in \operatorname{int} K$, every planar section of K through p is axially symmetric, then there is a planar section of K through p which is a disk. Unfortunately, the claim of Ódor (see [28]) that he proved this conjecture turned out to be too optimistic, his approach was found incomplete. The following theorem shows that the conjecture holds for the class of ball-polyhedra in $\mathbb{R}^3$ with the weaker condition in Montejano's result.

Theorem 10.1. *Let P be a ball-polyhedron in $\mathbb{R}^3$ and $p \in \operatorname{int} P$ with the property that any planar section of P through p is axially symmetric.*
Then P is either one point or a unit ball or the intersection of two unit balls.

Proof. Assume the contrary; that is, that the minimum number of unit balls needed to generate P is $k \geq 3$. Let the reduced family of generating unit spheres be $\mathbb{S}^{n-1}(c_1), \ldots, \mathbb{S}^{n-1}(c_k)$. Since P is generated by at least three unit balls, it has an edge. Let q_1 be any point in the relative interior of some edge e of P and let q_2 be a point in the relative interior of a facet F of P that does *not* contain e. By slightly moving q_1 on e and q_2 on F, we may assume that the plane H spanned by p, q_1 and q_2 does not contain any vertex of P and is neither parallel nor perpendicular to the line passing through c_i and c_j, for any $1 \leq i < j \leq k$.

Since F does not contain e, it follows that H intersects at least three edges of P. Thus, $H \cap P$ is a convex planar figure in H bounded by a closed curve that is a series of at least three circular arcs. Moreover, since H is neither parallel nor perpendicular to the line passing through c_i and c_j, for any $1 \leq i < j \leq k$, the radii of these arcs are pairwise distinct. This clearly contradicts our assumptions on P as such a planar figure is *not* axially symmetric. ☐

11. Illumination of Ball-polyhedra and Sets of Constant Width in $\mathbb{R}^3$

We consider the illumination problem for ball-polyhedra in $\mathbb{R}^3$ that contain the centers of their generating balls. We prove that such bodies are illuminated by three pairs of opposite directions that are mutually orthogonal. The method we use naturally extends to bodies obtained as intersections of infinitely many balls, hence it yields a proof of the known theorem (Lassak [37], and Weissbach [50]) that any set of constant width in $\mathbb{R}^3$ is illuminated by six light sources. For a survey on illumination see [39] and the new paper of K. Bezdek [6]; bodies of constant width are discussed in the surveys [15] and [40], see also the monograph [52].

Definition 11.1. Let $K \in \mathbb{R}^n$ be a convex body and $z \in \operatorname{bd} K$ a point on its boundary. We say that the direction $u \in \mathbb{S}^{n-1}(o)$ *illuminates* K at z if the ray $\{z + tu : t > 0\}$ intersects the interior of K. Furthermore, K is *illuminated* at $z \in \operatorname{bd} K$ *by a set* $A \subseteq \mathbb{S}^{n-1}(o)$ *of directions* if at least one direction from A illuminates K at x. Then K is illuminated by $A \subset \mathbb{S}^{n-1}(o)$ if K is illuminated by A at every boundary point of K.

Let K be a convex body and $z \in \operatorname{bd} K$. We denote by $G(z)$ the set of inward unit normal vectors of hyperplanes that support K at z. We note that $-G(z)$ is the Gauss image of z.

We denote the open hemisphere of $\mathbb{S}^{n-1}(o)$ with center $u \in \mathbb{S}^{n-1}(o)$ by $D(u)$ and its relative boundary (a great sphere of $\mathbb{S}^{n-1}(o)$) by $C(u)$. Then u, which is in $\mathbb{S}^{n-1}(o)$, illuminates K at $z \in \operatorname{bd} K$ if, and only if, $G(z) \subset D(u)$. This leads to the following observation which is an easy special case of the Separation Lemma in [7].

Observation 11.2. *The pair of directions $\{\pm u\} \subset \mathbb{S}^{n-1}(o)$ illuminates the convex body $K \subset \mathbb{R}^n$ at $z \in \operatorname{bd} K$ if, and only if, $G(z) \cap C(u) = \emptyset$.*

Now, we are ready to state the main result of this section.

Theorem 11.3. *Let $X \subset \mathbb{R}^3$ be a set of diameter at most one, and let $u \in \mathbb{S}^2(o)$ be given.*
Then there exist v and w in $\mathbb{S}^{n-1}(o)$ such that u, v and w are pairwise orthogonal, and the body $K := \mathbf{B}[X]$ is illuminated by the six directions $\{\pm u, \pm v, \pm w\}$.

Using the fact that a closed set $X \subset \mathbb{R}^n$ is of constant width one if, and only if, $\mathbf{B}[X] = X$, cf. [24], we obtain the following corollary.

Corollary 11.4. *Any set of constant width can be illuminated by three pairwise orthogonal pairs of opposite directions, one of which can be chosen arbitrarily.*

To prove the theorem we need the following lemma.

Lemma 11.5. *Let $X \subset \mathbb{R}^n$ be a set of diameter at most one and $z \in \operatorname{bd} \mathbf{B}[X]$. Then $G(z) \subset \mathbb{S}^{n-1}(o)$ is of spherical diameter not greater than $\frac{\pi}{3}$.*

Proof. We may assume that X is closed. It is easy to see that
$G(z) = \operatorname{Sconv}\left(\mathbb{S}^{n-1}(z) \cap X, \mathbb{S}^{n-1}(z)\right) - z$. So, we have to show that if $x_1, x_2 \in \mathbb{S}^{n-1}(z) \cap X$, then $\sphericalangle(x_1 z x_2) \leq \frac{\pi}{3}$. It is true, since the Euclidean isosceles triangle $\operatorname{conv}\{x_1, z, x_2\}$ has two legs $[z, x_1]$ and $[z, x_2]$ of length one, and base $[x_1, x_2]$ of length at most one, because the diameter of X is at most one. This proves the lemma. □

Proof of Theorem 11.3. Let the direction $u \in \mathbb{S}^2(o)$ be given. We will call u vertical, and directions perpendicular to u horizontal. We pick two pairwise orthogonal, horizontal directions, v_1 and v_2. Assume that the six directions $\{\pm u, \pm v_1, \pm v_2\} \subset \mathbb{S}^2$ do not illuminate K. According to Observation 11.2, there is a point $z \in \operatorname{bd} K$ such that $G(z)$ intersects each of the three great circles of $\mathbb{S}^2(o)$: $C(u), C(v_1)$ and $C(v_2)$. We choose three points of $G(z)$, one on each great circle: $y_0 \in G(z) \cap C(u), y_1 \in G(z) \cap C(v_1)$ and $y_2 \in G(z) \cap C(v_2)$. Note that each of the three great circles is dissected into four equal arcs (of length $\frac{\pi}{4}$) by the two other great circles.

By Lemma 11.5, $G(z) \subset \mathbb{S}^2(o)$ is a spherically convex set of spherical diameter at most $\frac{\pi}{3}$. However, $y_0, y_1, y_2 \in G(x)$, so the generalization of Jung's theorem for spherical space by Dekster [22] shows that y_0, y_1 and y_2 are the mid-points of the great circular arcs mentioned above. So, the only way that a point $z \in \operatorname{bd} K$ is not illuminated by any of the six directions $\{\pm u, \pm v_1, \pm v_2\}$ is the following. The set $G(z)$ contains a spherical equilateral triangle of spherical side length $\frac{\pi}{3}$ and the vertices of this spherical triangle lie on $C(u), C(v_1)$ and $C(v_2)$, respectively.

Furthermore, each vertex is necessarily the mid-point of the quarter arc of the great circle on which it lies, and $G(z)$ does not intersect either of the three great circles in any other point.

Since the set $\{G(z) : z \in \operatorname{bd} K\}$ is a tiling of $\mathbb{S}^2(o)$, there are only finitely many boundary points $z \in \operatorname{bd} K$ such that $G(z)$ contains an equilateral triangle of side length $\frac{\pi}{3}$ that has a vertex on $C(u)$. We call these tiles blocking tiles.

Now, by rotating v_1 and v_2 together in the horizontal plane, we can easily avoid all the blocking tiles; that is, we can find a rotation R about the line spanned by u such that none of the blocking tiles has a vertex on both circles $C(R(v_1))$ and $C(R(v_2))$. Now, $\pm u, \pm R(v_1)$ and $\pm R(v_2)$ are the desired directions finishing the proof of the theorem. □

We remark that in the theorem, we can "almost" choose the second direction arbitrarily, more precisely: Given any two orthogonal vectors $u, v_1 \in \mathbb{S}^2(o)$ and $\varepsilon > 0$, we may find two directions $v_1', v_2' \in \mathbb{S}^2(o)$ such that $\pm u, \pm v_1'$ and $\pm v_2'$ illuminate K, vu, v_1 and v_2 are pairwise orthogonal, and $\|v_1 - v_1'\| < \varepsilon$.
This statement may be derived from the last paragraph of the proof. The set of rotations about the line spanned by u is a one-parameter group parametrized by angle. Each blocking tile rules out at most four angles, and there are finitely many blocking tiles. This argument proves the following statement.

Theorem 11.6. *Let $X \subset \mathbb{R}^3$ be a set of diameter at most one. We choose three pairwise orthogonal directions u, v and w in $\mathbb{S}^2(o)$ randomly with a uniform distribution.*
Then the body $K := \mathbf{B}[X]$ is illuminated by $\{\pm u, \pm v, \pm w\}$ with probability one.

Problem 11.7. *Let $X \subset \mathbb{R}^3$ be a set of diameter at most one. Prove or disprove that $\mathbf{B}[X]$ is illuminated by four directions.*

12. Dowker-Type Isoperimetric Inequalities for Disk-Polygons

In this section, we examine theorems concerning disk-polygons that are analogous to those studied by Dowker in [23] and L. Fejes Tóth in [27] for polygons. The arguments are based on ([27], pp.162-170), but are adapted to the current setting using [10] and [20].

Let $x_0, x_1, \dots, x_n$ be points in the plane such that they are all distinct, except for x_0 and x_n, which are equal. Furthermore, suppose that the distance between each pair of consecutive points is at most two. Next, let $\widehat{x_i x_{i+1}}$ denote one of the two unit circle arcs of length at most π with endpoints x_i and x_{i+1}. A *circle-polygon* is the union of these unit circle arcs, $\widehat{x_0 x_1}, \widehat{x_1 x_2}, \dots, \widehat{x_{n-1} x_n}$. The points $x_0, x_1, \dots, x_n$ are the *vertices* of the circle-polygon and the unit circle arcs $\widehat{x_0 x_1}, \widehat{x_1 x_2}, \dots, \widehat{x_{n-1} x_n}$ are the *edges*, or more commonly the *sides*, of the circle-polygon. Finally, the *underlying polygon* is the polygon formed by joining the vertices, in order, by straight line segments. Observe that both a circle-polygon and its underlying polygon may have self-intersections.

The definition of a standard ball-polytope implies that a disk-polygon is standard if, and only if, there are at least three disks in the reduced family of generating disks. Such a disk-polygon P has well defined vertices and edges. Clearly, $\operatorname{bd} P$

is a circle-polygon with vertices and edges which coincide with those of P. The underlying polygon of an n-sided disk-polygon is just the boundary of the convex hull of the vertices. An n-sided disk-polygon is called *regular* if the underlying polygon is regular.

Let C be a circle of radius $r < 1$. A circle-polygon (resp. disk-polygon) P is *inscribed in* C if $P \subset \operatorname{conv} C$ and the vertices of P lie on C. A circle-polygon (resp. disk-polygon) P is *circumscribed about* C if $C \subset P$ and the interior of each edge of P is tangent to C. A standard compactness argument ensures the existence of an n-sided disk-polygon of largest (resp. smallest) perimeter, as well as one of largest (resp. smallest) area, inscribed in (resp. circumscribed about) C.

Lemma 12.1. *Let C be a circle of radius $r < 1$. Let P_n be an n-sided disk-polygon of largest perimeter inscribed in C. Then*

$$\text{Perimeter}(P_{n-1}) + \text{Perimeter}(P_{n+1}) < 2\,\text{Perimeter}(P_n), \text{ for all } n \geq 4. \tag{12.1}$$

Proof. Let Q be an $(n-1)$-sided disk-polygon and R be an $(n+1)$-sided disk-polygon, both inscribed in C. To prove the theorem we need only construct two n-sided disk-polygons S and T such that

$$\text{Perimeter}(Q) + \text{Perimeter}(R) \leq \text{Perimeter}(S) + \text{Perimeter}(T). \tag{12.2}$$

Without loss of generality we make the following assumptions. First, inscribe Q and R into C so that their respective vertices do not coincide. Second, any arc of C with length at least πr contains a vertex from each of Q and R. Otherwise, there exists an $(n-1)$-sided disk-polygon (resp. $(n+1)$-sided disk-polygon) with larger perimeter than Q (resp. R).

Let x_i and x_{i+1} be two consecutive vertices of a circle-polygon P inscribed in C. Suppose that one of the arcs of C from x_i to x_{i+1} contains neither x_{i-1} nor x_{i+2}. Let $\bar{C}$ denote this arc. A *cap* of C from x_i to x_{i+1}, denoted by $C(x_i, x_{i+1})$, is the segment of $\operatorname{conv} C$ bounded by $\bar{C}$ and the line segment through x_i and x_{i+1}. Now, suppose that there is a cap contained in another cap (see Figure 7). More precisely, there are four vertices a, b, l, m such that the vertices a and b (resp. l and m) form an edge $\widehat{ab}$ (resp. $\widehat{lm}$) and $C(l, m) \subset C(a, b)$.

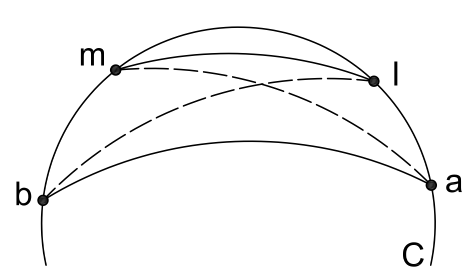

FIGURE 7

This configuration, where a cap is contained in another cap, may arise when two circle-polygons, say A and B, are inscribed in a single circle, or when a single self-intersecting circle-polygon, say U, is inscribed in a circle. Suppose that we start with the former (resp. latter) and as in the figure, assume that the cyclic ordering of the vertices is a, l, m, b. Choosing either of the unit circle arcs joining a to m, we obtain $\widehat{am}$. Similarly, we construct $\widehat{bl}$. After replacing the edges $\widehat{ab}$ and $\widehat{lm}$ with $\widehat{am}$ and $\widehat{bl}$, respectively, we obtain a single self-intersecting circle-polygon, which we call U (resp. two circle-polygons, which we call A and B). By the inequality mentioned in Corollary 2.11, the total perimeter of U (resp. A and B) is strictly larger than the total perimeter of A and B (resp. U).

Starting from the circle-polygons Q and R, we carry out the preceding algorithm for each cap contained in another cap. After every odd numbered iteration of the algorithm we obtain a single, self-intersecting circle-polygon and after every even numbered iteration we obtain two circle-polygons with no self-intersections. Since we have finitely many vertices and the perimeter increases strictly with each step, the algorithm terminates. Furthermore, when it does terminate, there is no cap contained within another cap.

A simple counting argument shows that the process terminates with two circle-polygons. Each one is an n-sided circle-polygon because no cap of one is contained within a cap of the other. Let us denote these n-sided circle-polygons by S^* and T^*. Since the perimeter increased at each step of the process,

$$\text{Perimeter}(Q) + \text{Perimeter}(R) < \text{Perimeter}(S^*) + \text{Perimeter}(T^*). \tag{12.3}$$

Now, it may be the case that some of the edges of S^* (resp. T^*) have relative interior points that meet the convex hull of the underlying polygon. We replace each such edge by the other shorter unit circle arc passing through the same two vertices. It is clear that this produces a disk-polygon with the same perimeter as S^* (resp. T^*). This is the desired S (resp. T). □

Theorem 12.2. *Let C be a circle of radius $r < 1$. Let P be an n-sided disk-polygon of largest perimeter that can be inscribed in C. Then P is regular.*

Proof. Suppose that P is not regular. Starting with P and a suitable rotation of P we modify the argument in the proof of the preceding lemma to construct two n-sided disk-polygons Q and R inscribed in C. By construction, $\text{Perimeter}(Q) + \text{Perimeter}(R) > 2\,\text{Perimeter}(P)$. Hence, one of Q or R has larger perimeter than P. □

Theorem 12.3. *Let C be a circle of radius $r < 1$. Let P be an n-sided disk-polygon of largest area that can be inscribed in C. Then P is regular.*

Proof. Suppose P is not regular. Let P_0 be the regular n-sided disk-polygon with the same perimeter as P. By the discrete isoperimetric inequality for circle-polygons proved in [20], $\text{Area}(P) < \text{Area}(P_0)$. Furthermore, by the preceding theorem, P_0 is inscribed in a circle C_0 with radius $r_0 < r$. Thus, P_1, the regular n-sided disk-polygon inscribed in C, clearly satisfies $\text{Area}(P_0) < \text{Area}(P_1)$ which completes the proof. □

In general, the behavior of the areas of disk-polygons inscribed in a circle is difficult to describe, but we do so in the following special case.

Lemma 12.4. *Let C be a circle of radius $r < 1$. Let P_n be an n-sided disk-polygon of largest area inscribed in C. Then*

$$\text{Area}(P_{n-1}) + \text{Area}(P_{n+1}) < 2\,\text{Area}(P_n), \text{ for all odd } n \text{ and } n \geq 5. \tag{12.4}$$

Proof. By Theorem 12.3, P_{n-1} and P_{n+1} are regular. Since n is odd, both P_{n-1} and P_{n+1} are symmetric about a line through opposite vertices. After an appropriate rotation, P_{n-1} and P_{n+1} share such a line of symmetry d which separates the symmetric sections of each of P_{n-1} and P_{n+1} (see Figure 8).

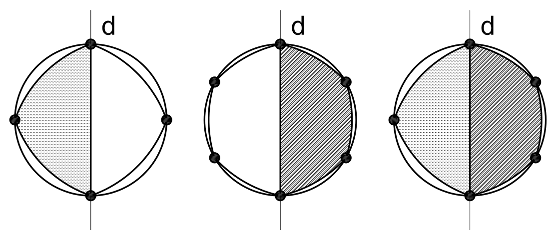

FIGURE 8

Let Q be one half of P_{n-1} lying on one side of d and R the half of P_{n+1} lying on the other side of d. The union of Q and R is an n-sided disk-polygon U inscribed in C. Clearly, $\text{Area}(P_{n-1}) + \text{Area}(P_{n+1}) = 2\,\text{Area}\,U < 2\,\text{Area}(P_n)$ □

There is no straight-forward method to generalize this result to all n-sided disk-polygon. The method described does not apply and no formula is known to describe this area. So, we make the following conjecture.

Conjecture 12.5. *Let C be a circle of radius $r < 1$. Let P_n be an n-sided disk-polygon of largest area inscribed in C. Then*

$$\text{Area}(P_{n-1}) + \text{Area}(P_{n+1}) < 2\,\text{Area}(P_n), \text{ for all } n \geq 4. \tag{12.5}$$

We now turn our attention to disk-polygons which are circumscribed about a circle. A modification of [27] p. 163 similar to the one described above provides the following theorem.

Theorem 12.6. *Let C be a circle of radius $r < 1$.*

(i) Let P_n be an n-sided disk-polygon of smallest area circumscribed about C. Then

$$\text{Area}(P_{n-1}) + \text{Area}(P_{n+1}) > 2\,\text{Area}(P_n), \text{ for all } n \geq 4. \tag{12.6}$$

Furthermore, P_i is regular for all $i \geq 3$.

(ii) Let P_n be an n-sided disk-polygon of smallest perimeter circumscribed about C. Then

$$\text{Perimeter}(P_{n-1}) + \text{Perimeter}(P_{n+1}) > 2\,\text{Perimeter}(P_n), \text{ for all } n \geq 4. \tag{12.7}$$

Furthermore, P_i is regular for all $i \geq 3$.

13. Erdős–Szekeres type Problems for Ball-polytopes

Definition 13.1. Let $A \subset \mathbb{R}^n$ be a finite set contained in a closed unit ball. Then $\operatorname{conv}_s A$ is called a ball-polytope.

In this section we are going to find analogues of some results about convex polytopes for ball-polytopes. We begin with two definitions.

Definition 13.2. Let $A \subset \mathbb{R}^n$ be a finite set. If $x \notin \operatorname{conv}(A \setminus \{x\})$ for any $x \in A$, we say that the points of A are in convex position.

Definition 13.3. For any $n \geq 2$ and $m \geq n+1$, let $f_n(m)$ denote the maximal cardinality of a set $A \subset \mathbb{R}^n$ that satisfies the following two conditions:

(i) any $n+1$ points of A are in convex position,
(ii) A does not contain m points that are in convex position.

In [25] and [26], Erdős and Szekeres proved the existence of $f(m) := f_2(m)$ for every m, and gave the estimates $2^{m-2} \leq f(m) \leq \binom{2m-4}{m-2}$. They conjectured that $f(m) = 2^{m-2}$. Presently, the best known upper bound is $f(m) \leq \binom{2m-5}{m-2} + 1$, given by Tóth and Valtr in [48]. We note that, if the projections of m points of $\mathbb{R}^n$ to an affine subspace are in convex position, then the original points are also in convex position. Thus, the results about $f_2(m)$ imply also that $f_n(m)$ exists, and $f_{n+1}(m) \leq f_n(m)$, for every n and m.

Definition 13.4. Let $A \subset \mathbb{R}^n$ be a finite set contained in a closed unit ball. If $x \notin \operatorname{conv}_s(A \setminus \{x\})$, for every $x \in A$, then we say that the points of A are in spindle convex position.

Definition 13.5. For $n \geq 2$ and $m \geq n+1$, let $g_n(m)$ be the maximal cardinality of a set $A \subset \mathbb{R}^n$ that is contained in a closed unit ball and satisfies the following properties:

(i) any $n+1$ points of A are in spindle convex position,
(ii) A does not contain m points in spindle convex position.

To show the importance of (i) in Definition 13.5, we provide the following example. Let $A := \{x_1, x_2, \ldots, x_k\}$, where $x_1, x_2, \ldots, x_k$ are points of an arc of radius $r > 1$ in this cyclic order. Then any $n+1$ points of A are affine independent whereas A does not contain three points in spindle convex position.

In the remaining part of this section, we show that $f_n(m) = g_n(m)$, for every n and m. Let us assume that $A \subset \mathbb{R}^n$ is a set that satisfies (i) and (ii) in Definition 13.3. Observe that, for a suitably small $\varepsilon > 0$, any $n+1$ points of εA are in spindle convex position. This implies that $f_n(m) \leq g_n(m)$. To show the inequality $f_n(m) \geq g_n(m)$, we prove the following stronger version of Theorem 5.7.

Theorem 13.6. *Let $P \subset \mathbb{R}^n$ be contained in a closed unit ball, and $p \in \operatorname{conv}_s P$. Then $p \in \operatorname{conv} P$ or $p \in \operatorname{conv}_s Q$, for some $Q \subset P$ with $\operatorname{card} Q \leq n$.*

Proof. We show that if $p \notin \operatorname{conv}_s Q$, for any $Q \subset P$ with $\operatorname{card} Q \leq n$, then $p \in \mathbf{B}^n[c, r]$, for any ball $\mathbf{B}^n[c, r]$ that contains P. Since $\operatorname{conv} P$ is the intersection of all the balls that contain P and have radii at least one, this will imply our statement.

We assume that there is a ball $B^n[q, r]$ with $r \geq 1$ that contains P but does not contain p.

If $p \in \operatorname{bd} \operatorname{conv}_s P$, our statement follows from Theorem 5.7. So, let us assume that $p \in \operatorname{int} \operatorname{conv}_s P$. From this and Lemma 3.1, we have $p \in \operatorname{int} \mathbf{B}^n[c, 1] = \mathbf{B}^n(c, 1)$ whenever $P \subset \mathbf{B}^n[c, 1]$. If, for every $r > 1$, there is a ball $\mathbf{B}^n[c_r, r]$ that contains P but does not contain p, then Blaschke's Selection Theorem guarantees the existence of a unit ball $\mathbf{B}^n[c, 1]$ such that $P \subset \mathbf{B}^n[c, 1]$ and $p \notin \mathbf{B}^n(c, 1)$, a contradiction. So, there is an $r > 1$ such that $P \subset \mathbf{B}^n[c, r]$ implies that $p \in \mathbf{B}^n[c, r]$. Clearly, if $1 < r_1 < r_2$ and r_2 satisfies this property then r_1 also satisfies it. Thus, there is a maximal value R satisfying this property. Corollary 3.4 suggests the notation

$$P(r) := \bigcap \{\mathbf{B}^n(c, r) : P \subset \mathbf{B}^n(c, r)\}. \tag{13.1}$$

Observe that $P(r_2) \subset P(r_1)$, for every $1 < r_1 < r_2$, and that $p \in \operatorname{bd} P(R)$. Hence, applying Corollary 3.4 and Theorem 5.7 for $\frac{1}{R}P$, we obtain a set $Q \subset P$ of cardinality at most n such that any ball of radius R that contains Q contains also p. We define $Q(r)$ similarly to $P(r)$ and so we have $Q(R) \subset Q(1) = \operatorname{conv}_s Q$, which implies our statement. $\square$

So, if $P \subset \mathbb{R}^n$ is contained in a closed unit ball, $\operatorname{card} P > f_n(m)$ and any $n + 1$ points of P are in spindle convex position, then P contains m points in convex position, which, according to our theorem, are in spindle convex position.

We note that our theorem implies the spindle convex analogues of numerous other Erdős–Szekeres type results. As examples, we mention [3], [14] and [49].

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